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Generalized Chebyshev polynomials

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Abstract

We generalize the first and second kind Chebyshev polynomials by using the concepts and the operational formalism of the Hermite polynomials of the Kampé de Fériet type. We will see how it is possible to derive integral representations for these generalized Chebyshev polynomials. Finally we will use these results to state several relations for Gegenbauer polynomials.

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1. Introduction

It is well known that the explicit form of the second kind Chebyshev polynomials [1] reads

$$U_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n-k)!(2x)^{n-2k}}{k!(n-2k)!} \tag{1.1}$$

In a previous paper [2] we have stated for these polynomials an integral representation of the type:

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t} t^n H_n\left(2x, -\frac{1}{t}\right) dt \tag{1.2}$$

where:

$$H_n(x, y) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{y^k x^{n-2k}}{k!(n-2k)!} \tag{1.3}$$

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are the two-variable Hermite polynomials of Kampé de Fériet [3, 4] type, with generating function given by the formula

$$e^{(xt+yt^2)} = \sum_{0}^{+\infty} \frac{t^n}{n!} H_n(x,y) \tag{1.4}$$

It is also possible to state a different representation for the second kind Chebyshev polynomials $U_n(x)$ by rearranging the argument of the $H_n(x,y)$ polynomials. In fact [5], by noting that

$$\begin{aligned} t^n H_n\left(2x, -\frac{1}{t}\right) &= n! t^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2x)^{n-2k}}{t^k k!(n-2k)!} = \\ &= n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k t^k (2xt)^{n-2k}}{k!(n-2k)!} = H_n(2xt, -t) \end{aligned} \tag{1.5}$$

and, from the fact that

$$(n-k)! = \int_0^{+\infty} e^{-t} t^{n-k} dt \tag{1.6}$$

we can immediately conclude with

$$U_n(x) = \frac{1}{n!} \int_0^{+\infty} e^{-t} H_n(2xt, -t) dt \tag{1.7}$$

The use of the above integral representations for the second kind Chebyshev polynomials can be used to introduce further generalized polynomial sets, including the two-variable Chebyshev polynomials [6, 7, 8] and the two-variable Gegenbauer polynomials.

2. Two-variable generalized Chebyshev polynomials

Before to proceed, we premise some relevant operational relations involving the generalized Hermite polynomials [5].

1. Proposition. The polynomials $H_n(x,y)$ solve the following partial differential equation:

$$\frac{\partial^2}{\partial x^2} H_n(x,y) = \frac{\partial}{\partial y} H_n(x,y) \tag{2.1}$$

Proof. By deriving, separately with respect to x and to y, in the (1.3), we obtain:

$$\begin{aligned} \frac{\partial}{\partial x} H_n(x,y) &= n H_{n-1}(x,y) \\ \frac{\partial}{\partial y} H_n(x,y) &= n(n-1) H_{n-2}(x,y). \end{aligned} \tag{2.2}$$

From the first of the above relations, by deriving again with respect to x and by noting the second relation in (2.2), we end up with the (2.1). The above results help us to derive an important operational rule. In fact, by considering the differential equation (2.1) as a linear ordinary one in the variable y and by noting that $H_n(x,0) = x^n$, we can immediately state that

$$H_n(x,y) = e^{y \frac{\partial^2}{\partial x^2}} x^n \tag{2.3}$$

□

2. Proposition. The two-variable Hermite polynomials satisfy the following relation

$$\left(x + 2y\frac{\partial}{\partial x}\right)^n = \sum_{s=0}^{n} \binom{n}{s} H_n(x,y)\,(2y)^{n-s}\,\frac{\partial^{n-s}}{\partial x^{n-s}} \tag{2.4}$$

Proof. By multiplying the l.h.s. of the above equation by $\frac{t^n}{n!}$ and then summing up, we find:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!}\left(x + 2y\frac{\partial}{\partial x}\right)^n = e^{t\left(x+2y\frac{\partial}{\partial x}\right)} \tag{2.5}$$

To develope the exponential in the r.h.s. of the (2.5) we need to apply the Weyl identity

$$e^{(A+B)} = e^A e^B e^{[A,B]/2} \tag{2.6}$$

and then we have to calculate the commutator of the two operators:

$$\left[tx, t2y\frac{\partial}{\partial x}\right] = -2t^2 y \tag{2.7}$$

which help us to write:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!}\left(x + 2y\frac{\partial}{\partial x}\right)^n = e^{xt+yt^2} e^{2ty\frac{\partial}{\partial x}}(1). \tag{2.8}$$

After expanding and manipulating the r.h.s. of the previous relation and by equating the like t powers we find immediately the (2.4).

