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Hypergroups and their pullback and pushout structures

Murat Alp* and Bijan Davvaz†‡

Abstract

Hypergroups in the sense of Marty are very important and a rather difficult subject to be understood since they do not generally have any identity or inverse element. Crossed modules are one of the most important tools to be applied on groups. In this study, we combine hypergroups and crossed modules to obtain the crossed modules of the hypergroups. We shortly present hypergroups with their properties and examples. In addition two important applications of crossed modules are given. These applications are about pullback and pushout crossed module of hypergroups and their properties. The definition of hypergroups generated by sets plays vital roles throughout the paper.

Keywords: Action, Crossed module, Hypergroup, Fundamental relation, Pullback, Pushout.

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1. Introduction

Crossed module is very fruitfull subject in the mathematical theories such as group theory and algebra theory. Crossed module first defined by Whitehead [28] and then many applicable examples were given by different authors such as actor [25], induced [9], pullback [8], pushout [20], polygroup [15] and hypergroups [5] crossed module. The other applications of crossed module were given by the authors as pullback and pushout crossed polymodule in [6] and algebroids in [3, 4]. Many good examples and properties of pullback and pushout crossed polymodule were given in [6]. In this paper, using the light of Brown and Higgins [8] and Korkes and Porter's ways [20], we present the pullback and pushout structures of crossed module of hypergroups. Let $\mathcal{X} = (C, H, \partial, \alpha)$ be a crossed module of hypergroup and $\iota : Q \to H$ be a morphism of hypergroups. Then

*Department of Mathematics, Niğde Ömer Halisdemir University, Niğde, Turkey, Email: muratalp@nigde.edu.tr

†Department of Mathematics, Yazd University, Yazd, Iran, Email: davvaz@yazd.ac.ir

‡Corresponding Author.

$\iota^{\bullet}\mathcal{X} = (\iota^{\bullet}C, Q, \partial^{\bullet}, \alpha^{\bullet})$ is the pullback of $\mathcal{X}$ by ι. The hypergroups action of Q on $\iota^{\bullet}C$ is given by

$$^{q}(q_1, c) \quad = \quad \{(x, y) \ \mid \beta^*_H(x) = \beta^*_H(q) \odot \beta^*_H(q_1) \odot \beta^*_H(q)^{-1},\ y \in {}^{\iota(q)}c\}.$$

To constitute pushout crossed module of hypergroups, we define hypergroups generated by sets which are the important part of this study. Indeed, some new examples of pullback crossed module of hypergroups are presented and pushout construction has been made very smoothly in this paper.

Let $(H, \circ)$, $(C, \star)$ and $(B, \cdot)$ be hypergroups. Let $\partial : C \to H$ and $\delta : K \to H$ be two crossed modules of hypergroups and let $(\phi, Id) : (\partial : C \to H) \to (\delta : K \to H)$ be a morphism of crossed modules of hypergroups. Then, defining a continuous K- action on C by ${}^{k}c = {}^{\delta(k)}c$ we have $\phi : C \to K$ is a pushout crossed module of hypergroups.

Pullback and pushout applications of hypergroups are different than polygroups applications, because of their definition actions on hypergroups are more different than the definition of the polygroup actions. These applications can allow us to obtain very different properties and examples from the polygroup applications [15]. Pullback and pushout applications are simple examples for crossed square according to [18]. After giving a brief introduction, Section 2 includes a brief presentations about hypergroups and fundamental relations. Section 3 describes crossed modules of hypergroups and their properties. Hypergroups and their pullback and pushouts properties are presented in Sections 4 and 5, respectively.

2. Hypergroups and fundamental relations

Hypergroup theory was born in 1934, after Marty [24] gave the definition of hypergroup, illustrated some applications and showed its utility in the study of groups, algebraic functions and relational fractions. Nowadays the hypergroups are theoretically studied for their applications on different subjects of pure and applied mathematics, such as geometry, topology, cryptography and coding theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, ethnology, etc. (see [1, 2, 11, 13, 17]).

Let H be a non-empty set and $\circ : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ \ A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

A hypergroupoid $(H, \circ)$ is called a *quasihypergroup* if for all a of H we have $a \circ H = H \circ a = H$. This condition is also called the *reproduction axiom*.

2.1. Definition. A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup*.

2.2. Definition. Let $(C, \star)$ and $(H, \circ)$ be two hypergroups. Let ∂ be a map from C into H. Then, ∂ is called

(1) an *inclusion homomorphism* if

$$\partial(x \star y) \subseteq \partial(x) \circ \partial(y), \text{ for all } x, y \in C;$$

(2) a *strong homomorphism* or a *good homomorphism* if

$$\partial(x \star y) = \partial(x) \circ \partial(y), \text{ for all } x, y \in C.$$

2.3. Remark. Every group is a hypergroup.

In a hypergroup $(H, \circ)$, an element $e \in H$ is called a *scalar identity element* if $e \circ x = x \circ e = \{x\} := x$, for all $x \in H$.

Here, we present two examples of hypergroups.

