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Functional equations related to weightable quasi-metrics

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Abstract

Starting from the definition of a weightable quasi-metric we observe that several functional equations are induced in a natural way. Studying these equations we characterize weightable quasi-metrics and show that they define representable total preorders. We also analyze how to retrieve weightable quasi-metrics from real-valued functions satisfying suitable functional equations.

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1. Introduction

In the last years, (weightable) quasi-metric spaces have proven to be useful in modeling many processes that arise in Theoretical Computer Science and that involve some situation of *asymmetry*. The aforementioned usefulness is due to the fact that quasi-metric spaces lack the symmetry and the Hausdorffness enjoyed by metric spaces. This fact allows to introduce techniques of measuring that, contrarily to the metric ones, reflect the asymmetry inherent to the computational process. It is then possible to develop a “metric” foundation for partial orders reasoning techniques in the spirit of D. Scott ([40, 34, 25]). Recent applications of the aforesaid metric tools based on the use

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of (weightable) quasi-metrics to Complexity Analysis of Algorithms, Denotational Semantics and Program Correctnes can be found in [33, 32, 34, 17, 23, 37, 38, 26] and [25].

Inspired in part by its utility in Theoretical Computer Science, we focus our attention on the definition of a (weightable) quasi-metric ([10, 21]) and we immediately encounter some functional equation that appears in a natural way. As a matter of fact weightable quasi-metrics are characterized as the quasi-metrics that satisfy a certain functional equation that we call circuit invariance. In addition, the disymmetry function (defined in Section 2) of a weightable quasi-metric satisfies Sincov's functional equation and induces a total preorder, different, in general, from the specialization order directly induced by the given quasi-metric. To conclude, we analyze the possibility of retrieving a weightable quasi-metric from a real-valued bivariate function that satisfies Sincov's functional equation. This allows us to establish a link between apparently disparate notions, namely: i) weightable quasi-metrics, ii) real-valued bivariate functions that satisfy Sincov's functional equation, and iii) total preorders that are representable through a real-valued utility function.

This possibility of relating weightable quasi-metrics, functional equations and representable total preorders is undoubtedly an important motivation, besides their aforementioned usefulness in Theoretical Computer Science, for the study of this particular kind of quasi-metrics.

The structure of the manuscript goes as follows.

The key definitions and notations are listed and discussed in Section 2. In Section 3 we consider and analyze different functional equations in two variables that are closely associated to the concept of a quasi-metric. In Section 4 we relate those functional equations to some kinds of orderings. In Section 5 we characterize when a positively weightable quasi-metric can be retrieved from a real-valued bivariate function that satisfies either the circuit invariance functional equation or Sincov's functional equation.

2. Preliminaries

In what follows, X will denote a nonempty set and $\mathbb{R}$ will stand for the set of real numbers.

The definition of a metric space is usually attributed to M. Fréchet (see [13]). However, asymmetric distances had already been implicitly considered by Pompeiu in [28], as mentioned in the seminal book by F. Hausdorff issued in 1914 (see [16]). Hausdorff introduced a wide sort of ideas in this direction. Having these ideas in mind, the formal definition of a quasi-metric space was issued by W. A. Wilson in 1931. (See [44, 18]). Other miscellaneous extensions, special cases and variations of the concept of a metric space (e.g. partial metric spaces, pseudo-metric spaces, etc.) are often encountered in the specialized literature [41, 25, 19].

2.1. Definition. Let X be a nonempty set. Following the modern terminology ([20]), by a *quasi-metric* on X we mean a function $d : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$ the following conditions hold:

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) + d(y, z) \geq d(x, z)$.

Of course a metric on a set X is a quasi-metric d on X satisfying, in addition, the following condition for all $x, y \in X$:

(iii) $d(x, y) = d(y, x)$.

By a *quasi-metric space* we mean a pair (X, d) such that X is a nonempty set and d is a quasi-metric on X.

If d is a quasi-metric on a set X, then the relation $\leq_d$ on X given by $x \leq_d y \Leftrightarrow d(x, y) = 0$, is an order on X (see Definition 4.1 in Section 4) called the specialization order of d.