The above result gives us another important operational rule for the generalized Hermite polynomials. By using in fact the identity stated in equation (2.4), we have

$$e^{y\frac{\partial^2}{\partial x^2}} x^n = \sum_{s=0}^{n} (2y)^s \binom{n}{s} H_n(x,y)\frac{\partial^s}{\partial x^s}(1) \tag{2.9}$$

Applying (2.4) at 1 and comparing with (2.5) gives

$$e^{y\frac{\partial^2}{\partial x^2}} x^n = \left(x + 2y\frac{\partial}{\partial x}\right)^n (1) \tag{2.10}$$

□

Finally, we can state

3. Proposition. The Hermite polynomials $H_n(x,y)$ solve the following differential equation:

$$2y\frac{\partial^2}{\partial x^2}H_n(x,y) + x\frac{\partial}{\partial x}H_n(x,y) = nH_n(x,y) \tag{2.11}$$

Proof. By using the results derived from the *Proposition 2*, we can easily write that:

$$\left(x + 2y\frac{\partial}{\partial x}\right) H_n(x,y) = H_{n+1}(x,y) \tag{2.12}$$

and from the first of the recurrence relations stated in (2.2):

$$\frac{\partial}{\partial x} H_n(x,y) = nH_{n-1}(x,y) \tag{2.13}$$

we have:

$$\left(x + 2y\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right) H_n(x,y) = nH_n(x,y) \tag{2.14}$$

which is the thesis.

From this statement can be also derived an important recurrence relation. By exploiting, in fact, the relation (2.12), we obtain:

$$H_{n+1}(x,y) = xH_n(x,y) + 2y\frac{\partial}{\partial x} H_n(x,y) \tag{2.15}$$

and then we can conclude with:

$$H_{n+1}(x,y) = xH_n(x,y) + 2nyH_{n-1}(x,y). \tag{2.16}$$

□

1. Definition. Let x, y be write real variables and let α a real parameter, we call generalized Chebyshev polynomials of second kind, the polynomials defined by the following relation:

$$U_n(x,y;\alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t} H_n(2xt, -yt)dt \tag{2.17}$$

By using the recurrence relations relevant to the two-variable Hermite polynomials, proved in first chapter, we can state the following:

4. Proposition. The generalized Chebyshev polynomials $U_n(x,y;\alpha)$ satisfy the following recurrence relations:

$$\begin{aligned} \frac{\partial}{\partial y} U_n(x,y;\alpha) &= \frac{\partial}{\partial \alpha} U_{n-2}(x,y;\alpha) \\ \frac{\partial}{\partial x} U_n(x,y;\alpha) &= -2\frac{\partial}{\partial \alpha} U_{n-1}(x,y;\alpha). \end{aligned} \tag{2.18}$$

Proof. By deriving with respect to y in the relation (2.17), we get:

$$\frac{\partial}{\partial y} U_n(x,y;\alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t} \frac{\partial}{\partial y} H_n(2xt, -yt)dt \tag{2.19}$$

and since:

$$\frac{\partial}{\partial y} H_n(2xt, -yt) = (-t)n(n-1)H_{n-2}(2xt, -yt) \tag{2.20}$$

we obtain:

$$\frac{\partial}{\partial y} U_n(x,y;\alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t}(-t)n(n-1)H_{n-2}(2xt, -yt)dt \tag{2.21}$$

which gives the first of the (2.18).

The second relation can be obtained in the same way, by noting that:

$$\frac{\partial}{\partial x} H_n(2xt, -yt) = 2tnH_{n-1}(2xt, -yt). \tag{2.22}$$

□

5. Proposition. The generalized Chebyshev polynomials $U_n(x, y; \alpha)$ satisfy the following Cauchy problem:

$$\begin{cases} \frac{\partial^2}{\partial x^2} U_n(x, y; \alpha) = 4\frac{\partial^2}{\partial \alpha \partial y} U_n(x, y; \alpha) \\ U_n(x, 0; \alpha) = \frac{(2x)^n}{\alpha^{n+1}} \end{cases}. \tag{2.23}$$

