European Journal of
Engineering and Applied Sciences

ISSN: 2651-3412 (Print) & 2667-8454 (Online)
Journal homepage: http://dergipark.gov.tr/EJEAS
Published by Çorlu Faculty of Engineering, Tekirdağ Namık Kemal University

European J. Eng. App. Sci. 3(2), 27-32, 2020

Research Article

Analysis of the HP Memristor and Capacitor (M-C) Series Circuit Using the Lambert W Function

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Received: 18.12.2020 *Accepted: 28.12.2020*

Abstract- Memristor is a new nonlinear circuit element. The HP memristor model is the first and the easiest model to understand memristor given in the literature. Examination of the usage of Memristor with other circuit elements is important for circuit analysis. There are publications in the literature that examine the use of memristor with various circuit elements and examine its usage areas. The circuits in which a capacitor and memristor are connected in series with and without a DC source have been studied in the literature. Lambert W function is a function encountered in system modeling in physics and engineering. In this study, it has been shown that the solution of the capacitor and memristor circuits connected in series with and without a DC source can be expressed using the Lambert W function.

Keywords: *Memristor, memristor-capacitor circuit, circuit analysis, Lambert W function.*

HP Memristör ve Kondansatör (M-C) Seri Devresinin Lambert W Fonksiyonu Kullanarak Analizi

Özet: Memristör yeni ve doğrusal olmayan devre elemanıdır. HP memristör model literatürde verilen ilk ve anlaması en kolay memristör modelidir. Memristörün diğer devre elemanları ile birlikte kullanımın incelenmesi Devre analizi açısından önemlidir. Literatürde memristörün türlü devre elemanları ile kullanımının incelendiği ve kullanım alanlarının incelendiği yayınlar mevcuttur. Bir kondansatör ile memristörün seri bağlı olduğu DC kaynaklı ve kaynaksız devreler literatürde incelenmiştir. Lambert W fonksiyonu fizikte ve mühendislikte system modellemede karşımıza çıkan bir fonksiyondur. Bu çalışmada bir kondansatör ile memristörün seri bağlı olduğu DC kaynaklı ve kaynaksız devrelerin çözümünün Lambert W fonksiyonu kullanarak ifade edilebileceği gösterilmiştir.

Anahtar kelimeler: *Memristör, memristör-kondansatör devresi, devre analizi, Lambert W fonksiyonu.*

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1. Introduction

The memristor's existence was theoretically predicted in [1] and a memristive system that behaves as a memristive system has been found in 2008 [2]. In the last decade, memristor and memristive systems have become a hot research area [4-9]. A memristor has a memory since its resistance which is also called memristance depends on its charge, which is described as its current's integration with respect to time [1]. Memristor has interesting characteristics [1,3,10]. It has potential for not only analog but also digital applications [4-9]. It is highly nonlinear and harder to comprehend compared to the other fundamental circuit element such as linear time-invariant resistor, capacitor, and inductor [1, 2,11, 12]. Its usage with other circuit elements must have also been examined to find new application areas and to optimize the existing ones. HP memristor model is given in [1] and very easy to understand. Its resistance is linearly dependent on charge. Its series combination with an inductor is examined in [13]. Its series usage with a capacitor has been analyzed in [12] without a DC source. The solution of an HP memristor-capacitor circuit is given and used to analyze a memristor-based relaxation generator [14]. Johann Heinrich Lambert first considered the related Lambert's Transcendental Equation in 1758 [15]. Leonard Euler has given the standard form of Lambert W function in [16]. This equation is found in scientific literature commonly. The following examples are given to imply its common occurrence and importance in different engineering problems. Granular and debris flow fronts and deposits, and the fronts of viscous fluids in natural events and in laboratory experiments can be written by using the Lambert W function [17]. It has been realized that the function can be used for linking cerebral blood flow and oxygen consumption changes in a brain voxel, to the corresponding blood oxygenation level-dependent signal in the neuroimaging area [18]. The function was employed for modeling the porous electrode film thickness in a glassy carbon-based supercapacitor for electrochemical energy storage in the field of chemical engineering. The function provides the exact solution for a gas phase thermal activation process where the growth of carbon film and combustion of the same film compete with each other [19-20]. The double-well Dirac delta function models a diatomic hydrogen molecule and its analytical solutions for the energy eigenvalues for the case of symmetric charges are given using Lambert W function [21]. The exact analytical solution for current flow through a diode with series resistance is given using the Lambert W function in [22]. Analytical modeling of magnetically saturated inductance is made with The Lambert W function in [23]. The memristor charge in the studies [12,14] has been found in implicit form, i.e., the memristor charge could not be written as a function of time. In this study, it has been shown that the implicit solutions of memristor charge given in [12,14] can be written using the Lambert W function.