2.4. Example. Suppose that $G = \{1, -1, i, -i, j, -j, k, -k\}$ the quaternion group of order 8. We define the following hyperoperation on G,

$$x \circ y = \{xy, \ xiy\}$$

for all $x, y \in G$. This hyperoperation is a P-hyperoperation with $P = \{1, i\}$. Then, $(G, \circ)$ is a hypergroup. Indeed, for all $x, y, z \in G$ we have

$$\begin{aligned}
(x \circ y) \circ z &= \{xy, \ xiy\} \circ z \\
&= xy \circ z \cup xiy \circ z \\
&= \{xyz, \ xyiz\} \cup \{xiyz, \ xiyiz\} \\
&= \{xyz, \ xyiz, \ xiyz, \ xiyiz\}, \\
x \circ (y \circ z) &= x \circ \{yz, yiz\} \\
&= x \circ yz \cup x \circ yiz \\
&= \{xyz, \ xiyz\} \cup \{xyiz, \ xiyiz\} \\
&= \{xyz, \ xiyz, \ xyiz, \ xiyiz\}.
\end{aligned}$$

Thus, $(x \circ y) \circ z = x \circ (y \circ z)$. Moreover, we have

$$x \circ G = \bigcup_{g \in G} x \circ g = \bigcup_{g \in G} \{xg, xig\} = G = \bigcup_{g \in G} \{gx, gix\} = G \circ x.$$

2.5. Example. If $(G, +)$ is an abelian group and ρ is an equivalence relation in G which has classes $\overline{x} = \{x, \ -x\}$, then for all $\overline{x}, \overline{y} \in G/\rho$, we define $\overline{x} \circ \overline{y} = \{\overline{x+y}, \ \overline{x-y}\}$. Then $(G/\rho, \circ)$ is a hypergroup [13, 14]. As an illustration of the above example, suppose that $G = \mathbb{Z}_{12}$, the abelian group modulo 12. Hence the equivalence classes are

$$\begin{aligned}
\overline{0} &= \{0\} \\
\overline{1} &= \{1, -1\} = \{1, 11\} = \overline{11} \\
\overline{2} &= \{2, -2\} = \{2, 10\} = \overline{10} \\
\overline{3} &= \{3, -3\} = \{3, 9\} = \overline{9} \\
\overline{4} &= \{4, -4\} = \{4, 8\} = \overline{8} \\
\overline{5} &= \{5, -5\} = \{5, 7\} = \overline{7} \\
\overline{6} &= \{6, -6\} = \{6\}
\end{aligned}$$

Then, we obtain the following hyperoperation on H.

$\circ$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{1}$	$\overline{1}$	$\overline{2}, \overline{0}$	$\overline{3}, \overline{1}$	$\overline{4}, \overline{2}$	$\overline{5}, \overline{3}$	$\overline{6}, \overline{4}$	$\overline{5}$
$\overline{2}$	$\overline{2}$	$\overline{3}, \overline{1}$	$\overline{4}, \overline{0}$	$\overline{5}, \overline{1}$	$\overline{6}, \overline{2}$	$\overline{5}, \overline{3}$	$\overline{4}$
$\overline{3}$	$\overline{3}$	$\overline{4}, \overline{2}$	$\overline{5}, \overline{1}$	$\overline{6}, \overline{0}$	$\overline{5}, \overline{1}$	$\overline{4}, \overline{2}$	$\overline{3}$
$\overline{4}$	$\overline{4}$	$\overline{5}, \overline{3}$	$\overline{6}, \overline{2}$	$\overline{5}, \overline{1}$	$\overline{4}, \overline{0}$	$\overline{3}, \overline{1}$	$\overline{2}$
$\overline{5}$	$\overline{5}$	$\overline{6}, \overline{4}$	$\overline{5}, \overline{3}$	$\overline{4}, \overline{2}$	$\overline{3}, \overline{1}$	$\overline{2}, \overline{0}$	$\overline{1}$
$\overline{6}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	$\overline{0}$

Therefore, $(\mathbb{Z}_{12}/\rho, \circ)$ is a hypergroup.

Let $(H, \circ)$ be a hypergroup and $n > 1$ be an integer. We say that

$$x\beta_n y \text{ if there exists } a_1, \ldots, a_n \text{ in } H, \text{ such that } \{x, y\} \subseteq \prod_{i=1}^{n} a_i.$$

Let $\beta_H = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$. Clearly, the relation β_H is reflexive and symmetric. Denote by β_H^* the transitive closure of β_H. The relation β_H^* is the smallest strongly regular relation on H, i.e.,

(1) β_H^* is a strongly regular relation on H;
(2) If R is a strongly regular relation on H, then $\beta_H^* \subseteq R$.

Thus, the relation β_H^* is the smallest equivalence relation on H such that the quotient H/β_H^* is a group. The relation β_H^* is called the *fundamental relation* on H and H/β_H^* is called the *fundamental group*. The product $\odot$ in H/β_H^* is defined as follows:

$$\beta_H^*(x) \odot \beta_H^*(y) = \beta_H^*(z), \text{ for all } z \in \beta_H^*(x) \circ \beta_H^*(y).$$

This relation is introduced by Koskas [21] and studied mainly by Corsini [10], Leoreanu-Fotea [22] and Freni [17] concerning hypergroups, Vougiouklis [27] concerning H_v-groups, Davvaz concerning polygroups [13], and many others. Freni proved that for hypergroups $\beta = \beta^*$ in [17]. The kernel of the *canonical map* $\varphi_H : H \longrightarrow H/\beta_H^*$ is called the *core* of H and is denoted by ω_H. Here we also denote by ω_H the unit of H/β_H^*. The heart of a hypergroup H is the intersection of all subhypergroups of H, which are complete parts.

