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Weak Semilocal Convergence Conditions for a Two-Step Newton Method in Banach Space

Ioannis K. Argyros[a] and Santhosh George[b*]

[a]*Department of Mathematical Sciences, Cameron University Lawton, OK 73505, USA*
[b]*Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India-757 025*
[*]*Corresponding author*

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Abstract

We present new sufficient convergence conditions for a two step Newton method (TSNM) to solve nonlinear equations in a Banach space setting. The new conditions depend on the center-Lipschitz constant instead of the Lipschitz constant. This way the applicability of (TSNM) is expanded in cases not covered before. Numerical examples are also provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.
Many problems from computational sciences and other disciplines can be brought in a form similar to equation (1.1) using mathematical modeling [5, 13, 18, 26, 29, 30]. The solution of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. In applied sciences the practice of Numerical Analysis for finding solutions x^* of equation (1.1) is essentially connected to variants of Newton's method [1]-[46].
The basic idea of Newton's method is linearization. Starting from an initial guess, we can have the linear approximation of $F(x)$ in the neighborhood of $x_0 : F(x_0+h) \approx F(x_0)+F'(x_0)h$, and solve the resulting linear equation $F(x_0)+F'(x_0)h = 0$, leading to the recurrent Newton method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{1.2}$$

for each $n = 0,1,2,\cdots$. This is Newton's method as proposed in 1669 by I.Newton (for polynomial only) defined on the real line. It was J. Raphson, who proposed the usage of Newton's method for general functions. That is why the method is often called the Newton-Raphson method. Later in 1818, Fourier proved that the method converges quadratically in a neighborhood of the root, while Cauchy (1829, 1847) provided the multidimensional extension of Newton's method (1.3). In 1948, L.V. Kantorovich published an important paper [26], extending Newton's method for functional spaces (the Newton-Kantorovich method (NKM)). Ever, since thousands of papers have been written in a Banach space setting for the (NM) as well as Newton-type methods, and their applications. We refer the reader to the publications [5, 13] for recent results (see also, the references therein).
The study about convergence matter of iterative procedures is usually based on two types: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls.

Email addresses and ORCID numbers: **ioannisa@cameron.edu, 0000-0002-9189-9298 (I. K. Argyros), sgeorge@nitk.ac.in, 0000-0002-3530-5539 (S. George)**

In order to increase the order of convergence the two step Newton method (TSNM)

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(y_n) \end{aligned}$$

for each $n = 0,1,2,\cdots$ has been used to approximate x^* [1]-[46]. (TSNM) has convergence order four. Let $U(x,r)$ denote an open ball with center $x \in X$ and of radius $r > 0$. Let $\overline{U(x,r)}$ denote the closure of $U(x,r)$. Let also $L(X,Y)$ denote the space of bounded linear operators from X into Y.
The Newton-Kantorovich theorem is a semilocal convergence result for solving nonlinear equations using (NM) or (TSNM) asserts that if there exist $x_0 \in D$, $\eta > 0$ and $L > 0$ such that

$$F'(x_0)^{-1} \in L(Y,X), \quad \|F'(x_0)^{-1}F'(x_0)\| \leq \eta \tag{1.3}$$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq L\|x-y\| \tag{1.4}$$

for each x and y in D

$$h = 2L\eta \leq 1 \tag{1.5}$$

and

$$\overline{U(x_0,R)} \subseteq D, \tag{1.6}$$

where

$$R = \frac{1-\sqrt{1-h}}{L} \tag{1.7}$$

then, (NM) or (TSNM) converges to x^*. Error estimates on the distances can be found in [5, 6, 13], [19]-[24] and the references therein. However, there are simple examples (see numerical examples at the end of the study) where (1.4) or (1.5) are not satisfied. In these cases the Newton-Kantorovich theorem cannot guarantee that (NM) or (TSNM) converge although these methods may converge. This is happening because (1.5) is only a sufficient convergence condition for (NM) or (TSNM). In the present paper we expand the applicability of (TSNM) by weakening (1.4) or (1.5). Relevant work for (NM) or (TSNM) can be found in [1]- [24].
In particular we replace Lipschitz condition (1.4) by the center-Lipschitz condition