This study is organized as follows. In the second section, a Lambert W function is briefly explained. In the third section, basic information about the HP memristor model is given. In the fourth section, the solutions of the series memristor-capacitor circuits with and without a DC voltage supply are given. In the fifth section, the solutions of the circuits are rewritten using the Lambert W function. The paper is concluded with the last section.

2. A Brief Explanation of Lambert W Function

The Lambert W function was introduced by Lambert in 1978 [24] and Euler transformed it more symmetrical form [25] and gave it as the inverse function of

$$f(W) = We^{W} \quad (1)$$

The Lambert W function is complex valued [26], with a complex argument, Z, and given as

$$Z = W(Z)e^{W(Z)} \quad (2)$$

The Lambert W function has a very complicated structure in the complex plane, but is simply equal to 1 for $R(Z) \geq 1$ and a slightly extended region above and below the real axis. With the domain: $[-1, \infty)$ and range: $\left[-\frac{1}{e}, \infty\right)$, the Lambert W function has the series expansion

$$W(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-2}}{(n-1)!} x^{n} \quad (3)$$

Since the Lambert W function is equal to the inverse function of, f, $W(x) = f^{-1}(x)$, then

$$W(f(x)) = W(xe^{x}) = x \quad (4)$$

$$f(W(x)) = W(x)e^{W(x)} = x \quad (5)$$

3. The HP Memristor Model

Memristor's symbol designated by Chua in [1] is shown in Figure 1. An ideal memristor model is described in [1] and its equations are shown below:

$$v(t) = M(q)i(t) \quad (6)$$

$$\frac{dq}{dt} = i(t) \quad (7)$$

where $v(t)$ is the memristor voltage, $i(t)$ is the memristor current, and $M(q)$ is the memristor's memristance or resistance.

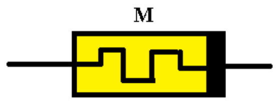

Figure 1. HP Memristor symbol [1]

In 2008, Stanley Williams et al in HP labs have found that a TiO_2 thin-film sandwiched between platinum contacts is a memristive system and behaves as a memristor for at least some part of its operation region [2]. They have given a first-order memristor model, a memristor model with linear dopant drift in [2]. The charge-dependent memristance equation of the TiO_2 memristor according to [2] is given as

$$M(q(t)) = R_{OFF}\left(1-\mu_v \frac{R_{ON}}{D^2} q(t)\right) \tag{8}$$

Where, μ_v is the mobility of oxygen atoms in the memristor, R_{OFF} is the resistance if the memristor region were fully undoped or the maximum resistance of the memristor, R_{ON} is the resistance if the memristor region were fully doped or the minimum resistance of the memristor, D is the total length of the memristor. $M(q(t))$ can also be written as

$$M(q(t)) = M_0 - K\, q(t) \tag{9}$$

Where, M_0 is the maximum memristance, K is the memristance charge coefficient, and *q(t)* is the memristance instantaneous charge.

The memristor charge can be given as the integration of the memristor current with respect to time;

$$q(t) = \int_0^t i(t)\, dt + q(0) \tag{10}$$

where $q(0)$ is the initial charge of the memristor.

When $q(t)$ is equal to q_{sat}, the memristor is under saturation. In this case, the memristance is equal to

$$M_{sat} = M(q_{sat}) = M_0 - K\, q_{sat} \tag{11}$$

4. M-C Circuits

In this section, the M-C circuits given in [12,14] and their solutions are summarized.

4.1 M-C Series Circuit without a Source

There are several studies in the literature on the M-C series circuits and most of them are about filters. Joglekar et al have studied an M-C series circuit without a source with the HP memristor model [12]. Joglekar has made that work with the HP memristor model. Here, an analysis of an M-C circuit without a source is made by Joglekar. The circuit examined in [12] is shown in Figure 2.

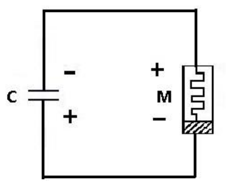

Figure 2. The M-C Series Circuit without a source.

The differential equation describing the circuit is obtained using Kirchoff's Laws as

$$M(q(t))\frac{dq(t)}{dt} + \frac{q(t)}{c} = 0 \tag{12}$$

where *q(t)* is the memristor charge at time *t* and *C* is the capacitor's capacitance.

Joglekar has given the solution of Eq. (12) in [12] as

$$q(t)\, e^{\frac{\eta \Delta R q(t)}{R_F Q_0}} = q(0)\, e^{-\frac{t}{R_F C}}\, e^{\frac{\eta \Delta R q(0)}{R_F Q_0}} \tag{13}$$

where R_F is the initial memristance value when the initial charge of memristor is $q(0)$ at t=0 and η is the memristor polarity variable.

