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Fuzzy integro-differential equations with compactness type conditions

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Abstract

In the paper fuzzy integro-differential equations with almost continuous right hand sides are studied. The existence of solution is proved under compactness type conditions.

Keywords: Fuzzy integro-differential equation; Measure of noncompactness.

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1. Introduction

Many problems in modeling as well as in medicines are described by fuzzy integro-differential equations, which are helpful in studying the observability of dynamical control systems. This is the main reason to study these equations extensively. We mention the papers [1] and [2], where nonlinear integro-differential equations are studied in Banach spaces and in fuzzy space respectively. In [3], existence result for nonlinear fuzzy Volterra-Fredholm integral equation is proved. In [14], fuzzy Volterra integral equations are studied using fixed point theorem, while in [10], the method of successive approximation is used, when the right hand side satisfies Lipschitz condition. In [15] Kuratowski measure of noncompactness as well as imbedding map from fuzzy to Banach space is used to prove existence of solutions. In [11] existence and uniqueness result for fuzzy Volterra integral equation with Lipschitz right hand side and with infinite delay is proved using successive approximations method. We also refer to [4] where existence of solution of functional integral equation under compactness condition is proved.

In the paper we study the following fuzzy integro-differential equation:

$$\dot{x}(t) = F(t, x(t), (Vx)(t)),\ x(0) = x_0,\ t \in I = [0, T], \tag{1.1}$$

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where $(Vx)(t) = \int_0^t K(t,s)x(s)ds$ is an integral operator of Volterra type.

2. Preliminaries

In this section we give our main assumptions and preliminary results needed in the paper.

The fuzzy set space is denoted by $\mathbb{E}^n = \{x : \mathbb{R}^n \to [0,1];\ x \text{ satisfies } 1) - 4)\}$.

1) x is normal i.e. there exists $y_0 \in \mathbb{R}^n$ such that $x(y_0) = 1$,

2) x is fuzzy convex i.e. $x(\lambda y + (1-\lambda)z) \geq \min\{x(y), x(z)\}$ whenever $y, z \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

3) x is upper semicontinuous i.e. for any $y_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta(y_0, \varepsilon) > 0$ such that $x(y) < x(y_0) + \varepsilon$ whenever $|y - y_0| < \delta$ and $y \in \mathbb{R}^n$,

4) The closure of the set $\{y \in \mathbb{R}^n;\ x(y) > 0\}$ is compact.

The set $[x]^\alpha = \{y \in \mathbb{R}^n;\ x(y) \geq \alpha\}$ is called α-level set of x.

It follows from 1) – 4) that the α-level sets $[x]^\alpha$ are convex compact subsets of $\mathbb{R}^n$ for all $\alpha \in (0,1]$. The fuzzy zero is

$$\hat{0}(y) = \begin{cases} 0 \ if \ y \neq 0, \\ 1 \ if \ y = 0. \end{cases}$$

Evidently $\mathbb{E}^n$ is a complete metric space equipped with metric

$$D(x,y) = \sup_{\alpha \in (0,1]} D_H([x]^\alpha, [y]^\alpha),$$

where $D_H(A,B) = \max\{\max_{a\in A}\min_{b\in B}|a-b|, \max_{b\in B}\min_{a\in A}|a-b|\}$ is the Hausdorff distance between the convex compact subsets of $\mathbb{R}^n$. From Theorem 2.1 of [7], we know that $\mathbb{E}^n$ can be embedded as a closed convex cone in a Banach space $\mathbb{X}$. The embedding map $j : \mathbb{E}^n \to \mathbb{X}$ is isometric and isomorphism.

The function $g : I \to \mathbb{E}^n$ is said to be simple function if there exists a finite number of pairwise disjoint measurable subsets $I_1, \ldots, I_n$ of I with $I = \bigcup_{k=1}^{n} I_k$ such that $g(\cdot)$ is constant on every I_k.

The map $f : I \to \mathbb{E}^n$ is said to be strongly measurable if there exists a sequence $\{f_m\}_{m=1}^\infty$ of simple functions $f_m : I \to \mathbb{E}^n$ such that $\lim_{m\to\infty} D(f_m(t), f(t)) = 0$ for a.a $t \in I$.