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x-x_0\| \tag{1.8}$$

for each $x \in D$, (1.5) by

$$h_0 = (9+4\sqrt{5})L_0\eta \leq 1 \tag{1.9}$$

and (1.6) by

$$\overline{U(x_0,r)} \subseteq D, \tag{1.10}$$

where

$$r = \frac{2\eta}{1+L_0\eta+\sqrt{(1+L_0\eta)^2-20L_0\eta}}. \tag{1.11}$$

Note that

$$L_0 \leq L \tag{1.12}$$

holds in general and $\frac{L_0}{L}$ can be arbitrarily small (see Example 3.3).
We also have

$$h \leq 1 \text{ and } (9+4\sqrt{5})L_0 \leq 2L \Rightarrow h_0 \leq 1 \tag{1.13}$$

and

$$\frac{h_0}{h} \to 0 \text{ as } \frac{L_0}{L} \to 0. \tag{1.14}$$

Hence, in these cases the applicability of (TSNM) is extended under weaker hypotheses since the computation of constant L_0 is less expensive than the computation of constant L. Note also that we may have

$$h_0 \leq 1 \text{ and } (9+4\sqrt{5})L_0 \geq 2L \Rightarrow h \leq 1 \tag{1.15}$$

Therefore, in practice we shall choose the condition that is satisfied (if any).
The paper is organized as follows. In Section 2 we present the semilocal convergence of (TSNM). The numerical examples are given in Section 3.

2. Semilocal Convergence of (TSNM)

We need the following Ostrowski-type representations for (TSNM).

Lemma 2.1. *Suppose that sequences $\{x_n\}, \{y_n\}$ generated by (TSNM) are well defined for each $n = 0, 1, 2, \cdots$. Then, the following assertions hold for each $n = 1, 2, \cdots$*

$$x_n - y_{n-1} = \Gamma_1 + \Gamma_2, \tag{2.1}$$

where

$$\begin{aligned}\Gamma_1 &= F'(y_{n-1})^{-1}[F'(y_{n-1})(y_{n-1} - x_{n-1}) - F(y_{n-1}) + F(x_{n-1})] \\ &= F'(y_{n-1})^{-1}\int_0^1 [F'(y_{n-1} + t(x_{n-1} - y_{n-1})) - F'(y_{n-1})] \\ &\quad \times (x_{n-1} - y_{n-1})dt \qquad (2.2)\\ \Gamma_2 &= F'(y_{n-1})^{-1}(F'(y_{n-1}) - F'(x_{n-1}))(x_{n-1} - y_{n-1}) \qquad (2.3)\end{aligned}$$

and

$$y_n - x_n = \Gamma_3 + \Gamma_4, \tag{2.4}$$

where

$$\begin{aligned}\Gamma_3 &= F'(x_{n-1})^{-1}[F'(x_n)(x_n - y_{n-1}) - F(x_n) + F(y_{n-1})] \\ &= F'(x_{n-1})^{-1}\int_0^1 [F'(x_n + t(y_{n-1} - x_n)) - F'(x_n)] \\ &\quad \times (y_{n-1} - x_n)dt \qquad (2.5)\\ \Gamma_4 &= F'(x_n)^{-1}(F'(x_n) - F'(y_{n-1}))(y_{n-1} - x_n). \qquad (2.6)\end{aligned}$$

Proof. Use (TSNM) and Taylor's formula.
We can show the main semilocal convergence results for (TSNM).