Proof. By deriving with respect to x in the second identity of (2.18), we find:

$$\frac{\partial^2}{\partial x^2} U_n(x, y; \alpha) = 4\frac{\partial}{\partial \alpha}\left(\frac{\partial}{\partial \alpha} U_{n-2}(x, y; \alpha)\right) \tag{2.24}$$

and then, since:

$$\frac{\partial}{\partial \alpha} U_{n-2}(x, y; \alpha) = \frac{\partial}{\partial y} U_n(x, y; \alpha) \tag{2.25}$$

we obtain:

$$\frac{\partial^2}{\partial x^2} U_n(x, y; \alpha) = 4\frac{\partial^2}{\partial \alpha \partial y} U_n(x, y; \alpha) \tag{2.26}$$

By setting $y = 0$ in the relation (2.17), we have:

$$U_n(x, 0; \alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t} H_n(2xt, 0)dt \tag{2.27}$$

and since:

$$H_n(2xt, 0) = (2xt)^n \tag{2.28}$$

we find:

$$U_n(x, 0; \alpha) = \frac{(2x)^n}{n!}\int_0^{+\infty} e^{-\alpha t} t^n dt \tag{2.29}$$

that is:

$$U_n(x, 0; \alpha) = \frac{(2x)^n}{\alpha^{n+1}}. \tag{2.30}$$

The partial differential equation, stated in (2.26), can be viewed as a first order ordinary differential equation for the variable y; and then by using the initial condition founded through the (2.30), we can state the solution:

$$U_n(x, y; \alpha) = e^{\frac{y}{4}\widehat{D}_\alpha^{-1}\frac{\partial^2}{\partial x^2}}\frac{(2x)^n}{\alpha^{n+1}} \tag{2.31}$$

which completely prove the proposition.

The symbol $\widehat{D}_\alpha^{-1}$ denotes the inverse of the derivative [9], defined by

$$(2.32)\qquad \widehat{D}_\alpha^{-1} f(x) = -\int_x^{+\infty} f(t)dt$$

In Definition (2.17), we have introduced the generalized Chebyshev polynomials $U_n(x, y, \alpha)$ by using a specific integral form of the standard second kind Chebyshev polynomials.

By using the same procedure, it is possible to obtain similar integral representations for the first kind Chebyshev polynomials. There are some relevant applications in the field of electromagnetics, and in particular they are an important tool to solve integral equations [10]. In fact, since their explicit form is [1, 2]:

$$(2.33)\qquad T_n(x) = \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n-k-1)!(2x)^{n-2k}}{k!(n-2k)!},$$

we can immediately derive that

$$(2.34)\qquad T_n(x) = \frac{1}{2(n-1)!}\int_0^{+\infty} e^{-t} t^{n-1} H_n\left(2x, -\frac{1}{t}\right) dt\,.$$

We have also introduced [1, 2, 6] Chebyshev-like polynomials by using the method of integral representation:

$$(2.35)\qquad W_n(x) = \frac{2}{(n+1)!}\int_0^{+\infty} e^{-t} t^{n+1} H_n\left(2x, -\frac{1}{t}\right) dt\,.$$

We can now generalize the above Chebyshev polynomials.

□

2. Definition. Let x, y real variables and let α a real parameter, we define the following three polynomials sets

$$(2.36)\qquad U_n(x, y; \alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t} t^n H_n\left(2x, -\frac{y}{t}\right) dt\,,$$

$$(2.37)\qquad T_n(x, y; \alpha) = \frac{1}{2(n-1)!}\int_0^{+\infty} e^{-\alpha t} t^{n-1} H_n\left(2x, -\frac{y}{t}\right) dt$$

and:

$$(2.38)\qquad W_n(x, y; \alpha) = \frac{1}{(n+1)!}\int_0^{+\infty} e^{-\alpha t} t^{n+1} H_n\left(2x, -\frac{y}{t}\right) dt\,.$$

It is immediate to note the following relations.

6. Proposition. The generalized Chebyshev polynomials satisfy the following recurrence relations:

$$(2.39)\qquad \begin{aligned} \frac{\partial}{\partial\alpha} U_n(x, y; \alpha) &= -\frac{1}{2}(n+1) W_n(x, y; \alpha) \\ \frac{\partial}{\partial\alpha} T_n(x, y; \alpha) &= -\frac{n}{2} U_n(x, y; \alpha)\,. \end{aligned}$$

Proof. By deriving with respect to α in the relation (2.17), we find:

$$\frac{\partial}{\partial\alpha}U_n(x,y;\alpha) = -\frac{1}{n!}\int_0^{+\infty} e^{-\alpha t}t^{n+1}H_n\left(2x,-\frac{y}{t}\right)dt \tag{2.40}$$

and then the first of equations (2.39), immediatly follows.