In the fuzzy set literature starting from [12] the integral of fuzzy functions is defined levelwise, i.e. there exists $g(t) \in \mathbb{E}^n$ such that $[g]^\alpha(t) = \int_0^t [f]^\alpha(s)ds$.

Now if $g(\cdot) : I \to \mathbb{E}^n$ is strongly measurable and integrable then $j(g)(\cdot)$ is strongly measurable and Bochner integrable and

$$j\left(\int_0^t g(s)ds\right) = \int_0^t j\,(g)\,(s)ds \text{ for all } t \in I. \tag{2.1}$$

We recall some properties of integrable fuzzy set valued mapping from [7].

2.1. Theorem. *Let $G, K : I \to \mathbb{E}^n$ be integrable and $\lambda \in \mathbb{R}$ then*

(i) $\int_I (G(t) + K(t))dt = \int_I G(t)dt + \int_I K(t)dt$,
(ii) $\int_I \lambda G(t)dt = \lambda \int_I G(t)dt$,
(iii) $D(G, K)$ *is integrable,*
(iv) $D(\int_I G(t)dt, \int_I K(t)dt) \leq \int_I D(G(t), K(t))dt$.

A mapping $F : I \to \mathbb{E}^n$ is said to be differentiable at $t \in I$ if there exists $\dot{F}(t) \in \mathbb{E}^n$ such that the limits $\lim_{h\to 0^+} \frac{F(t+h)-F(t)}{h}$ and $\lim_{h\to 0^+} \frac{F(t)-F(t-h)}{h}$ exist, and are equal to $\dot{F}(t)$. At the end point of I we consider only the one sided derivative.

Notice that $\mathbb{E}^n$ is not locally compact (cf. [13]). Consequently we need compactness type assumptions to prove existence of solution, we refer the interested reader to [5] and the references therein.

Let Y be complete metric space with metric $\varrho_Y(\cdot,\cdot)$. The Hausdorff measure of noncompactness $\beta : Y \to \mathbb{R}$ for the bounded subset A of Y is defined by

$$\beta(A) := \inf\{d > 0 : A \text{ can be covered by finite many balls with radius } \leq d\}$$

and "Kuratowski measure" of noncompactness $\rho : Y \to \mathbb{R}$ for the bounded subset A of Y is defined by

$$\rho(A) := \inf\{d > 0 : A \text{ can be covered by finite many sets with diameter } \leq d\},$$

where for any bounded set $A \subset Y$, we denote $\mathrm{diam}(A) = \sup_{a,b\in A} \varrho_Y(a, b)$. It is well known that $\rho(A) \leq \beta(A) \leq 2\rho(A)$ (cf. [8] p.116).

Let $\gamma(\cdot)$ represent the both $\rho(\cdot)$ and $\beta(\cdot)$, then some properties of $\gamma(\cdot)$ are listed below:

(i) $\gamma(A) = 0$ if and only if A is precompact, i.e. its closure $\bar{A}$ is compact,
(ii) $\gamma(A + B) = \gamma(A) + \gamma(B)$ and $\gamma(\overline{co}A) = \gamma(A)$,
(iii) If $A \subset B$ then $\gamma(A) \leq \gamma(B)$,
(iv) $\gamma(A \bigcup B) = max(\gamma(A), \gamma(B))$,
(v) $\gamma(\cdot)$ is continuous with respect to the Hausdorff distance.

The following theorem of Kisielewicz can be found e.g. in [8].

2.2. Theorem. *Let X be separable Banach space and let $\{g_n(\cdot)\}_{n=1}^{\infty}$ be an integrally bounded sequence of measurable functions from I into X, then $t \to \beta\{g_n(t), n \geq 1\}$ is measurable and*

$$\beta\left(\int_t^{t+h} \left\{\bigcup_{i=1}^{\infty} g_i(s)\right\} ds\right) \leq \int_t^{t+h} \beta\left\{\bigcup_{i=1}^{\infty} g_i(s)\right\} ds, \tag{2.2}$$

where $t, t + h \in I$.