Given a quasi-metric d on X, and an ordered pair $(x, y) \in X \times X$, the real number $F(x, y) = d(x, y) - d(y, x)$ is said to be the *disymmetry of the pair* (x, y). The function $F : X \times X \to \mathbb{R}$ defined by $F(x, y) = d(x, y) - d(y, x) \quad (x, y \in X)$ is said to be the *disymmetry function* associated to the quasi-metric d on X.

2.2. Remark. The original definition of a quasi-metric, due to Wilson ([44]) is a bit more restrictive than Definition 2.1 above. Namely, in the sense of Wilson ([44]), a *quasi-metric* d on X is a quasi-metric in the sense of Definition 2.1 which satisfies in addition that $d(x, y) = 0 \Leftrightarrow x = y$ for every $x, y \in X$. Obviously condition (i) in Definition 2.1 is less restrictive than the preceding condition (see also Example 2.3 below, due to Hausdorff [16]). Nowadays, according, for instance, to [29], quasi-metrics in the sense of Wilson are called T_1 *quasi-metrics.* As a matter of fact, any quasi-metric generates a topology in a natural way. This topology will satisfy the separation axiom T_1 if and only if the given quasi-metric is a quasi-metric in the sense of Wilson. This is the reason why quasi-metrics in the sense of Wilson are called T_1-quasi-metrics.

2.3. Example. ([16]) Let $\mathcal{H}$ denote the family of non-empty compact sets of the real plane $\mathbb{R}^2$. Let d_E denote the usual Euclidean distance on the real plane $\mathbb{R}^2$. Given $A, B \in \mathcal{H}$, consider the non-negative real number $d_H(A, B)$ defined as follows:

$$d_H(A, B) = \max_{a \in A}\{\min_{b \in B} d_E(a, b)\}.$$

It is well-known that d_H is a quasi-metric (in the sense of Definition 2.1 above) on the real plane. Moreover, it is not a metric, that is $d_H(A, B)$ could be different from $d_H(B, A) \quad (A, B \in \mathcal{H})$. This quasi-metric d_H is said to be the *Hausdorff quasi-metric* on $\mathcal{H}$. By the way, note that this quasi-metric is not T_1, i.e., d_H does not satisfy Wilson's original definition, since if $A \subsetneq B \in \mathcal{H}$, we have that $d_H(A, B) = 0$ but $d_H(B, A) \neq 0$ as well.

(A very important use in Pure Mathematics of the Hausdorff quasi-metric d_H appears in the definition and study of *fractal sets*. See Chapter II, Section 6 in [6] for further details).

2.4. Example. Let $d_S : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ be the function defined by

$$d_S(x, y) = \begin{cases} \min\{y - x, 1\} & \text{if } x \leq y \\ 1 & \text{if } x > y \end{cases}.$$

It is easy to check that d_S is a quasi-metric ([29]), known as the *Sorgenfrey quasi-metric*, which is T_1.

2.5. Definition. ([10, 25, 20]) Let X be a nonempty set. A quasi-metric d on X, as well as the associated quasi-metric space (X, d), are said to be *weightable* if there exists a function $w : X \to \mathbb{R}$ such that $d(x, y) + w(x) = d(y, x) + w(y)$ holds for every $x, y \in X$. The function w is called a *weighting function* for d.

In the particular case in which there is at least one weighting function that only takes non-negative values ($w(X) \subseteq [0, +\infty)$) we say that the quasi-metric d is *positively weightable*. (See Example 2.8 below).

2.6. Example. Let $d : \mathbb{R}\times\mathbb{R} \to [0, +\infty)$ be the function given by $d(x, y) = \max\{y-x, 0\}$ for all $x, y \in \mathbb{R}$. It is clear that $(\mathbb{R}, d)$ is a quasi-metric space which is weightable with weighting function $w : \mathbb{R} \to \mathbb{R}$ given by $w(x) = x$ for all $x \in \mathbb{R}$.

2.7. Remarks.

(1) Note that if (X, d) is a weightable quasi-metric space with weighting function w, then the disymmetry function F associated to d is given by $F(x, y) = w(y)-w(x)$ for all $x, y \in X$.