η is equal to +1 for forward direction and -1 for reverse direction. Also, the initial memristance of the memristor is given as $R_F = R_0 - (\eta \Delta R q_0 / Q_0)$.

where $Q_0 = D^2/\mu_D R_{ON}$ and Q_0 is the charge that is required to pass through the memristor for the dopant boundary to move through distance *D*.

4.2 M-C series circuit with a DC voltage source

Another study on the analysis of an M-C circuit excited by a constant voltage source is done by Mutlu [14]. Its circuit diagram is given in Figure 3.

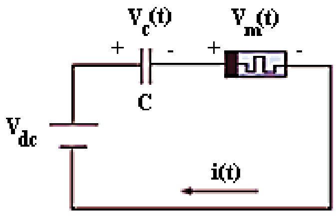

Figure 3. The M-C Series Circuit supplied by a constant voltage source [14].

The differential equation describing the circuit is obtained using Kirchoff's Laws as

$$V_{dc} = M(q(t))\frac{dq(t)}{dt} + \frac{q(t)}{c} \tag{14}$$

Mutlu has given the solution of Eq. (14) in [14], if the memristor is not saturated, as

$$t = C(M_0 - KB - KCV_{dc})\ln\left(\frac{CV_{dc} - q(0) + B}{CV_{dc} - q + B}\right) + KC(q - q(0)) \tag{15}$$

where M_0 is the maximum memristance and K is the memristor's charge coefficient.

In [14], the relationship between the capacitor charge and the unsaturated memristor is given as

$$q(t) = q_c(t) + B \tag{16}$$

where B is the integration constant and equal to

$$B = q(0) - q_c(0) \tag{17}$$

where $q(0)$ and $q_c(0)$ are equal to the initial memristor charge and the initial capacitor charge at t=0, respectively.

5. Solutions of the Joglekar's and Mutlu's M-C Circuits Using Lambert W Function

In this section, the implicit solutions of the circuits in the previous section are rewritten using the Lambert W function. The solution given in Eq. (13), can be written as using the Lambert W function as follows. Replacing $\frac{\eta\Delta R}{R_F Q_0} = A$, Eq. (13)

turns into:

$$q(t)\, e^{\frac{\eta \Delta R q(t)}{R_F Q_0}} = q(0) e^{-\frac{t}{R_F C}} e^{\frac{\eta \Delta R q(0)}{R_F Q_0}} \tag{18}$$

$$q(t) e^{Aq(t)} = q(0) e^{-\frac{t}{R_F C}} e^{Aq(0)} \tag{19}$$

By multiplying both sides of Eq. (19) by A, then we get:

$$Aq(t) e^{Aq(t)} = Aq(0) e^{-\frac{t}{R_F C}} e^{Aq(0)} \tag{20}$$

By using Lambert W function:

$$W\left(Aq(t) e^{Aq(t)}\right) = W\left(Aq(0) e^{-\frac{t}{R_F C}} e^{Aq(0)} \right) \tag{21}$$

$$Aq(t) = W\left(Aq(0) e^{-\frac{t}{R_F C}} e^{Aq(0)} \right) \tag{22}$$

Therefore, the memristor charge of the M-C circuit without a voltage source is found as

$$q(t) = \frac{1}{A} W\left(Aq(0) e^{-\frac{t}{R_F C}} e^{Aq(0)} \right) \tag{23}$$

$$q(t) = \frac{R_F Q_0}{\eta \Delta R} W\left(\frac{\eta \Delta R q(0)}{R_F Q_0} e^{-\frac{t}{R_F C}} e^{\frac{\eta \Delta R q(0)}{R_F Q_0}} \right) \tag{24}$$

Eq. (24) is plotted and shown in Figure 4 for $\eta = 1$, $\Delta R = 100\,\Omega$, $R_F = 1000\,\Omega$, $Q_0 = 1.5\,\mu C$, $q(0) = Q_0/2$, C=500 nF, and $V_C(0)$=10 V.

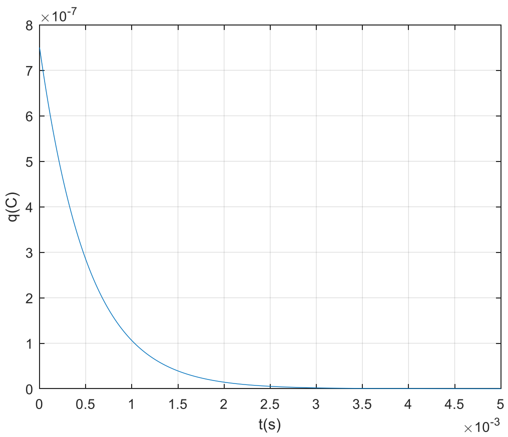

Figure 4. The memristor charge vs. time for $\eta = 1$, C=500 nF, $\Delta R = 100\,\Omega$, $R_F = 1000\,\Omega$, $Q_0 = 1.5\,\mu C$, $q(0) = Q_0/2$, and $V_C(0)$=10 V.