In the same way, by following a similar procedure, by using the identity (2.37), we have:

$$\frac{\partial}{\partial\alpha}T_n(x,y;\alpha) = -\frac{1}{2(n-1)!}\int_0^{+\infty} e^{-\alpha t}t^{n}H_n\left(2x,-\frac{y}{t}\right)dt \tag{2.41}$$

and then the thesis.

□

3. Generalized Gegenbauer polynomials

It is worth noting that the Chebyshev polynomials can be viewed as a particular case of the Gegenbauer polynomials [6, 11].

3. Definition. Let x and μ real variables, be $n-th$ order Gegenbauer polynomials [11], the polynomials defined by the follow relation:

$$C_n^{(\mu)}(x) = \frac{1}{\Gamma(\mu)}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2x)^{n-2k}\Gamma(n-k+\mu)}{k!(n-2k)!} \tag{3.1}$$

where $\Gamma(\mu)$ is the Euler function.

By recalling the integral representation of the above Euler function:

$$\Gamma(\mu) = \int_0^{+\infty} e^{-t}t^{\mu-1}dt \tag{3.2}$$

and by using the same arguments exploited for the Chebyshev case, we can state the integral representation for the Gegenbauer polynomials:

$$C_n^{(\mu)}(x) = \frac{1}{n!\Gamma(\mu)}\int_0^{+\infty} e^{-t}t^{n+\mu-1}H_n\left(2x,-\frac{1}{t}\right)dt\,. \tag{3.3}$$

We can also generalize the Gegenbauer polynomials by using their integral representation.

4. Definition. Let x, y be write real variables and let α a real parameter, we call generalized Gegenbauer polynomials, the polynomials defined by the following relation:

$$C_n^{(\mu)}(x,y;\alpha) = \frac{1}{n!\Gamma(\mu)}\int_0^{+\infty} e^{-\alpha t}t^{n+\mu-1}H_n\left(2x,-\frac{y}{t}\right)dt\,. \tag{3.4}$$

The above integral representation is a very flexible tool; in fact it can be exploited to derive interesting relations regarding the Gegenbauer polynomials and also the Chebyshev polynomials [4, 6].

7. Proposition. Let $\xi \in \mathbf{R}$, such that $|\xi| < 1$, $\mu \neq 0$. The generating function of the polynomials $C_n^{(\mu)}(x,y;\alpha)$ is given by:

$$\sum_{n=0}^{+\infty}\xi^n C_n^{(\mu)}(x,y;\alpha) = \frac{1}{[\alpha-2x\xi+y\xi^2]^{\mu}}\,. \tag{3.5}$$

Proof. By multiplying both sides of the identity (3.4), by ξ^n and by summing up over n, we get:

$$\sum_{n=0}^{+\infty} \xi^n C_n^{(\mu)}(x,y;\alpha) = \int_0^{+\infty} \sum_{n=0}^{+\infty} \frac{\xi^n t^n}{n!\Gamma(\mu)} e^{-\alpha t} t^{\mu-1} H_n\left(2x, -\frac{y}{t}\right) dt \tag{3.6}$$

and by noting that:

$$\sum_{n=0}^{+\infty} \frac{(\xi t)^n}{n!} H_n\left(2x, -\frac{y}{t}\right) = \exp\left[\xi\,(2xt) + \xi^2(-yt)\right] \tag{3.7}$$

we can write:

$$\sum_{n=0}^{+\infty} \xi^n C_n^{(\mu)}(x,y;\alpha) = \int_0^{+\infty} \frac{1}{\Gamma(\mu)} e^{-\alpha t} e^{\xi(2xt)+\xi^2(-yt)} t^{\mu-1}\, dt\,. \tag{3.8}$$

Finally, by integrating over t , by using the integral representation of the Euler function, we obtain the thesis.

□

8. Proposition. The generalized second kind Chebyshev polynomials and the generalized Gegenbauer polynomials satisfy the following recurrence relation:

$$(-1)^m \frac{\partial^m}{\partial \alpha^m} U_n(x,y;\alpha) = m! C_n^{(m+1)}(x,y;\alpha)\,. \tag{3.9}$$

Proof. By deriving with respect to α in the relation (2.36), m-times, we get:

$$\frac{\partial^m}{\partial \alpha^m} U_n(x,y;\alpha) = \frac{(-1)^m}{n!} \int_0^{+\infty} e^{-\alpha t} t^{n+m} H_n\left(2x, -\frac{y}{t}\right) dt\,. \tag{3.10}$$

The r.h.s. of the above identity can be written in the form:

$$\frac{(-1)^m}{n!} \int_0^{+\infty} e^{-\alpha t} t^{n+m} H_n\left(2x, -\frac{y}{t}\right) dt = \frac{(-1)^m m!}{n! m!} \int_0^{+\infty} e^{-\alpha t} t^{n+m} H_n\left(2x, -\frac{y}{t}\right) dt \tag{3.11}$$

and then the thesis follows.