(2) The original definition of a weightable quasi-metric ([10, 25]) does not force the weighting functions to take non-negative values. However, inspired by the applications in Theoretical Computer Science, other authors (see e.g. [21]) define a weightable quasi-metric by imposing the weighting functions to be non-negative. By this reason, we have pointed out this nuance, distinguishing accordingly between “weightable quasi-metrics" and “positively weightable quasi-metrics" in Definition 2.5.

In the next examples we provide a few weightable quasi-metrics which play a central role in several fields of Theoretical Computer Science.

2.8. Examples. We introduce now several well-known examples of weightable quasi-metrics.

1. The *domain of words* Σ^∞ (see e.g. [20, 25, 31, 35]) consists of all finite and infinite sequences over a nonempty set Σ, ordered by $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y, where we assume that the empty sequence ϕ is an element of Σ^∞.

   For each $x, y \in \Sigma^\infty$ denote by $x \sqcap y$ the longest common prefix of x and y, and for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of x. Thus $\ell(x) \in [1, \infty]$ whenever $x \neq \phi$, and $\ell(\phi) = 0$. Then ([20, 25]) the function $d : \Sigma^\infty \times \Sigma^\infty \to [0, +\infty)$ given by

   $$d(x, y) = 2^{-\ell(x\sqcap y)} - 2^{-\ell(x)},$$

   is a positively weightable quasi-metric on Σ^∞ with weighting function w given by $w(x) = 2^{-\ell(x)}$ for all $x \in \Sigma^\infty$. Note that the specialization order $\leq_d$ coincides with $\sqsubseteq$. Moreover, the disymmetry function associated to d is given by $F(x, y) = 2^{-\ell(y)} - 2^{-\ell(x)}$ for all $x, y \in \Sigma^\infty$.

2. The *interval domain* $I([0, 1])$ ([11, 22, 25]) consists of the nonempty closed intervals of $[0, 1]$ ordered by reverse inclusion, i.e., $[a, b] \sqsubseteq [c, d] \Leftrightarrow [a, b] \supseteq [c, d]$. In particular, points of [0,1] are identified with the singleton intervals. Then, the function d defined on $I([0, 1]) \times I([0, 1])$ by

   $$d([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\} - (b - a),$$

   is a weightable quasi-metric on $I([0, 1])$ with weighting function w given by $w([a, b]) = b - a$, for all $[a, b] \in I([0, 1])$ (see e.g. [25, 31, 35]). The specialization order $\leq_d$ coincides with $\sqsubseteq$. Moreover, the disymmetry function associated to d is given by $F([a, b], [c, d]) = d + a - b - c$ for all $[a, b], [c, d] \in I([0, 1])$.

3. Denote by ω the set of non-negative integer numbers. The *complexity quasi-metric space* [34] is the pair $(\mathcal{C}, d_{\mathcal{C}})$, where

   $$\mathcal{C} = \{f : \omega \to (0, +\infty] \mid \sum_{n=0}^{+\infty} 2^{-n} \frac{1}{f(n)} < +\infty\},$$

and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ defined by

$$d_{\mathcal{C}}(f,g) = \sum_{n=0}^{+\infty} 2^{-n} \left[\max\left(\frac{1}{g(n)} - \frac{1}{f(n)}, 0 \right) \right].$$

Furthermore, $(\mathcal{C}, d_{\mathcal{C}})$ is weightable with weighting function $w_{\mathcal{C}}$ given by $w_{\mathcal{C}}(f) = \sum_{n=0}^{+\infty}(2^{-n}/f(n))$ for all $f \in \mathcal{C}$. The specialization order of $d_{\mathcal{C}}$ coincides with the pointwise order of $\mathcal{C}$. Moreover, the disymmetry function associated to $d_{\mathcal{C}}$ is given by $F(f,g) = \sum_{n=0}^{+\infty} 2^{-n}[1/g(n) - 1/f(n)]$ for all $f, g \in \mathcal{C}$.

3. Functional equations defined through quasi-metrics

Let us see now how the definition of a weightable quasi-metric gives rise to the consideration of several functional equations. To this end, let us denote by $\mathbb{N}$ the set of positive integer numbers. The following Lemma 3.1 hangs from a well known result of the classical theory of functional equations in two variables.