We have seen so far two different ways to define cyclic hypergroups:

(1) (Vougiouklis, [26]) For every integer $n > 0$ and for every $x \in H$, we get the powers of x as follows:

$$x' = \{x\}, \ x^{n+1} = x^n \circ x \subseteq H.$$

A hypergroup $(H, \circ)$ is called cyclic if $H = x' \cup x^2 \cup \cdots \cup x^n \cup \cdots$, for some $x \in H$. If there exists an integer $n > 0$, the minimum one has the following property

$$H = x' \cup x^2 \cup \cdots \cup x^n,$$

then we call H cyclic hypergroup with finite period and we call x generator of H with period n. If there exists an integer $n > 0$, the minimum one with the following property

$$H = x^n,$$

then we call H single-power cyclic hypergroup and x generator of H with period n.

(2) (Karimian and Davvaz, [19]) Let $(H, \circ)$ be a hypergroup and $\phi : H \to H/\beta_H^*$ be the canonical projection. A hypergroup $(H, \circ)$ is called cyclic hypergroup with generator x if $\phi(H)$ is a cyclic group generated by $\phi(x)$. Suppose that $(H, \circ)$ is a hypergroup and K is a subhypergroup of H. We say that K is a cyclic subhypergroup of H with generator x if $\phi(K)$ is a subgroup of H/β_H^*.

In the following we use a way similar to the second view to define a hypergroup generated by a set.

2.6. Definition. Let H be a hypergroup and K be a subhypergroup of H. We say that K is generated by a non-empty subset X of H if $\phi(K)$ is a subgroup of H/β_H^* generated by $\phi(X)$.

3. Crossed module of hypergroups

Some authors [12, 23, 29] considered the actions of algebraic hyperstructures. In [23], Madanshekaf and Ashrafi considered a generalized action of a hypergroup H on a non-empty set X and obtained some results in this respect. For the definition of crossed modules of hypergroups, we need the notion of hypergroup action. So, we recall the following definition from [23].

3.1. Definition. Let $(H, \circ)$ be a hypergroup and X be a non-empty set. A map $\alpha : H \times X \to \mathcal{P}^*(X)$ is called a *generalized action* of H on X, if the following axiom hold:

(1) $\alpha(g \star h, x) \subseteq \alpha(g, \alpha(h, x))$, for all $g, h \in H$ and $x \in X$, where

$$\alpha(g \star h, x) = \bigcup_{k \in g \star h} \alpha(k, x).$$

(2) For all $h \in H$, $\alpha(h, X) = X$, where

$$\alpha(h, X) = \bigcup_{x \in X} \alpha(h, x).$$

If the equality holds in the axiom (1) of Definition 3.1, the action is called *strong generalized action*. Moreover, if H has the scalar identity element e, then the following condition must be satisfied too,

(3) $\alpha(e, x) = \{x\} := x$, for all $x \in X$.

3.2. Example. [23]

(1) For any hypergroup $(H, \star)$ and any non-empty set X, the map $\alpha : H \times X \to \mathcal{P}^*(X)$, given by $\alpha(h, x) = X$ is a strong generalized action of H on X. If we define $\alpha(h, x) = \{x\}$, then this map is also a strong generalized action of H on X.
(2) Let $(H, \star)$ be a hypergroup. Then, the map $\alpha : H \times H \to \mathcal{P}^*(H)$, given by $\alpha(h, x) = h \star x$ is a strong generalized action of H on H.

3.3. Example. [23] Let X be a non-empty set, $f \in M_\theta$ and $H = M_f$. Then, the map $\alpha : H \times X \to \mathcal{P}^*(X)$, defined by $\alpha(h, x) = h(x)$ is a strong generalized action of H on X.

For $x \in X$, we put ${}^h x := \alpha(h, x)$. Then, for a strong generalized action, we have

(1) ${}^g({}^h x) = {}^{g \star h} x$, for all $g, h \in H$ and $x \in X$.
(2) $\bigcup\limits_{x \in X} {}^h x = X$, for all $h \in H$.

3.4. Definition. [5] A *crossed module of hypergroups* $\mathcal{X} = (C, H, \partial, \alpha)$ consists of hypergroups $(C, \star)$ and $(H, \circ)$ together with a strong homomorphism $\partial : C \to H$ and a strong generalized action $\alpha : H \times C \to \mathcal{P}^*(C)$ on C, satisfying the conditions:

(1) $h \circ \partial(c) \subseteq \partial({}^h c) \circ h$, for all $c \in C$ and $h \in H$.
(2) $c \star c' \subseteq {}^{\partial(c)} c' \star c$, for all $c, c' \in C$.