The solution given in Eq. (15) can be written using Lambert W function as the follows. Replacing

$$C(M_0 - KB - KCV_{dc}) = D \tag{25}$$

and

$$CV_{dc} + B = E \tag{26}$$

then

$$t = C(M_0 - KB - KCV_{dc}) \ln\left(\frac{CV_{dc} - q(0) + B}{CV_{dc} - q(t) + B} \right) + KC(q(t) - q(0)) \tag{27}$$

$$t = D \ln\left(\frac{E - q(0)}{E - q} \right) + KC(q - q(0)) \tag{28}$$

$$t = D\ln(E - q(0)) - D\ln(E - q) + KC(q - q(0)) \tag{29}$$

$$D\ln(E - q) = D\ln(E - q(0)) - t + KCq - KCq(0) \tag{30}$$

Replacing

$$D\ln(E - q(0)) - t - KCq(0) = F \tag{31}$$

Then,

$$D\ln(E - q) == F + KCq \tag{32}$$

$$e^{D\ln(E-q)} = e^{F + KCq} \tag{33}$$

$$e^{D} e^{\ln(E-q)} = e^{F} e^{KCq} \tag{34}$$

$$e^{D} (E - q) e^{-KCq} = e^{F} \tag{35}$$

$$KC(E - q) e^{-KCq} = KC\, e^{F-D} \tag{36}$$

$$KC(E - q) e^{-KCq} e^{KCE} = KC\, e^{F-D} e^{KCE} \tag{37}$$

$$KC(E - q) e^{KC(E-q)} = KC\, e^{F-D+KCE} \tag{38}$$

By using Lambert W function,

$$W\left(KC(E - q) e^{KC(E-q)}\right) = W\left(KC\, e^{F-D+KCE}\right) \tag{39}$$

$$KC(E - q) = W\left(KC\, e^{F-D+KCE}\right) \tag{40}$$

$$q = E - \frac{1}{KC} W\left(KC\, e^{F-D+KCE}\right) \tag{41}$$

Therefore, the memristor charge of the M-C circuit with a DC voltage source is found as

$$q(t) = E - \frac{1}{KC} W\left[KC\, e^{D\ln(E-q(0))} - t + KCq(0) - C(M_0 - KB - KCV_{dc}) + KC(CV_{dc} + B) \right] \tag{42}$$

Eq. (42) is plotted for $= 900000000\,\Omega/\mathrm{C}$, $\mathrm{C} = 500$ nF, $M_0 = 1500\,\Omega$, q(0)= 0 C, $q_{\mathrm{sat}} = 1.5\,\mu$C, $V_C(0)$=5V and V_{dc} =10 V and shown in Figure 5.

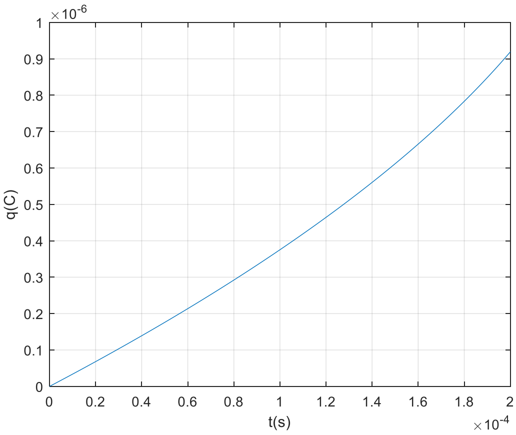

Figure 5. The memristor charge vs. time for $C = 500$ nF, $K = 900000000\ \Omega/C$, $M_0 = 1500\ \Omega$, q(0)= 0 C, $q_{sat} = 1.5$ µC, $V_C(0)$=5 V, and V_{dc} =10 V.

6. Conclusions

The Lambert W function is not commonly known among electrical and electronics engineers since it is not part of their curriculum. In literature, solutions to some electrical problems have already made use of the Lambert W function [22-23]. The M-C series circuits have already been examined using the HP memristor model and the solutions of memristor charge have already been given implicitly in [12, 14]. In this study, it has been shown that the solutions can be written using Lambert W function and, therefore, a new example for the usage of Lambert W function to model physical systems is added to Literature. We suggest the analysis of combinations of linear circuit elements with other circuit elements memory such as memcapacitor and meminductor using the Lambert W function as a future study. In literature, various memristor-based low-pass and high-pass filters are examined [27-29]. The square-wave response of the filters given in [27-28] may be found using Lambert W function for small signals when the memristor is not under saturation since the memristor and capacitor are in series in these filters. We suggest their analysis as a future study.

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