3.1. Lemma. *Let (X,d) be a quasi-metric space. Assume that the quasi-metric d satisfies the functional equation of the 3-circuit, namely $d(x,y) + d(y,z) + d(z,x) = d(x,z) + d(z,y) + d(y,x)$, for every $x, y, z \in X$. Then d is weightable.*

Proof. By hypothesis we observe that the disymmetry function $F : X \times X \to \mathbb{R}$ given by $F(x,y) = d(x,y) - d(y,x)$ $(x, y \in X)$ satisfies $F(x,y) + F(y,z) = F(x,z), (x,y,z \in X)$. It is well known (see e.g. [4, 5, 15]) that in this case there exists a function $w : X \to \mathbb{R}$ such that $F(x,y) = w(y) - w(x) = d(x,y) - d(y,x)$, for every $x, y \in X$. Therefore d is a weightable quasi-metric. □

The converse of Lemma 3.1 is also true, as well as some other equivalences stated in the following Theorem 3.2.

3.2. Theorem. *Let (X,d) be a quasi-metric space. The following statements are equivalent:*

i) *The quasi-metric d is weightable.*
ii) *The quasi-metric d satisfies the functional equation of the 3-circuit, namely $d(x,y) + d(y,z) + d(z,x) = d(x,z) + d(z,y) + d(y,x)$, for every $x, y, z \in X$.*
iii) *For every $n \geq 3$, $n \in \mathbb{N}$, the quasi-metric d satisfies the functional equation of the n-circuit, namely $d(x_1,x_2)+d(x_2,x_3)+\ldots+d(x_{n-1},x_n)+d(x_n,x_1) = d(x_1,x_n)+d(x_n,x_{n-1}) + \ldots + d(x_3,x_2) + d(x_2,x_1)$, for every $x_1, x_2, x_3, \ldots, x_n \in X$.*
iv) *For some $k \geq 3$, $k \in \mathbb{N}$, the quasi-metric d satisfies the functional equation of the k-circuit, namely $d(x_1,x_2)+d(x_2,x_3)+\ldots+d(x_{k-1},x_k)+d(x_k,x_1) = d(x_1,x_k)+d(x_k,x_{k-1}) + \ldots + d(x_3,x_2) + d(x_2,x_1)$, for every $x_1, x_2, x_3, \ldots, x_k \in X$.*

Proof. *i) ⇒ iii)*:

Since d is weightable by hypothesis, there exists a function $w : X \to \mathbb{R}$ such that $d(x,y) + w(x) = d(y,x) + w(y)$, for every $x, y \in X$. Thus, for every $x_1, x_2, x_3, \ldots, x_n \in X$ we have that $[d(x_1,x_2) + w(x_1)] + [d(x_2,x_3) + w(x_2)] + \ldots + [d(x_{n-1},x_n) + w(x_{n-1})] + [d(x_n,x_1) + w(x_n)] = [d(x_2,x_1) + w(x_2)] + [d(x_3,x_2) + w(x_3)] + \ldots + [d(x_n,x_{n-1}) + w(x_n)] + [d(x_1,x_n) + w(x_1)]$. Hence $d(x_1,x_2) + d(x_2,x_3) + \ldots + d(x_{n-1},x_n) + d(x_n,x_1) = d(x_2,x_1) + d(x_3,x_2) + \ldots + d(x_n,x_{n-1}) + d(x_1,x_n) = d(x_1,x_n) + d(x_n,x_{n-1}) + \ldots + d(x_3,x_2) + d(x_2,x_1)$.

iii) ⇒ ii) and *iii) ⇒ iv)*:

These implications are obvious.

iv) ⇒ ii):

It follows immediately, by taking $x_3 = x_4 = \ldots = x_k$.

ii) $\Rightarrow$ *i*): This has already been stated in Lemma 3.1.
This finishes the proof. $\square$

Now we introduce a necessary definition concerning functional equations.

3.3. Definition. Let X be a nonempty set.

a) A bivariate function $F : X \times X \longrightarrow \mathbb{R}$ is said to satisfy *Sincov's* functional equation if $F(x, y) + F(y, z) = F(x, z)$ holds for every $x, y, z \in X$.
b) A bivariate function $F : X \times X \longrightarrow \mathbb{R}$ is said to satisfy the *separability* equation if $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ holds for every $x, y, z \in X$.

Observe that the notion of Sincov's functional equation is already involved in the previous Theorem 3.2.