Theorem 2.2. *Let $F : D \to Y$ be Fréchet differentiable. Suppose that (1.3), (1.8)- (1.11) hold. Then, sequence $\{x_n\}, \{y_n\}$ generated by (TSNM) are well defined, remain in $\overline{U(x_0, r)}$ for each $n = 0, 1, 2, \cdots$ and converge to a solution $x^* \in \overline{U(x_0, r)}$ of equation $F(x) = 0$. Moreover, the following estimates hold for each $n = 1, 2, \cdots$*

$$\begin{aligned}\|x_n - y_{n-1}\| &\le \frac{L_0}{2(1 - L_0\|y_{n-1} - x_0\|)}[5\|y_{n-1} - x_0\| \\ &\quad + 3\|x_{n-1} - x_0\|]\|x_{n-1} - y_{n-1}\|, \qquad (2.7)\\ \|x_n - x_{n-1}\| &\le [1 + \frac{L_0}{2(1 - L_0\|y_{n-1} - x_0\|)}(5\|y_{n-1} - x_0\| \\ &\quad + 3\|x_{n-1} - x_0\|)]\|x_{n-1} - y_{n-1}\|, \qquad (2.8)\\ \|x_n - y_n\| &\le \frac{L_0}{2(1 - L_0\|x_{n-1} - x_0\|)}[5\|x_{n-1} - x_0\| \\ &\quad + 3\|y_{n-1} - x_0\|]\|x_n - y_{n-1}\|, \qquad (2.9)\end{aligned}$$

and

$$\|x^* - x_n\| \le \frac{(1 + b)b^{2n}}{1 - b^2}\eta, \tag{2.10}$$

where

$$b = b(r) = \frac{4L_0 r}{1 - L_0 r} < 1. \tag{2.11}$$

Furthermore, if there exists $R \ge r$ such that

$$\overline{U(x_0, R)} \subseteq D \tag{2.12}$$

and

$$L_0(R + r) < 2, \tag{2.13}$$

then the solution x^* is unique in $\overline{U(x_0, R)}$.

Proof. We have by (TSNM) for $n=0$, (1.3) and (1.10) that $\|y_0-x_0\|\le\eta\le r$, which shows that $y_0\in\overline{U(x_0,r)}$. Let $x\in\overline{U(x_0,r)}$. Then, using the center-Lipschitz condition we get that

$$\|F'(x_0)^{-1}(F'(x)-F'(x_0))\|\le L_0\|x-x_0\|\le L_0r<1 \tag{2.14}$$

by the choice of r. It follows from (2.14) and Banach Lemma on invertible operators [5]-[26] that $F'(x)^{-1}\in L(Y,X)$ and

$$\|F'(x)^{-1}F'(x_0)\|\le\frac{1}{1-L_0\|x-x_0\|}\le\frac{1}{1-L_0r}. \tag{2.15}$$

In particular (2.15) holds for $x=y_0$. Then, by (TSNM) x_1 is well defined. Let us assume that $x_k,y_k\in\overline{U(x_0,r)}$. Then, we have using Lemma 2.1, the center-Lipschitz condition and (2.15) that

$$\begin{aligned}
\|\Gamma_1\| &\le \|F'(y_{k-1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}\int_0^1\{[F'(x_{k-1}+t(y_{k-1}-x_{k-1}))\\
&\quad -F'(x_0)]+[F'(x_0)-F'(y_{k+1})]\}(x_{k-1}-y_{k-1})dt\|\\
&\le \frac{L_0}{1-L_0\|y_{k-1}-x_0\|}[\frac{\|y_{k-1}-x_0\|+\|x_{k-1}-x_0\|}{2}\\
&\quad +\|y_{k-1}-x_0\|]\|x_{k-1}-y_{k-1}\|\\
&\le \frac{L_0}{2(1-L_0\|y_{k-1}-x_0\|)}(3\|y_{k-1}-x_0\|+\|x_{k-1}-x_0\|)\\
&\quad \|x_{k-1}-y_{k-1}\|,\\
\|\Gamma_2\| &\le \|F'(y_{k-1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}(F'(y_{k-1})-F'(x_{k-1}))\|\\
&\quad \|x_{k-1}-y_{k-1}\|\\
&\le \frac{L_0}{1-L_0\|y_{k-1}-x_0\|}[\|y_{k-1}-x_0\|+\|x_{k-1}-x_0\|]\\
&\quad \|x_{k-1}-y_{k-1}\|,
\end{aligned}$$