3.5. Example. Suppose that H is a non-empty set. We define the hyperoperation $\circ$ on H by

$$h_1 \circ h_2 = \{h_1,\ h_2\}, \text{ for all } h_1, h_2 \in H.$$

Then, $(H, \circ)$ is a hypergroup. Suppose that C is a subhypergroup of H and $\partial : C \to H$ is the identity map. The map $\alpha : H \times C \to \mathcal{P}^*(C)$ is defined by ${}^h c := C$ is a strong generalized action. Moreover,

(1) For all $c \in C$ and $h \in H$, we have

$$h \circ \partial(c) = h \circ c = \{h,\ c\} \subseteq C \cup \{h\} = C \circ h = \partial(C) \circ h = \partial({}^h c) \circ h.$$

(2) For all $c, c' \in C$, we have

$$c \circ c' = \{c,\ c'\} \subseteq C = C \circ c = {}^{c}c' \circ c = {}^{\partial(c)}c' \circ c.$$

Therefore, $\mathcal{X} = (C, H, \partial, \alpha)$ is a crossed module of hypergroups.

3.6. Example. Suppose that G is an abelian group and P a non-empty subset of G. We consider the P-hyperoperation $\star_P$ on G as follows:

$$x \star_P y = xyP, \text{ for all } x, y \in G.$$

Then, $(G, \star_P)$ is a hypergroup. Suppose that $\partial : G \to G$ is the identity map. The map $\alpha : G \times G \to \mathcal{P}^*(G)$ is defined by ${}^{g}x := \{x\}$ is a strong generalized action. Moreover,

(1) For all $x, y \in G$, we have

$$g \star_P \partial(x) = g \star_P x = gxP = xgP = x \star_P g = \partial(x) \star_P g = \partial({}^{g}x) \star_P g$$

(2) For all $x, y \in G$, we have

$$x \star_P y = xyP = yxP = y \star_P x = {}^{x}y \star_P x = {}^{\partial(x)}y \star_P x.$$

Therefore, $\mathcal{X} = ((G, \star_P), (G, \star_P), \partial, \alpha)$ is a crossed module of hypergroups.

3.7. Example. The direct product of $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed modules of hypergroups has source $C_1 \times C_2$, range $H_1 \times H_2$ and boundary homomorphism $\partial_1 \times \partial_2$ with $H_1 \times H_2$ acting trivially on $C_1 \times C_2$.

3.8. Definition. Let $\mathcal{X} = (C, P, \partial, \alpha)$ and $\mathcal{X}' = (C', P', \partial', \alpha')$ be two crossed modules of hypergroups. A *crossed module of hypergroups morphism*

$$< \theta, \phi >: (C, H, \partial, \alpha) \to (C', H', \partial', \alpha')$$

is a commutative diagram of strong homomorphisms of hypergroups

$$\begin{tikzcd} C \arrow[r, "\theta"] \arrow[d, "\partial"'] & C' \arrow[d, "\partial'"] \\ H \arrow[r, "\phi"'] & H' \end{tikzcd}$$

such that for all $h \in H$ and $c \in C$, we have

$$\theta({}^{h}c) = {}^{\phi(h)}\theta(c).$$

We say that $< \theta, \phi >$ is an *isomorphism* if θ and ϕ are both isomorphisms. Similarly, we can define *monomorphism*, *epimorphism* and *automorphism* of crossed modules of hypergroups.

The following example give us another crossed module structure on the fundamental groups.

3.9. Example. Suppose that $(H, \circ)$ is a hypergroup. Then, H/β_H^* is a group. Suppose that $Aut\,(H/\beta_H^*)$ its group of automorphisms. There is a trivial action α of $Aut\,(H/\beta_H^*)$ on H/β_H^*, and a group homomorphism $\partial : H/\beta_H^* \to Aut\,(H/\beta_H^*)$ sending each $\beta_H^*(h) \in P/\beta_P^*$ to the inner automorphism of conjugation by $\beta_P^*(p)$. These together form a crossed module $(H/\beta_H^*, Aut\,(H/\beta_H^*)\,, \partial, \alpha)$.

4. Pullback crossed module of hypergroups

In this section we define pullback crossed module of hypergroups.

4.1. Lemma. *Let $(H, \circ)$ be a hypergroup and β_H^* be its fundamental relation. For every $q, q' \in H$, we have*

$$\{x \mid \beta_H^*(x) = \beta_H^*(q') \odot \beta_H^*(q) \odot \beta_H^*(q')^{-1}\} \circ q' \supseteq q' \circ q.$$

Proof. Suppose that $y \in q' \circ q$ is arbitrary. Then

$$\beta_H^*(y) \quad = \quad \beta_H^*(q') \odot \beta_H^*(q) \tag{4.1}$$

On the other hand, since $y \in H \circ q'$, it follows that there exists $z \in H$ such that $y \in z \circ q'$. So,

$$\beta_H^*(y) \quad = \quad \beta_H^*(z) \odot \beta_H^*(q') \tag{4.2}$$

By Equations (4.1) and (4.2) we obtain

$$\beta_H^*(z) \odot \beta_H^*(q') \quad = \quad \beta_H^*(q') \odot \beta_H^*(q)$$

or

$$\beta_H^*(z) \quad = \quad \beta_H^*(q') \odot \beta_H^*(q) \odot \beta_H^*(q')^{-1}$$

Thus $y \in \{x \mid \beta_H^*(x) = \beta_H^*(q') \odot \beta_H^*(q) \odot \beta_H^*(q')^{-1}\} \circ q'$. □