The following result is well-known (see e.g. [5], pp. 122 and ff.).

3.4. Proposition. *Let X be a nonempty set. A bivariate function $F : X \times X \longrightarrow \mathbb{R}$ satisfies the separability equation if and only if $F(x, y) = G(x) + H(y) \quad (x, y \in X)$, for some functions $G, H : X \to \mathbb{R}$ that depend of only one variable.*

3.5. Theorem. *Let (X, d) be a quasi-metric space. Let F be the dysymmetry function associated to d. The following statements are equivalent:*

i) *The quasi-metric d is weightable.*
ii) *The disymmetry function F associated to d satisfies Sincov's functional equation $F(x, y) + F(y, z) = F(x, z)$, for every $x, y, z \in X$.*
iii) *The disymmetry function F satisfies the functional equation of separability $F(x, y)+ F(y, z) = F(x, z) + F(y, y)$, for every $x, y, z \in X$.*
iv) *The disymmetry function F satisfies the functional equation $F(x, y) + F(y, z) = F(x, z) + F(t, t)$, for every $x, y, z, t \in X$.*

Proof. *i*) $\Leftrightarrow$ *ii*):

By Theorem 3.1, d is weightable, if and only if $d(x, y) + d(y, z) + d(z, x) = d(x, z) + d(z, y) + d(y, x)$ holds for every $x, y, z \in X$. But this is equivalent to say that $[d(x, y) - d(y, x)] + [d(y, z) - d(z, y)] = [d(x, z) - d(z, x)]$, or, just changing the notation, $F(x, y) + F(y, z) = F(x, z) \quad (x, y, z \in X)$.

ii) $\Leftrightarrow$ *iii*) $\Leftrightarrow$ *iv*):

Just notice that, for all $x \in X$, $F(x, x) = d(x, x) - d(x, x) = 0$ holds. $\square$

4. Orderings induced by functional equations related to weightable quasi-metrics

In this Section 4 we study orderings that are induced in a natural way by weightable quasi-metrics. First we recall some basic definitions concerning orderings.

4.1. Definition. A *preorder* $\precsim$ on an arbitrary nonempty set X is a binary relation on X which is reflexive and transitive. If $\precsim$ is a preorder on X, then the pair $(X, \precsim)$ is said to be a preordered set. An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set X is a preorder such that $[x \precsim y] \vee [y \precsim x]$ holds for every $x, y \in X$.

4.2. Definition. Let X be a nonempty set. Let $\prec$ be an asymmetric binary relation defined on X. Associated to $\prec$ we define the reflexive and total binary relation $\precsim$ given by $x \precsim y \Leftrightarrow \neg(y \prec x) \quad (x, y \in X)$.

An *interval order* $\prec$ is an asymmetric binary relation such that $[(x \prec y) \wedge (z \prec t)] \Rightarrow [(x \prec t) \vee (z \prec y)]$ $(x, y, z, t \in X)$. An interval order $\prec$ is said to be a *semiorder* if $[(x \prec y) \wedge (y \prec z)] \Rightarrow [(x \prec w) \vee (w \prec z)]$, for every $x, y, z, w \in X$.

4.3. Remark. Interval orders are perhaps the best class of ordered structures to build models of uncertainty or to represent and manipulate vague or imperfectly described pieces of knowledge. The notion of an interval order was introduced[¶] by Peter C. Fishburn (see [12]), in order to study models of preference or measurement orderings whose associated indifference may fail to be transitive.

The concept of a semiorder was introduced in [24] to deal with innacuracies in measurements where a nonnegative threshold of discrimination is involved. Semiordered structures are often encountered in a wide range of applications (see e.g. [2] for further details).