so, since $\|x_k-y_{k-1}\|\le\|\Gamma_1\|+\|\Gamma_2\|$, is obtained by adding the preceding two inequalities. To show (2.8) we use (2.7) and the triangle inequality

$$\|x_k-x_{k-1}\|\le\|x_k-y_{k-1}\|+\|y_{k-1}-x_{k-1}\|.$$

As in the computation of $\|\Gamma_1\|$ and $\|\Gamma_2\|$, we get in turn that

$$\begin{aligned}
\|\Gamma_3\| &\le \frac{L_0}{2(1-L_0\|x_{k-1}-x_0\|)}(3\|x_{k-1}-x_0\|+\|y_{k-1}-x_0\|)\\
&\quad \|x_k-y_{k-1}\|,\\
\|\Gamma_4\| &\le \frac{L_0}{1-L_0\|x_{k-1}-x_0\|}[\|x_k-x_0\|+\|y_{k-1}-x_0\|]\\
&\quad \|x_k-y_{k-1}\|
\end{aligned}$$

and since $\|y_k-x_k\|\le\|\Gamma_3\|+\|\Gamma_4\|$, we obtain (2.9). It then follows from (2.9) and (2.11) that

$$\begin{aligned}
\|y_k-x_k\| &\le \frac{L_0}{1-L_0r}8r\|x_k-y_{k-1}\|\\
&\le b^2\|x_{k-1}-y_{k-1}\|\le b^{2k}\|x_0-y_0\|\\
&\le b^{2k}\eta.
\end{aligned}$$

We also have that for $m=0,1,2,\cdots$

$$\|x_{k+m}-x_k\|\le\|x_{k+m}-x_{m+k-1}\|+\|x_{k+m-1}-x_{m+k-2}\|+\cdots+\|x_{k+1}-x_k\|$$

but

$$\|x_{k+m}-x_{m+k-1}\|\le(1+b)\|x_{k+m-1}-y_{m+k-1}\|\le(1+b)b^{2(k+m-1)}\eta,$$

so,

$$\begin{aligned}
\|x_{k+m}-x_k\| &\le (1+b)(b^{2(k+m-1)}+\cdots+b^{2k})\eta\\
&\le (1+b)b^{2k}\frac{1-b^{2m}}{1-b^2}\eta\le\frac{\eta}{1-b}=r.
\end{aligned}$$

It follows that sequence $\{x_k\}$ is complete in a Banach space X and as such it converges to some $x^*\in\overline{U(x_0,r)}$ (since $\overline{U(x_0,r)}$ is closed). By letting $m\to\infty$ in the preceding inequality we get (2.10). In particular the preceding inequality for $k=0$ gives that sequence $x_m\in\overline{U(x_0,r)}$ for each $m=0,1,2\cdots$. We also have that

$$\begin{aligned}
\|y_k-x_0\| &\le \|y_k-x_k\|+\|x_k-x_0\|\\
&\le b^{2k}\eta+(1+b)\frac{1-b^{2k}}{1-b^2}\eta\\
&\le \frac{1-b^{2k+1}}{1-b}\eta<\frac{\eta}{1-b}=r.
\end{aligned}$$

That is $y_k \in \overline{U(x_0,r)}$ for each $k = 1,2,\cdots$.
In order for us to show that $F(x^*) = 0$, we use the approximation

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)(y_k - x_k) \\ &= (F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k)) - F'(x_k)(y_k - x_{k+1}), \end{aligned}$$

so

$$\begin{aligned} F'(x_0)^{-1}F(x_{k+1}) &= F'(x_0)^{-1}\int_0^1 (F'(x_k + t(x_{k+1} - x_k)) - F'(x_k))(x_{k+1} - x_k)dt \\ &\quad + (I - F'(x_0)^{-1}(F'(x_k) - F'(x_0)))(y_k - x_{k+1}). \end{aligned}$$