The map $t \to \{\bigcup_{i=1}^{\infty} g_i(t)\}$ is a set valued (multifunction). The integral is defined in Auman sense, i.e. union of the values of the integrals of all (strongly) measurable selections.

2.3. Remark. Since the imbedding map $j : \mathbb{E}^n \to \mathbb{X}$ is isometry and isomorphism, one has that it preserve diameter of any closed subset i.e. $\rho(A) = \rho(j(A))$, for any closed and bounded set $A \in \mathbb{E}^n$.

2.4. Theorem. *Let $\{f_n(\cdot)\}_{n=1}^{\infty}$ be a (integrally bounded) sequence of strongly measurable fuzzy functions defined from I into $\mathbb{E}^n$. Then $t \to \rho(\{f_m(t), m \geq 1\})$ is measurable and*

$$\rho\left(\int_a^b \bigcup_{m=1}^{\infty} f_m(s)ds\right) \leq 2\int_a^b \rho\left(\bigcup_{m=1}^{\infty} f_m(s)\right) ds. \tag{2.3}$$

Proof. Since f_m are strongly measurable, one has that $j(f_m)(\cdot)$ are also strongly measurable and hence almost everywhere separably valued.

Thus there exists a separable Banach space $Y \subset X$ such that $j(f_m)(I \setminus N) \subset Y$, where $N \subset I$ is a null set.

Furthermore without loss of generality from Theorem 2.2 and Remark 2.3, we have

$$\begin{aligned} \rho\left(\int_a^b \left(\bigcup_{m=1}^{\infty} f_m(s)\right) ds\right) &= \rho\left(\int_a^b \left(\bigcup_{m=1}^{\infty} j(f_m(s))\right) ds\right) \\ \leq \beta\left(\int_a^b \left(\bigcup_{m=1}^{\infty} j\left(f_m(s)\right)\right) ds\right) &= \int_a^b \beta\left(\bigcup_{m=1}^{\infty} j(f_m(s))\right) ds \\ \leq 2\int_a^b \rho\left(\bigcup_{m=1}^{\infty} j(f_m(s))\right) ds &= 2\int_a^b \rho\left(\bigcup_{m=1}^{\infty} f_m(s)\right) ds. \end{aligned}$$

Consequently, we get (2.3). □

2.5. Remark. Evidently one can replace $\rho(\cdot)$ by $\beta(\cdot)$ in *(2.3)*. It would be interesting to see is it possible to replace 2 in the right hand side by smaller constant, using the special structure of the fuzzy set space, i.e. is it true that

$$\beta\left(\int_a^b \bigcup_{m=1}^{\infty} f_m(s)ds\right) \leq C\int_a^b \beta\left(\bigcup_{m=1}^{\infty} f_m(s)\right) ds,$$

for some $1 \leq C < 2$?

3. Main Results

In this section we prove the existence of solution of (1.1). The following hypotheses will be used;

(H1) $F : I \times \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^n$ is such that
(i) $t \to F(t, x, y)$ is strongly measurable for all $x, y \in \mathbb{E}^n$,
(ii) $(x, y) \to F(t, x, y)$ is continuous for almost all $t \in I$.

Suppose there exist $a(\cdot), b(\cdot) \in L^1(I, \mathbb{R}_+)$ such that:

(H2) $\rho(F(t, A, B)) \leq \lambda(t)(\rho(A)+\rho(B))$, for all non empty bounded subsets $A, B \in \mathbb{E}^n$ and $\lambda(\cdot) \in L^1(I, \mathbb{R}_+)$,

(H3) $D\left(F(t, x, y), \hat{0}\right) \leq a(t) + b(t)\left[D(x, \hat{0}) + D(y, \hat{0})\right]$,

(H4) $K : \triangle = \{(t, s); 0 \leq s \leq t \leq a\} \to \mathbb{R}_+$ is a continuous function.