We can generate total preorders, interval orders and semiorders from particular solutions of suitable functional equations, as stated in the next straightforward Proposition 4.4, whose proof is omitted for the sake of brevity. (For similar results see e.g. [3])

4.4. Proposition. *Let X be a nonempty set.*

i) *If $F : X \times X \to \mathbb{R}$ satisfies Sincov's functional equation, then the binary relation $\precsim$ defined on X by $x \precsim y \Leftrightarrow F(y, x) \leq 0 \quad (x, y \in X)$ is a total preorder.*
ii) *If $F : X \times X \to \mathbb{R}$ satisfies the separability equation and, in addition $F(t, t) \leq 0 \quad (t \in X)$, then the binary relation defined on X by $x \prec y \Leftrightarrow F(x, y) > 0 \quad (x, y \in X)$ is an interval order.*
iii) *If $F : X \times X \to \mathbb{R}$ satisfies the separability equation and, in addition, there exists a non-positive real constant $K \leq 0$ such that $F(t, t) = K \quad (t \in X)$, then the binary relation defined on X by $x \prec y \Leftrightarrow F(x, y) > 0 \quad (x, y \in X)$ is a semiorder.*

The following result is a direct consequence of Proposition 4.4 (part *i*), Theorem 3.2 and Theorem 3.5 in Section 3 above.

4.5. Corollary. *Let X be a nonempty set. Let d be a weightable quasi-metric defined on X. Let F be the dysymmetry function associated to d. Then, the binary relation $\precsim_F$ defined on X as $x \precsim_F y \Leftrightarrow F(y, x) \leq 0 \Leftrightarrow F(x, y) \geq 0 \quad (x, y \in X)$ is a total preorder.*

4.6. Remark. Observe that given a quasi-metric space (X, d), then the following easy relationship is satisfied: If $x \leq_d y$, then $x \precsim_F y \Leftrightarrow x = y$. Moreover, note that the order relation $\precsim_F$ is total whereas that the specialization oder $\leq_d$ is not. This property could be an advantage to model certain processes in applied contexts in the sense that the order relation $\precsim_F$ allows to compare elements of X that are not comparable with respect to $\leq_d$. For instance, coming back to Example 2.8 (2) we observe that $[a, b] \leq_d [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$, whereas $[a, b] \precsim_F [c, d] \Leftrightarrow b + c \leq a + d$.

4.7. Remark. Suppose that we want to induce interval orders or semiorders from the disymmetry function of a quasi-metric. Taking into account Proposition 4.4, we should look for disymmetry functions that satisfy the separability equation. However, if $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ holds for every $x, y, z \in X$, we immediately get Sincov's functional equation $F(x, y) + F(y, z) = F(x, z) \quad (x, y, z \in X)$ because by definition of F, we have that $F(y, y) = d(y, y) - d(y, y) = 0$, for every $y \in X$. Thus, even if the separability equation is accomplished, we would induce an interval order or a semiorder that coincides with the asymmetric part of a total preorder. That is, if $\prec$ is the induced interval order or semiorder, in this case we have that the binary relation $\precsim$ given by

[¶]Under a different name, the concept of an interval order, as well as the concept of a semiorder, was already implicit much earlier, in the work of Norbert Wiener. (See e.g. [43]).

$x \precsim y \Leftrightarrow \neg(y \prec x) \ \ (x, y \in X)$ is actually a total preorder‖. Notice also that this fact is the “expected one", taking into account the result stated in Theorem 3.5.

Furthermore, if the disymmetry function F of a quasi-metric satisfies the separability equation $F(x,y)+F(y,z) = F(x,z)+F(y,y) \ \ (x,y,z \in X)$, by Proposition 3.4 we have that F can be decomposed as $F(x,y) = G(x)+H(y)$ for every $x, y \in X$. But here we have that $F(x,y) = -F(y,x)$. Hence $G(x)+H(y) = -G(y)-H(x)$, for every $x, y \in X$. Thus $G(x)+H(x) = G(y)+H(y)$, for every $x, y \in X$. But $G(x)+H(x) = F(x,x) = d(x,x)-d(x,x) = 0$, for every $x \in X$. Therefore $G(x) = -H(x)$, for every $x \in X$. Thus we have that $F(x,y) = G(x)-G(y) = H(y)-H(x) \Leftrightarrow d(x,y)-d(y,x) = H(y)-H(x) \Leftrightarrow d(x,y)+H(x) = d(y,x)+H(y) \ \ (x,y \in X)$. This is an alternative argument to show that the quasi-metric d must be weightable, as stated in Theorem 3.5.