Then, we get that

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{k+1})\| &\leq \frac{L_0}{2}(\|x_{k+1} - x_0\| + 3\|x_k - x_0\|)\|x_{k+1} - x_k\| \\ &\quad + (1 + L_0\|x_k - x_0\|)\|x_{k+1} - y_k\| \\ &\leq 2L_0 r\|x_{k+1} - x_k\| + (1 + L_0 r)\|x_{k+1} - y_k\|. \end{aligned}$$

But $\{x_k\}$ is a complete sequence and $\|x_{k+1} - y_k\| \to 0$ as $k \to \infty$ (by (2.7)). Hence, $\|F'(x_0)^{-1}F(x_{k+1})\| \to 0$ as $k \to \infty$. That is $F(x^*) = 0$.
Finally, to show the uniqueness part, let $y^* \in \overline{U(x_0,R)}$ be such that $F'(y^*) = 0$. Let $M = \int_0^1 F'(y^* + t(x^* - y^*))dt$. Then, we have that

$$\begin{aligned} \|F'(x_0)^{-1}(M - F'(x_0))\| &\leq L_0\int_0^1 \|y^* + t(x^* - y^*) - x_0\|dt \\ &\leq \frac{L_0}{2}(\|x^* - x_0\| + \|y^* - x_0\|) \\ &\leq \frac{L_0}{2}(r + R) < 1. \end{aligned}$$

It follows that $M^{-1} \in L(Y,X)$. Then, in view of the identity

$$0 = F(x^*) - F(y^*) = M(x^* - y^*)$$

we deduce that $x^* = y^*$. The proof of the Theorem is complete.

Remark 2.3. *The Newton-Kantorovich hypothesis (1.5) has been used in the literature [1]-[46], as the sufficient convergence condition for both (NM) and the modified Newton's method (MNM)*

$$z_{n+1} = z_n - F'(z_0)^{-1}F(z_n) \quad \text{for each} \quad n = 0,1,2,\cdots.$$

In [6] we showed that (1.5) can be replaced by

$$h_1 = 2L_0\eta \leq 1 \tag{2.16}$$

and

$$\overline{U(x_0,r_1)} \subseteq D, \tag{2.17}$$

where

$$r_1 = \frac{1 - \sqrt{1 - h_1}}{L_0}. \tag{2.18}$$

Note that

$$h_0 \leq 1 \Rightarrow h_1, \quad h \leq 1 \Rightarrow h_1 \leq 1 \tag{2.19}$$

and

$$\frac{h_1}{h} \to 0 \ \text{ as } \ \frac{L_0}{L} \to 0. \tag{2.20}$$

Hence, in cases (1.5) or (1.9) are not satisfied but (2.16) is satisfied we can start with linearly convergent (MNM) until a certain iterate z_n (or (1.5)) is satisfied, then we continue with faster (TSNM) [6].

Remark 2.4. *The convergence order of (TSNM) is expected to be four. In Theorem 2.2 the error bounds are too pessimistic. That is why in practice we shall use the computational order of convergence (COC) (see eg. [14]) defined by*

$$\rho \approx \ln\left(\frac{\|x_{n+1} - x_\alpha^\delta\|}{\|x_n - x_\alpha^\delta\|}\right) / \ln\left(\frac{\|x_n - x_\alpha^\delta\|}{\|x_{n-1} - x_\alpha^\delta\|}\right).$$

The (COC) ρ will then be close to 4 which is the order of convergence of (TSNM).