4.2. Definition. Let $\mathfrak{X} = (C, H, \partial, \alpha)$ be a crossed module of hypergroups and $\iota : Q \to H$ be a strong homomorphism of hypergroups. Then, $\iota^\bullet \mathfrak{X} = (\iota^\bullet C, Q, \partial^\bullet, \alpha^\bullet)$ is the pullback of $\mathfrak{X}$ by ι, where $\iota^\bullet C = \{(q, c) \in Q \times C \mid \iota(q) = \partial(c)\}$ and $\partial^\bullet(q, c) = q$. The hypergroup action of Q on $\iota^\bullet C$ is given by

$${}^{q}(q_1, c) \quad = \quad \{(x, y) \mid \beta_H^*(x) = \beta_H^*(q) \odot \beta_H^*(q_1) \odot \beta_H^*(q)^{-1},\ y \in {}^{\iota(q)}c\}.$$

4.3. Lemma. *The following diagram is commutative, i.e., $\partial\lambda = \iota\partial^\bullet$.*

$$\begin{tikzcd}
\iota^\bullet C \arrow[r, "\lambda"] \arrow[d, "\partial^\bullet"'] & C \arrow[d, "\partial"] \\
Q \arrow[r, "\iota"'] & H
\end{tikzcd}$$

Proof. It is clear. □

Note that the above definition is a generalization of pullbacks of crossed modules [7].

4.4. Theorem. *Every pullback crossed module is a pullback crossed module of hypergroup.*

Proof. It is straightforward. □

4.5. Theorem. $\iota^\bullet \mathfrak{X} = (\iota^\bullet C, Q, \partial^\bullet, \alpha^\bullet)$ *is a crossed module of hypergroups.*

Proof. We denote the hyperoperation on $\iota^\bullet C$ by $\diamond$. We investigate the condition of Definition 3.4. For the first condition, we have

(1)

$$\begin{aligned}
&\partial^{\bullet}({}^{q'}(q,c))\circ q' \\
&\quad = \ \partial^{\bullet}(\{(x,y)\mid \beta_H^*(x)=\beta_H^*(q')\odot\beta_H^*(q)\odot\beta_H^*(q')^{-1},\ y\in {}^{\iota(q')}c\})\circ q' \\
&\quad = \ \{x\mid \beta_H^*(x)=\beta_H^*(q')\odot\beta_H^*(q)\odot\beta_H^*(q')^{-1}\}\circ q' \\
&\quad \supseteq \ q'\circ q \ \text{(by Lemma 4.1).}
\end{aligned}$$

The verification of the second condition of Definition 3.4 is given as follows:

$$\begin{aligned}
&{}^{\partial^{\bullet}(q',c')}(q,c)\diamond(q',c') \\
&\quad = \ \{(x,y)\mid (x,y)\in {}^{\partial^{\bullet}(q',c')}(q,c)\}\diamond(q',c') \\
&\quad = \ \{(x,y)\mid (x,y)\in {}^{q'}(q,c)\}\diamond(q',c'),\quad \text{by definition of } \partial^{\bullet} \\
&\quad = \ \{(x,y)\mid \beta_H^*(x)=\beta_H^*(q')\odot\beta_H^*(q)\odot\beta_H^*(q')^{-1}, y\in {}^{\iota q'}c)\}\diamond(q',c') \\
&\quad = \ \{(x,y)\mid \beta_H^*(x)=\beta_H^*(q')\odot\beta_H^*(q)\odot\beta_H^*(q')^{-1}, y\in {}^{\partial c'}c\}\diamond(q',c') \\
&\quad = \ \bigcup_{\beta_H^*(x)=\beta_H^*(q')\odot\beta_H^*(q)\odot\beta_H^*(q')^{-1}} x\circ q'\times({}^{\partial c'}c * c') \\
&\quad \supseteq \ (q'\circ q)\times(c' * c) \\
&\quad = \ \{(x,y)\ \mid\ x\in q'\circ q,\ y\in c' * c \\
&\quad = \ (q',c')\diamond(q,c)
\end{aligned}$$

where $(q,c),(q',c')\in\iota^*C$. □

The universal property of induced crossed module of hypergroups is similar to the universal property of induced crossed module [9] as well as induced crossed polymodule [6]. Let $\mathcal{X}=(\mu: M\to Q)$ be a crossed module of hypergroup and let $\iota^{\bullet\bullet}\mathcal{X}=(\delta:\iota^{\bullet\bullet}M\to H)$ be induced by the strong homomorphism $\iota: Q\to H$. In the diagram

$$\begin{tikzcd}
 & M \arrow[r, "\mu"] \arrow[d, "\bar{\iota}"] \arrow[ddl, "f"'] & Q \arrow[d, "\iota"] \\
 & \iota^{\bullet\bullet}M \arrow[r, "\delta"] \arrow[dl, "g"] & H \\
G \arrow[urr, "\gamma"'] & &
\end{tikzcd}$$

the pair $(\bar{\iota},\iota)$ is a morphism of crossed module of hypergroups such that for any crossed module of hypergroups $\mathcal{Y}=(\gamma: G\to H)$ and any morphism of crossed modules of hypergroups $(f,\iota):\mathcal{X}\to\mathcal{Y}$ there is a unique morphism $(g,1):\iota^{\bullet\bullet}\mathcal{X}\to\mathcal{Y}$ of crossed modules of hypergroups such that $g\bar{\iota}=f$.