Inspired by Proposition 4.4, we may pay attention to the following important detail: the total preorders associated to a weightable quasi-metric can be framed by means of the disymmetry function F as well as by the weighting function w. Indeed, the fact $x \precsim_F y \Leftrightarrow F(x,y) \geq 0 \Leftrightarrow w(x) \leq w(y) \ \ (x,y \in X)$ is crucial. This inspires the following definition.

4.8. Definition. Let $\precsim$ denote a total preorder defined on a nonempty set X. We say that $\precsim$ is *representable* if there exists a function $u : X \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow u(x) \leq u(y)$ for every $x, y \in X$. The function u is called a *numerical isotony*, or, mainly in contexts coming from Economics, a *utility function*.

The kind of numerical representation involved in Definition 4.8 is actually equivalent to a representation that uses a bivariate function accomplishing Sincov's functional equation, as the next well-known result shows. (See e.g. Theorem 1 in [7]).

4.9. Proposition. *Let $\precsim$ be a total preorder defined on a nonempty set X. Then $\precsim$ is representable if and only if there exists a bivariate function $F : X \times X \to \mathbb{R}$ such that F satisfies Sincov's functional equation and, in addition, $x \precsim y \Leftrightarrow F(x,y) \geq 0$ holds for every $x, y \in X$.*

The following definition is inspired by Proposition 4.9, and it is equivalent to Definition 4.8.

4.10. Definition. Let $\precsim$ be a representable total preorder defined on a nonempty set X. A bivariate function $F : X \times X \to \mathbb{R}$ satisfying Sincov's functional equation, and such that $x \precsim y \Leftrightarrow F(x,y) \geq 0$ holds for every $x, y \in X$, is called a *bivariate numerical representation* of $\precsim$.

4.11. Remark. Not every total preorder is representable. A well known example is the lexicographic order $\precsim_L$ on the real plane $\mathbb{R}^2$: Given $(a,b), \ (c,d) \in \mathbb{R}^2$, then $(a,b) \precsim_L (c,d) \Leftrightarrow [(a < c) \vee (a = c, \ b \leq d)]$. (See e.g [9] for further details).

Looking again at Corollary 4.5 we may observe that from a weightable quasi-metric we get a representable total preorder. Looking for a converse result, we may start from a representable total preorder $\precsim$ defined on a nonempty set X, and search for a weightable quasi-metric whose disymmetry function constitutes a bivariate numerical representation of $\precsim$. We get a positive answer to this question, as the next Proposition 4.12 states.

4.12. Proposition. *Let X be a nonempty set. Let $\precsim$ be a representable total preorder defined on X. Then there exists a positively weightable quasi-metric $d : X \times X \to [0, +\infty)$*

‖In this case the corresponding interval order or semiorder is said to be *degenerate* or non-typical. An interval order or semiorder $\prec$ defined on a set X is said to be *typical* provided that its a associated symmetric part $\precsim$ fails to be transitive.

whose disymmetry function F is a bivariate numerical representation of the given total preorder $\precsim$.

Proof. Since $\precsim$ is representable, there exists a function $u : X \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow u(x) \leq u(y)$. We may assume without loss of generality that u takes strictly positive values: indeed, if $h(t) = 3 + \frac{2}{\pi} arctan(t)$ for $t \in \mathbb{R}$, we have that h is a strictly increasing function whose range is $(2, 4)$, and such that $x \precsim y \Leftrightarrow h(u(x)) \leq h(u(y))$, for every $x, y \in X$, so that the composition $h \circ u$ is another utility function that represents the total preorder $\precsim$. Thus, already assuming that $u : X \to (0, +\infty)$ is a strictly positive utility representation for $\precsim$, given $x, y \in X$ we may define $d(x, y) = 0$ if $x = y$ and $d(x, y) = u(y)$ otherwise. It is straightforward to see that d is a (T_1) quasi-metric. Moreover, it is positively weightable: just observe that $d(x, y) + u(x) = d(y, x) + u(y)$ for every $x, y \in X$. Finally, its disymmetry function F satisfies that $F(x, y) = d(x, y) - d(y, x) = u(y) - u(x)$ for every $x, y \in X$, so that $x \precsim y \Leftrightarrow u(x) \leq u(y) \Leftrightarrow u(y) - u(x) \geq 0 \Leftrightarrow F(x, y) \geq 0$. Therefore F is a bivariate numerical representation for $\precsim$. $\square$

4.13. Remark. Notice that the quasi-metric d that appears in the statement of Proposition 4.12 is not unique, in general. As a matter of fact, to get d we may use *any* strictly positive utility function u that represents the total preorder $\precsim$.