3.1. Theorem. *If **(H1)**– **(H4)** hold, then problem (1.1) has at least one solution on $[0, T]$.*

Proof. First, we will show that a solution of (1.1) is bounded. Indeed, we have

$$
\begin{aligned}
D(x(t),\hat{0}) &= D(x_0,\hat{0}) + D\left(\int_0^t F(s,x(s),(Vx)(s))ds,\hat{0}\right)\\
&\le D(x_0,\hat{0}) + \int_0^t D\left(F(s,x(s),(Vx)(s)),\hat{0}\right)ds
\end{aligned}
$$

$$
\le D(x_0,\hat{0}) + \int_0^t \left(a(s) + b(s)\left[D(x(s),\hat{0}) + D(\int_0^s K(s,\tau)x(\tau)d\tau,\hat{0})\right]\right)ds
$$

$$
\le D(x_0,\hat{0}) + \int_0^t \left(a(s) + b(s)D(x(s),\hat{0}) + K_\Delta b(s)\int_0^s D(x(\tau)d\tau,\hat{0})\right)ds,
$$

where $K_\Delta = \max_{(t,s)\in\Delta}|K(t,s)|$.

Therefore, if we denote $m(t) = D(x(t),\hat{0})$, then we obtain

$$
m(t) = m(0) + \int_0^t \left(a(s) + b(s)m(s) + K_\Delta b(s)\int_0^s m(\tau)d\tau\right)ds.
$$

By Pachpatte's inequality (see Theorem 1 in [9]), we get that there exists $M_0 > 0$ such that $m(t) = D(x(t),\hat{0}) \le M_0$ for all $t \in [0,T]$.

Moreover, we obtain that

$$
\begin{aligned}
D((Vx)(t),\hat{0}) &= D(\int_0^t K(t,s)x(s)ds,\hat{0})\\
&\le \int_0^t D(K(t,s)x(s),\hat{0})ds
\end{aligned}
$$

$$
\le K_\Delta \int_0^t D(x(s),\hat{0})ds \le K_\Delta M_0 T \doteq M_1.
$$

It follows that

$$
D\left(F(t,x(t),(Vx)(t)),\hat{0}\right) \le a(t) + Mb(t) \doteq \mu(t),
$$

where $M = M_0 + M_1$. Since $a(\cdot), b(\cdot) \in L^1(I,\mathbb{R}_+)$, one has that $\mu(\cdot) \in L^1(I,\mathbb{R}_+)$ Let $c = \int_0^T \mu(s)ds + 1$. We define

$$
\Omega = \left\{x(\cdot) \in C([0,T],\mathbb{E}^n) : \sup_{t\in[0,T]} D(x(t),x_0) \le c\right\}.
$$

Clearly, Ω closed, bounded and convex set. We also define the operator $P : C[[0,T],\mathbb{E}^n] \to C[[0,T],\mathbb{E}^n]$ by

$$
(Px)(t) = x_0 + \int_0^t F(s,x(s),(Vx)(s))ds,\ t \in [0,T].
$$

Therefore

$$
\begin{aligned}
D((Px)(t),x_0) &= D\left(\int_0^t F(s,x(s),(Vx)(s))ds,\hat{0}\right)\\
&\le \int_0^t D\left(F(s,x(s),(Vx)(s)),\hat{0}\right)ds\\
&\le \int_0^T \mu(s)ds < c
\end{aligned}
$$

for $x \in \Omega$ and $t \in [0,T]$. Thus $P(\Omega) \subset \Omega$ and $P(\Omega)$ is uniformly bounded on $[0,T]$.

Next we have to show that P is a continuous operator on Ω. For this, let $x_n(\cdot) \in \Omega$ such that $x_n(\cdot) \to x(\cdot)$. Then

$$D((Px_n)(t),(Px)(t)) = D\left(\int_0^t F(s,x_n(s),(Vx_n)(s))ds, \int_0^t F(s,x(s),(Vx)(s))ds\right)$$
$$\leq \int_0^t D\left(F(s,x_n(s),(Vx_n)(s)),F(s,x(s),(Vx)(s))\right)ds.$$

Also, $V:\Omega\to\mathbb{E}^n$ defined by $(Vx)(t)=\int_0^t K(t,s)x(s)ds$ is a continuous operator, because

$$D((Vx_n)(t),(Vx)(t)) = D\left(\int_0^t K(t,s)x_n(s)ds, \int_0^t K(t,s)x(s)ds\right)$$
$$\leq \int_0^t D\left(K(t,s)x_n(s),K(t,s)x(s)\right)ds$$
$$\leq K_\Delta \int_0^t D(x_n(s),x(s))ds \to 0 \text{ as } n\to\infty.$$

Thus by **(H1)**, it follows that $D((Px_n)(t),(Px)(t))\to 0$ as $n\to\infty$ uniformly on $[0,T]$, so P is a continuous operator on $[0,T]$.