By using *Proposition 3*, it is easy to note that:

$$\left[(2x) + \left(-\frac{y}{t}\right)\frac{\partial}{\partial x}\right] H_n\left(2x, -\frac{y}{t}\right) = H_{n+1}\left(2x, -\frac{y}{t}\right) \tag{3.12}$$

which can be used to derive the following results:

□

3.1. Theorem. *The generalized Gegenbauer polynomials* $C_n^{(\mu)}(x,y;\alpha)$ *satisfy the recurrence relations:*

$$\frac{n+1}{2\mu} C_{n+1}^{(\mu)}(x,y;\alpha) = x C_n^{(\mu+1)}(x,y;\alpha) - y C_{n-1}^{(\mu+1)}(x,y;\alpha) \tag{3.13}$$

and:

(3.14) $$\frac{\partial}{\partial y} C_n^{(\mu)}(x,y;\alpha) = -\mu C_{n-2}^{(\mu+1)}(x,y;\alpha)\,.$$

Proof. By using the relation (3.12), we can write the generalized Gegenbauer polynomial of order $n+1$, in the form:

(3.15) $$C_{n+1}^{(\mu)}(x,y;\alpha) = \\ = \frac{1}{(n+1)!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu} \left[(2x) + \left(-\frac{y}{t}\right)\frac{\partial}{\partial x}\right] H_n\left(2x, -\frac{y}{t}\right) dt\,.$$

After exploiting the r.h.s of the above identity, we get:

(3.16) $$C_{n+1}^{(\mu)}(x,y;\alpha) =$$

(3.17) $$= \frac{1}{(n+1)!\Gamma(\mu)} \left[\int_0^{+\infty} e^{-\alpha t} t^{n+\mu} (2x) H_n\left(2x, -\frac{y}{t}\right) dt - \\ + \int_0^{+\infty} e^{-\alpha t} t^{n-1+\mu} y(2n) H_{n-1}\left(2x, -\frac{y}{t}\right) dt\right]$$

and then:

(3.18) $$C_{n+1}^{(\mu)}(x,y;\alpha) =$$

(3.19) $$= \frac{2x}{(n+1)!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu} H_n\left(2x, -\frac{y}{t}\right) dt - \\ + \frac{2yn}{(n+1)!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n-1+\mu} H_{n-1}\left(2x, -\frac{y}{t}\right) dt\,.$$

We can rearrange the above relation in the form:

(3.20) $$\frac{n+1}{2} C_{n+1}^{(\mu)}(x,y;\alpha) =$$

(3.21) $$= x\frac{1}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu} H_n\left(2x, -\frac{y}{t}\right) dt - \\ + y\frac{1}{(n-1)!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n-1+\mu} H_{n-1}\left(2x, -\frac{y}{t}\right) dt$$

and finally:

(3.22) $$\frac{n+1}{2\mu} C_{n+1}^{(\mu)}(x,y;\alpha) =$$

(3.23) $$= x\frac{1}{n!\Gamma(\mu+1)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu} H_n\left(2x, -\frac{y}{t}\right) dt - \\ + y\frac{1}{(n-1)!\Gamma(\mu+1)} \int_0^{+\infty} e^{-\alpha t} t^{n-1+\mu} H_{n-1}\left(2x, -\frac{y}{t}\right) dt$$

which proves the (3.13).

To show the recurrence relation in the (3.14), it is important to note that:

(3.24) $$\frac{\partial}{\partial y} H_n\left(2x, -\frac{y}{t}\right) = -\frac{n(n-1)}{t} H_{n-2}\left(2x, -\frac{y}{t}\right)\,.$$

In fact, by deriving respct to y in the (3.14), we get:

(3.25) $$.\frac{\partial}{\partial y} C_n^{(\mu)}(x,y;\alpha) = \frac{1}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu-1} \frac{\partial}{\partial y} H_n\left(2x, -\frac{y}{t}\right) dt$$

and by using the (3.24), we can write:

(3.26) $$\frac{\partial}{\partial y} C_n^{(\mu)}(x,y;\alpha) = -\frac{n(n-1)}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n-2+\mu} H_{n-2}\left(2x, -\frac{y}{t}\right) dt$$

which immediately gives the thesis.

□

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