4.6. Proposition. [5] *Let $(C,\star)$ and $(H,\circ)$ be two hypergroups and let $\partial: C\to H$ be a strong homomorphism. Then, ∂ induces a group homomorphism $\mathcal{D}: C/\beta_C^*\to H/\beta_H^*$ by setting*

$$\mathcal{D}(\beta_C^*(c))=\beta_H^*(\partial(c)),\ \text{forall } c\in C.$$

We say the action of H on C is *productive*, if for all $c\in C$ and $h\in H$ there exist $c_1,\ldots,c_n$ in C such that ${}^{h}c=c_1\star\ldots\star c_n$.

Let $(C,\star)$ and $(H,\circ)$ be two hypergroups and let $\alpha : H\times C \to \mathcal{H}^*(C)$ be a productive action on C. We define the map $\psi : H/\beta_H^* \times H/\beta_C^* \to \mathcal{H}^*(H/\beta_C^*)$ as usual manner:

$$\psi(\beta_H^*(h),\beta_C^*(c)) = \{\beta_C^*(x) \mid x \in \bigcup_{\substack{y\,\in\,\beta_C^*(c)\\ z\,\in\,\beta_H^*(h)}} {}^{z}y\}.$$

By definition of β_C^*, since the action of H on C is productive, we conclude that $\psi(\beta_H^*(h),\beta_C^*(c)$ is singleton, i.e., we have

$$\psi : H/\beta_H^* \times H/\beta_C^* \to H/\beta_C^*,$$
$$\psi(\beta_H^*(h),\beta_C^*(c)) = \beta_C^*(x),\ \text{for all } x \in \bigcup_{\substack{y\,\in\,\beta_C^*(c)\\ z\,\in\,\beta_H^*(h)}} {}^{z}y.$$

We denote $\psi(\beta_H^*(h),\beta_C^*(c)) = {}^{[\beta_H^*(h)]}[\beta_C^*(c)]$.

4.7. Proposition. [5] *Let $(C,\star)$ and $(P,\circ)$ be two hypergroups and let $\alpha : H\times C\to \mathcal{H}^*(C)$ be a productive action on C. Then, ψ is an action of the group H/β_H^* on the group H/β_C^*.*

4.8. Theorem. [5] *Let $\mathcal{X} = (C,H,\partial,\alpha)$ be a crossed hypergroup such that the action of H on C is productive. Then, $\mathcal{X}_{\beta^*} = (C/\beta_C^*, H/\beta_H^*, \mathcal{D},\psi)$ is a crossed module.*

4.9. Corollary. *Let $\mathcal{X} = (C,H,\partial,\alpha)$ be a crossed module of hypergroups such that the action of H on C is productive and $\iota : Q \to H$ be a strong homomorphism of hypergroups. Then, $(\iota^*)^\bullet = \big((\iota^*)^\bullet(C/\beta_C^*),\ Q/\beta_Q^*, \mathcal{D}^\bullet,\ \psi^*\big)$ is the pullback of $\mathcal{X}_{\beta^*} = (C/\beta_C^*, H/\beta_H^*, \mathcal{D},\psi)$ by ι^*, where*

$$\iota^* : Q/\beta_Q^* \to H/\beta_H^*,\quad \iota_Q^*(\beta^*(q)) = \beta_P^*(\iota(q)),$$
$$(\iota^*)^\bullet(C/\beta_C^*) = \{(\beta_Q^*(q),\beta_C^*(c)) \mid \iota^*(\beta^*(q)) = \mathcal{D}(\beta_C^*(c))\},$$
$$\mathcal{D}^\bullet(\beta_Q^*(q),\beta_C^*(c)) = \beta_Q^*(q).$$

Now, we conclude the following theorem.

4.10. Theorem. $(\iota^*)^\bullet = \big((\iota^*)^\bullet(C/\beta_C^*),\ Q/\beta_Q^*, \mathcal{D}^\bullet,\ \psi^*\big)$ *is a crossed module.*

Proof. For the first axiom of crossed module, we have

$$\begin{aligned}\mathcal{D}^\bullet\Big({}^{[\beta_Q^*(q)]}[(\beta_Q^*(q),\beta_C^*(c))]\Big) &= \mathcal{D}^\bullet\Big(\beta_Q^*(q')\oslash\beta_Q^*(q)\oslash\beta_Q^*(q')^{-1},\ {}^{[\iota^*(\beta_Q^*(q))]}[\beta_C^*(c)]\Big)\\ &= \beta_Q^*(q')\oslash\beta_Q^*(q)\oslash\beta_Q^*(q')^{-1}\\ &= \beta_Q^*(q')\oslash\mathcal{D}^\bullet\Big(\beta_Q^*(q),\beta_C^*(c)\Big)\oslash\beta_Q^*(q')^{-1}.\end{aligned}$$