3. Numerical Examples

Example 3.1. *Let $X=Y=\mathbb{R}$, $D=[0,\infty), x_0=1$ and define function F on D by*

$$F(x)=\frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}}+c_1x+c_2, \tag{3.1}$$

where c_1,c_2 are real parameters and $i>2$ an integer. Then $F'(x)=x^{1/i}+c_1$ is not Lipschitz on D. However central Lipschitz condition $(C1)'$ holds for $L_0=1$.
Indeed, we have

$$\begin{aligned}\|F'(x)-F'(x_0)\| &= |x^{1/i}-x_0^{1/i}|\\ &= \frac{|x-x_0|}{x_0^{\frac{i-1}{i}}+\cdots+x^{\frac{i-1}{i}}}\\ &\leq L_0|x-x_0|.\end{aligned}$$

Example 3.2. *We consider the integral equations*

$$u(s)=f(s)+\tau\int_a^b G(s,t)u(t)^{1+1/n}dt, \quad n\in\mathbb{N}. \tag{3.2}$$

Here, f is a given continuous function satisfying $f(s)>0, s\in[a,b], \tau$ is a real number, and the kernel G is continuous and positive in $[a,b]\times[a,b]$.
For example, when $G(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \tau u^{1+1/n} \tag{3.3}$$
$$u(a) = f(a), u(b)=f(b). \tag{3.4}$$

These type of problems have been considered in [5], [9]-[13], [18]-[22].
Equation of the form (3.2) generalize equations of the form

$$u(s)=\int_a^b G(s,t)u(t)^n dt \tag{3.5}$$

studied in [5], [13], [20]. Instead of (3.2) we can try to solve the equation $F(u)=0$ where

$$F:\Omega\subseteq C[a,b]\to C[a,b], \Omega=\{u\in C[a,b]: u(s)\geq 0, s\in[a,b]\},$$

and

$$F(u)(s)=u(s)-f(s)-\tau\int_a^b G(s,t)u(t)^{1+1/n}dt.$$

The norm we consider is the max-norm.
The derivative F' is given by

$$F'(u)v(s)=v(s)-\tau(1+\frac{1}{n})\int_a^b G(s,t)u(t)^{1/n}v(t)dt, \quad v\in\Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b]=[0,1], G(s,t)=1$ and $y(t)=0$. Then $F'(y)v(s)=v(s)$ and

$$\|F'(x)-F'(y)\|=|\tau|(1+\frac{1}{n})\int_a^b x(t)^{1/n}dt.$$

If F' were a Lipschitz function, then

$$\|F'(x)-F'(y)\|\leq L_1\|x-y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n}dt\leq L_2\max_{x\in[0,1]}x(s), \tag{3.6}$$

would hold for all $x\in\Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t)=\frac{t}{j}, \quad j\geq 1, \quad t\in[0,1].$$

If these are substituted into (3.6)

$$\frac{1}{j^{1/n}(1+1/n)}\leq\frac{L_2}{j}\Leftrightarrow j^{1-1/n}\leq L_2(1+1/n), \quad \forall j\geq 1.$$

This inequality is not true when $j\to\infty$.

Therefore, condition (3.6) is not satisfied in this case. However, condition (1.8) holds. To show this, let $x_0(t) = f(t)$ *and* $\gamma = \min_{s\in[a,b]} f(s), \alpha > 0$. *Then for* $v \in \Omega$,

$$\begin{aligned} \|[F'(x)-F'(x_0)]v\| &= |\tau|(1+\frac{1}{n}) \max_{s\in[a,b]} |\int_a^b G(s,t)(x(t)^{1/n} - f(t)^{1/n})v(t)dt| \\ &\leq |\tau|(1+\frac{1}{n}) \max_{s\in[a,b]} G_n(s,t) \end{aligned}$$

where $G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n} f(t)^{1/n} + \cdots + f(t)^{(n-1)/n}} \|v\|$. *Hence,*

$$\begin{aligned} \|[F'(x)-F'(x_0)]v\| &= \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b G(s,t)dt \|x-x_0\| \\ &\leq L_0\|x-x_0\|, \end{aligned}$$

where $L_0 = \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}} N$ *and* $N = \max_{s\in[a,b]} \int_a^b G(s,t)dt$.