The function $t\to\int\limits_0^t \mu(\cdot)ds$ is uniformly continuous on the closed set $[0,T]$, i.e. there exist $\eta>0$ such that $\left|\int_s^t \mu(\tau)d\tau\right| \leq \frac{\varepsilon}{2}$ for all $t,s\in[0,T]$ with $|t-s|<\eta$.

Further, for each $m\geq 1$, we divide $[0,T]$ into m subintervals $[t_i,t_{i+1}]$ with $t_i=\frac{iT}{m}$.

$$x_m(t)=\begin{cases} x_0 & \text{if t} \in [0,\text{t}_1],\\ (Px_m)(t-t_i) & \text{if t} \in [\text{t}_\text{i},\text{t}_{\text{i}+1}].\end{cases}$$

Then $x_m(\cdot)\in\Omega$ for every $m\geq 1$. Moreover, for $t\in[0,t_1]$, we have

$$D\left((Px_m)(t),x_m(t)\right) = D\left(\int_0^t F(s,x_m(s),(Vx_m)(s)),\hat{0}\right)ds$$
$$\leq \int_0^{t_1} D\left(F(s,x_m(s),(Vx_m)(s)),\hat{0}\right)ds \leq \int_0^{t_1}\mu(s)ds,$$

and for $t\in[t_i,t_{i+1}]$, we have $t-t_i\leq\frac{T}{m}$ and hence

$$D\left((Px_m)(t),x_m(t)\right) = D\left((Px_m)(t),(Px_m)(t-t_i)\right)$$
$$= D\left(\int_0^t F(s,x_m(s),(Vx_m)(s))ds, \int_0^{t_i} F(s,x_m(s),(Vx_m)(s))ds\right)$$
$$= D\left(\int\limits_{t-t_i}^{t} F(s,x_m(s),(Vx_m)(s))ds,\hat{0}\right)$$
$$\leq \int\limits_{t-T/m}^{t} D\left(F(s,x_m(s),(Vx_m)(s))ds,\hat{0}\right)ds$$

$$\leq \int_{t-T/m}^{t} \mu(s)ds.$$

Therefore $\lim_{m\to\infty} D((Px_m)(t), x_m(t)) = 0$ on [0,T]. Let $A = \{x_m(\cdot); m \geq 1\}$. We claim that A is equicontinuous on $[0, T]$. If $t, s \in [0, T/m]$, then $D(x_m(t), x_m(s)) = 0$. If $0 \leq s \leq T/m \leq t \leq T$, then

$$\begin{aligned} D\left(x_m(t), x_m(s)\right) &= D\left(x_0 + \int_0^{t-T/m} F(\sigma, x_m(\sigma), (Vx_m)(\sigma))d\sigma, x_0\right) \\ &\leq \int_0^{t-T/m} D\left(F(\sigma, x_m(\sigma), (Vx_m)(\sigma)), \hat{0}\right) d\sigma \\ &\leq \int_0^{t-T/m} \mu(\sigma)d\sigma \leq \int_0^{t} \mu(\sigma)d\sigma < \epsilon/2, \end{aligned}$$

for $|t-s| < \eta$. If $T/m \leq s \leq t \leq T$, then

$$D\left(x_m(t), x_m(s)\right) < \epsilon/2 \quad \text{when} \quad |t-s| < \epsilon.$$

Therefore A is equicontinuous on [0,T]. Set $A(t) = \{x_m(t); m \geq 1\}$ for $t \in [0, T]$. We are to show that $A(t)$ is precompact for each $t \in [0, T]$. We have

$$\rho(A(t)) \leq \rho\left(\int_0^{t-T/m} F(s, A(s), (VA)(s))ds\right) + \rho\left(\int_{t-T/m}^{t} F(s, A(s), (VA)(s))ds\right).$$