For the second axiom of crossed module, we have

$$
\begin{aligned}
&\Big(\beta_Q^*(q'),\beta_C^*(c')\Big)^{-1}\circledast\Big(\beta_Q^*(q),\beta_C^*(c)\Big)\circledast\Big(\beta_Q^*(q'),\beta_C^*(c')\Big)\\
&=\Big(\beta_Q^*(q')^{-1},\beta_C^*(c')^{-1}\Big)\circledast\Big(\beta_Q^*(q),\beta_C^*(c)\Big)\circledast\Big(\beta_Q^*(q'),\beta_C^*(c')\Big)\\
&=\Big(\beta_Q^*(q')^{-1}\oslash\beta_Q^*(q)\oslash\beta_Q^*(q'),\ \beta_C^*(c')^{-1}\otimes\beta_C^*(c)\otimes\beta_C^*(c')\Big)\\
&=\Big(\ {}^{[\beta_Q^*(q')]}[\beta_Q^*(q)],\ \ {}^{[\mathcal{D}\beta_C^*(c')]}[\beta_C^*(c)]\Big)\\
&=\ {}^{[\beta_Q^*(q')]}[\beta_Q^*(q),\beta_C^*(c)]\\
&=\ {}^{\left[\mathcal{D}^{\bullet}\left(\beta_Q^*(q),\beta_C^*(c)\right)\right]}\Big[\Big(\beta_Q^*(q),\beta_C^*(c)\Big)\Big]
\end{aligned}
$$

□

5. Pushouts of crossed module of hypergroups

Let $\mathcal{X}=(C,H,\partial,\alpha)$ be a crossed module of hypergroups and let $(K,\cdot)$ be a hypergroup. Note that K/β_K^* is a group, and so we can consider it as a hypergroup too. We denote the multiplication in K/β_K^* by $\boxdot$.

5.1. Definition. Let $\gamma: H\to K$ be a strong homomorphism of hypergroups. Consider the hypergroup $\iota_\bullet(C)$ generated by $C\times K$ with relations

(1) $(c_1,k)*(c_2,k)=\{(c,k)\ \mid\ \beta_C^*(c)=\beta_C^*(c_1)\otimes\beta_C^*(c_2)\}$,

(2) $({}^{h}c,k)=(c,k\cdot\gamma(h))$, that is,

$$\{(c',k)\mid c'\in\ {}^{h}c\}=\{(c,k')\ \mid\ k'\in k\cdot\gamma(h)\},$$

(3) $(c_1,k_1)*(c_2,k_2)\subseteq\{(c_2,k')\mid\beta_K^*(k')=\beta_K^*(k_1)\boxdot\beta_K^*(\gamma\partial(c_1))\boxdot\beta_K^*(k_1))^{-1}\boxdot\beta_K^*(k_2))\}*(c_1,k_1)$,

for all $k,k_1,k_2\in K, c,c_1,c_2\in C$ and $h\in H$.

Define a homomorphism $\partial_\bullet:\iota_\bullet(C)\to K/\beta_K^*$ by extending

$$\partial_\bullet(c,k)=\beta_K^*(k)\boxdot\beta_K^*(\gamma\partial(c))\boxdot\beta_K^*(k)^{-1}$$

to the whole of $\iota_\bullet(C)$ and define a K/β_K^*-hypergroup action on the left of $\iota_\bullet(C)$ by ${}^{\beta_K^*(k)}(c,k_1)=\{(c,k')\mid\beta_K^*(k')=\beta_K^*(k)\boxdot\beta_K^*(k_1)\}$, for $k,k_1\in K, c\in C$ and a strong homomorphism $\psi: C\to\iota_\bullet(C)$ by $\psi(c)=(c,e_0)$, where e_0 is a fix element of ω_K.

5.2. Lemma. *The following diagram is commutative, i.e., $\partial_\bullet\psi=\phi_K\gamma\partial$.*

$$
\begin{tikzcd}
C \arrow[r, "\psi"] \arrow[d, "\partial"'] & \iota_\bullet(C) \arrow[d, "\partial_\bullet"] \\
H \arrow[r, "\phi_K\gamma"'] & K/\beta_K^*
\end{tikzcd}
$$

where $\phi_K: K\to K/\beta_K^*$ is the canonical projection.

Proof. We have

$$\begin{aligned}\partial_\bullet\psi(c) &= \partial_\bullet(c,e_0)\\ &= \beta_K^*(e_0)\boxdot\beta_K^*(\gamma\partial(c))\boxdot\beta_K^*(e_0)^{-1}\\ &= \omega_K\boxdot\beta_K^*(\gamma\partial(c))\boxdot\omega_K\\ &= \beta_K^*(\gamma\partial(c))\\ &= \phi_K(\gamma\partial(c))\\ &= \phi_K\gamma\partial(c).\end{aligned}$$

This completes the proof. □

5.3. Proposition. *With the notation above $\partial_\bullet:\iota_\bullet(C)\to K/\beta_K^*$ is a crossed module of hypergroups*

Proof. We check the axioms of crossed module of hypergroups as follows.