Example 3.3. *Let* $X = D(F) = \mathbb{R}, x_0 = 0$, *and define function* F *on* $D(F)$ *by*

$$F(x) = d_0 x + d_1 + d_2 \sin e^{d_3 x}, \tag{3.7}$$

where $d_i, i = 0,1,2,3$ *are given parameters. Then, it can easily be seen that for* d_3 *sufficiently large and* d_2 *sufficiently small,* $\frac{L_0}{L}$ *can be arbitrarily small.*

Example 3.4. *Let* $X = Y = \mathbb{R}, x_0 = 1, D = \overline{U(x_0, 1-p)}$ *for* $p \in (0, \frac{1}{2})$ *and define* F *on* D *by*

$$F(x) = x^3 - p. \tag{3.8}$$

Then, using (1.3), (1.4), (1.8) and (3.8) we obtain that

$$\eta = \frac{1-p}{3}, L_0 = 3 - p < L = 2(2-p).$$

Hence, there is no guarantee that (TSNM) converges to x^* *in these cases. Then, by (1.5) and (1.9) we get that* $h > 1$ *and* $h_0 > 1$ *for each* $p \in (0, \frac{1}{2})$. *However, using (2.16) we get that*

$$h_1 \leq 1 \quad \text{for each} \quad p \in [0.418861170, 0.5).$$

Note that we have that [6]

$$\|z_{n+1} - z_n\| \leq q\|z_n - z_{n-1}\| \quad \text{for each} \quad n = 1, 2, \cdots,$$

where $q = 1 - \sqrt{1-h_1}$.
Let us choose $p = 0.48$. *Then, we have that* $L_0 = 2.52, L = 3.04, \eta = 0.17333\cdots, h_1 = 0.8735999\cdots$ *and* $q = 0.644472221$. *Using the estimates*

$$\begin{aligned} \|F'(z_N)^{-1}(F'(x)-F'(y))\| &\leq \|F'(z_N)^{-1}F'(z_0)\| \\ &\quad \times \|F'(z_0)^{-1}(F'(x)-F'(y))\| \\ &\leq \frac{L}{1-L_0\|z_N - z_0\|}\|x-y\| \\ &\leq \frac{L}{1-L_0 r_1}\|x-y\| \\ &= \frac{L}{\sqrt{1-h_1}}\|x-y\| \end{aligned}$$

and

$$\begin{aligned} \|F'(z_N)^{-1}(F'(x)-F'(z_0))\| &\leq \|F'(z_N)^{-1}F'(z_0)\| \\ &\quad \times \|F'(z_0)^{-1}(F'(x)-F'(z_0))\| \\ &\leq \frac{L}{1-L_0\|z_N - z_0\|}\|x-z_0\| \\ &\leq \frac{L}{1-L_0 r_1}\|x-z_0\| \\ &= \frac{L}{\sqrt{1-h_1}}\|x-z_0\| \end{aligned}$$

Therefore, we can set

$$\bar{L} = \frac{L}{\sqrt{1-h_1}} \quad \text{and} \quad \bar{L_0} = \frac{L_0}{\sqrt{1-h_1}}.$$

we then have $\bar{L} = 8.550668013$ *and* $\bar{L_0} = 7.088053748$. *Hence, estimates (1.5) and (1.9) hold, respectively, if*

$$\bar{h} = 2\bar{L}\eta q^N \leq 1$$

and

$$\bar{h_0} = (9+4\sqrt{5})\bar{L_0}\eta q^N \leq 1.$$

These inequalities are satisfied, respectively for $N = 3$ and $N = 8$, since they become

$$\bar{h} = 0.793459449 < 1$$

and

$$\bar{h_0} = 0.656097755 < 1.$$

Hence, we must choose $x_0 = z_3$ (under (1.5)) or $x_0 = z_8$ (under (1.9)).

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