Given $\epsilon > 0$, we can find $m(\epsilon) > 0$, such that $\int_{t-T/m}^{t} \mu(s)ds < \epsilon/2$, for all $t \in [0, T]$ and $m \geq m(\epsilon)$. Hence

$$\begin{aligned} &\rho\left(\int_{t-T/m}^{t} F(s, A(s), (VA)(s))ds\right) \\ &= \rho\left(\left\{\int_{t-T/m}^{t} F(s, x_m(s), (Vx_m))ds; m \geq n(\epsilon)\right\}\right) \\ &\leq 2\int_{t-T/m}^{t} \mu(s)ds < \epsilon. \end{aligned}$$

It follows that

$$\begin{aligned} \rho(A(t)) \leq \rho\left(\int_0^t F(s, A(s), (VA)(s))ds\right) &\leq 2\int_0^t \rho\left(F(s, A(s), (VA)(s))\right) ds \\ &\leq 2\int_0^t \lambda(s)[\rho(A(s)) + \rho((VA)(s))]ds. \end{aligned}$$

However,

$$\rho(VA(s)) = \rho\left(\int_0^t K(t,s)A(s)ds\right) = \rho\left(\left\{\int_0^t K(t,s)x_m(s)ds;\, m \geq 1\right\}\right)$$
$$\leq 2\int_0^t \rho\left(\{K(t,s)x_m(s);\, m \geq 1\}\right)ds \leq 2\int_0^t K_\Delta \rho\left(\{x_m(s);\, m \geq 1\}\right)ds$$
$$= 2\int_0^t K_\Delta \rho(A(s))ds$$

and

$$\int_0^t \rho\left(VA(s)\right)ds \leq \int_0^t 2\int_0^s K_\Delta \rho\left(A(\tau)\right)d\tau\, ds$$
$$= 2\int_0^t \int_\tau^t K_\Delta \rho\left(A(\tau)\right)dsd\tau$$
$$= 2\int_0^t K_\Delta (t-\tau)\rho(A(\tau))d\tau \leq K_\Delta T\int_0^t \rho(A(\tau))d\tau.$$

Therefore we obtain that

$$\rho(A(t)) \leq 2\int_0^t \lambda(s)[\rho\left(A(s)\right) + K_\Delta T\rho\left(A(s)\right)]ds.$$

Let $R = e^{2(1+K_\Delta T)\int_0^T \lambda(t)dt}$. Due to Gronwall inequality

$$\rho(A(t)) \leq R\int_0^t \rho\left(A(s)\right)ds.$$

Therefore $\rho(A(t)) = 0$ and hence $A(t)$ is precompact for every $t \in [0,T]$. Since A is equicontinuous and $A(t)$ is precompact, one has that Arzela-Ascoli theorem holds true in our case. Thus (passing to subsequences if necessary) the sequence $\{x_n(t)\}_{n=1}^\infty$ converges uniformly on $[0,T]$ to a continuous function $x(\cdot) \in \Omega$. Due to the triangle inequality

$$D\left((Px)(t), x(t)\right) \leq D\left((Px)(t), (Px_n)(t)\right)$$
$$+D\left((Px_n)(t), x_n(t)\right) + D\left(x_n(t), x(t)\right) \to 0,$$

we have $(Px)(t) = x(t)$ for all $t \in [0,T]$, i.e. $x(\cdot)$ is a solution of (1.1). □

3.2. Remark. From Theorem 3.1 it is easy to see that the solution set of (1.1) denoted by

$$\Omega = \left\{x(\cdot) \in C([0,T], \mathbb{E}^n) : \sup_{t\in[0,T]} D(x(t), x_0) \leq c\right\}$$

is compact.

4. Conclusion

We pay our attention to find existence of solution of fuzzy integro-differential equations under mild assumption as compared with the already existing results in the literature, To overcome some difficulties as lack of compactness and other restrictive properties of fuzzy space $\mathbb{E}^n$, we use Kuratowski measure of non compactness, which enables us to use Arzela-Ascoli theorem.

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