(1) We have

$$\begin{aligned}&\partial_\bullet\left({}^{\beta_K^*(k)}(c,k_1)\right)\boxdot\beta_K^*(k)\\ &= \partial_\bullet(\{(c,k')\ \mid\ \beta_K^*(k')=\beta_K^*(k)\boxdot\beta_K^*(k_1)\})\boxdot\beta_K^*(k)\\ &= \{\partial_\bullet(c,k')\ \mid\ \beta_K^*(k')=\beta_K^*(k)\boxdot\beta_K^*(k_1)\})\boxdot\beta_K^*(k)\\ &= \{\beta_K^*(k')\boxdot\beta_K^*((\gamma\partial(c)))\boxdot\beta_K^*(k')^{-1}\ \mid\ \beta_K^*(k')=\beta_K^*(k)\boxdot\beta_K^*(k_1)\})\boxdot\beta_K^*(k)\\ &= \beta_K^*(k)\boxdot\beta_K^*(k_1)\boxdot\beta_K^*((\gamma\partial(c)))\boxdot(\beta_K^*(k)\boxdot\beta_K^*(k_1))^{-1}\boxdot\beta_K^*(k)\\ &= \beta_K^*(k)\boxdot\beta_K^*(k_1)\boxdot\beta_K^*((\gamma\partial(c)))\boxdot(\beta_K^*(k_1)^{-1}\boxdot\beta_K^*(k)^{-1}\boxdot\beta_K^*(k)\\ &= \beta_K^*(k)\boxdot\beta_K^*(k_1)\boxdot\beta_K^*((\gamma\partial(c)))\boxdot(\beta_K^*(k_1)^{-1}\boxdot\omega_K\\ &= \beta_K^*(k)\boxdot\beta_K^*(k_1)\boxdot\beta_K^*((\gamma\partial(c)))\boxdot(\beta_K^*(k_1)^{-1}\\ &= \beta_K^*(k)\boxdot\partial_\bullet\left((c,k_1)\right).\end{aligned}$$

(2) We have

$$\begin{aligned}&{}^{\partial_\bullet(c,k)}(c_1,k_1)*(c,k)\\ &= {}^{\beta_K^*(k)\boxdot(\gamma\partial(c))\boxdot\beta_K^*(k)^{-1}}(c_1,k_1)*(c,k)\\ &= \{(c_1,k')\ \mid\ \beta_K^*(k')=\beta_K^*(k)\boxdot\beta_K^*(\gamma\partial(c))\boxdot\beta_K^*(k)^{-1}\boxdot\beta_K^*(k_1)\}*(c,k)\\ &\supseteq (c,k)*(c_1,k_1).\end{aligned}$$

This completes the proof. □

5.4. Proposition. *Let $(H,\circ)$, $(C,\star)$ and $(K,\cdot)$ be hypergroups. Let $\partial:C\to H$ and $\delta:K\to H$ be two crossed modules of hypergroups and let $(\phi,Id):(\partial:C\to H)\to(\delta:K\to H)$ be a morphism of crossed modules of hypergroups. Then, the defining a K-action on C by ${}^kc={}^{\delta(k)}c$ we have $\phi:C\to K$ is a crossed module of hypergroups.*

Proof. We can show two crossed modules of hypergroups as follows:

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[d, "\partial"'] & K \arrow[d, "\delta"] \\ H \arrow[r, "Id"'] & H \end{tikzcd}$$

where $\partial = \delta\phi$ and $\phi({}^{h}c) = {}^{h}\phi(c)$. We can verify the axioms of crossed module of hypergroups as follows:

(1) $\phi({}^{k}c) \cdot k = \phi({}^{\delta(k)}c) \cdot k = {}^{\delta(k)}(\phi(c)) \cdot k \supseteq = k \cdot \phi(c)$.
(2) ${}^{\phi(c_2)}c_1 \star c_2 = {}^{\delta(\phi(c_2))}c_1 \star c_2 = {}^{\delta\phi(c_2)}(c_1) \star c_2 = {}^{\partial(c_2)}c_1 \star c_2 \supseteq c_2 \star c_1$.

□

5.5. Corollary. *If we consider the pullback and pushout diagrams together, we get the following commutative diagram.*

$$\begin{tikzcd} \iota^{\bullet}C \arrow[r, "\lambda"] \arrow[d, "\partial^{\bullet}"'] & C \arrow[r, "\psi"] \arrow[d, "\partial"] & \iota_{\bullet}C \arrow[d, "\partial_{\bullet}"] \\ Q \arrow[r, "\iota"'] & H \arrow[r, "\phi_K\gamma"'] & K/\beta_K^* \end{tikzcd}$$

5.6. Corollary. *We have the following commutative diagram.*

$$\begin{tikzcd} \iota^{\bullet}C \arrow[r, "\psi\lambda"] \arrow[d, "\partial^{\bullet}"'] & \iota_{\bullet}C \arrow[d, "\partial_{\bullet}"] \\ Q \arrow[r, "\phi_K\gamma\iota"'] & K/\beta_K^* \end{tikzcd}$$

Proof. We have

$$\begin{aligned} \phi_K\gamma\iota\partial^{\bullet}(q,c) \quad &= \phi_K\gamma\iota(q) \\ &= \phi_K\gamma\partial(c) \quad (\text{since } \iota(q) = \partial(c)) \\ &= \partial_{\bullet}(h(q,c), e_0) \\ &= \beta_K^*(\gamma\partial(c)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \partial_{\bullet}\psi\lambda(q,c) \quad &= \partial_{\bullet}\psi(c) \\ &= \partial_{\bullet}(c, e_0) \quad (\text{where } e_0 \in \omega_K) \\ &= \beta_H^*(e_0) \boxdot \beta_H^*(\gamma\partial(c)) \boxdot \beta_H^*(e_0)^{-1} \\ &= \omega_K \boxdot \beta_H^*(\gamma\partial(c)) \boxdot \omega_K \\ &= \beta_K^*(\gamma\partial(c)). \end{aligned}$$

Therefore, we obtain $\phi_K\gamma\iota\partial^{\bullet} = \partial_{\bullet}\psi\lambda$. This completes the proof. □

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