To summarize this Section 4, we may notice that in Corollary 4.5 we get a representable total preorder from a weightable quasi-metric, whereas in Proposition 4.12 we retrieve a positively weightable quasi-metric from a representable total preorder. By Theorem 3.5, in both results the disymmetry functions associated to the weightable quasi-metrics involved satisfy Sincov's functional equation.

To complete the panorama, we may wonder if it is also possible to retrieve a (positively) weightable quasi-metric directly from a bivariate function that satisfies Sincov's functional equation. We answer this question throughout the next Section 5.

5. Retrieving positively weightable quasi-metrics from functional equations

The main questions to be analyzed throughout this Section 5 are the following:

i) Suppose that X is a nonempty set and $D : X \times X \to \mathbb{R}$ is a bivariate function that satisfies the 3-circuit invariance functional equation. Can we induce from D, in a natural way, a positively weightable quasi-metric** on X?

ii) Suppose that $F : X \times X \to \mathbb{R}$ is a bivariate function that satisfies Sincov's functional equation. Can we induce from F a positively weightable quasi-metric on X whose disymmetry function is F?

To study the former question, let X denote a nonempty set, and let $D : X \times X \to \mathbb{R}$ be a bivariate function such that $D(a, b) + D(b, c) + D(c, a) = D(a, c) + D(c, b) + D(b, a)$ holds for every $a, b, c \in X$. Define $F : X \times X \to \mathbb{R}$ by declaring that $F(x, y) = D(x, y) - D(y, x)$ for every $x, y \in X$. We immediately realize that F satisfies Sincov's functional equation $F(a, b) + F(b, c) = F(a, c)$, for every $a, b, c \in X$. Therefore, we pass to consider the latter question, since its solution would immediately lead to a solution for the former one.

Next Lemma 5.1, Theorem 5.2 and their subsequent corollaries provide us with a positive answer.

5.1. Lemma. *Let X be a nonempty set. Let $F : X \times X \to \mathbb{R}$ be a bivariate function that satisfies Sincov's functional equation. Suppose, in addition, that there exists a strictly positive function $G : X \to (0, +\infty)$ such that $F(x, y) = G(y) - G(x)$, for every $x, y \in$*

**Notice that we are not imposing D to be the solution, not even to be a quasi-metric.

X. Then there exists a positively weightable quasi-metric $d : X \times X \to [0, +\infty)$ whose disymmetry fuction is F.

Proof. Given $x, y \in X$, we define $d(x, y) = 0$ if $x = y$ and $d(x, y) = G(y)$ otherwise. Notice that both $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ hold by definition of d. To check the triangle inequality, given $x, y, z \in X$ we distinguish the following cases:

Case 1: $x = y$. In this case we have that $0 = d(x, y) \leq d(x, z) + d(z, y)$ because, by definition of d, we have that $d(x, z) \geq 0$ and $d(z, y) \geq 0$.
Case 2: $x \neq y$; $y = z$. In this case we have that $G(y) = d(x, y) \leq d(x, z) + d(z, y) = d(x, y) + d(y, y)$ since $d(y, y) = 0$ and $d(x, y) = G(y)$ by definition of d.
Case 3: $x \neq y$; $x = z$. In this case the proof runs as in Case 2.
Case 4: $x \neq y$; $y \neq z$. In this case we have that $G(y) = d(x, y) \leq G(y) + d(x, z) = d(z, y) + d(x, z) = d(x, z) + d(z, y)$ since $d(x, z) \geq 0$ and $d(x, y) = d(z, y) = G(y)$ by definition of d.

Therefore d is a quasi-metric on X.

It is straightforward to check that F is the disymmetry function associated to d, so that G is a weighting function for d. Hence d is positively weightable. □

From Lemma 5.1, we finally reach the main result in this Section 5, namely Theorem 5.2, which is a characterization of real-valued bivariate functions that can be identified to the disymmetry function of some positively weightable quasi-metric.

5.2. Theorem. *Let X be a nonempty set. Let $F : X \times X \to \mathbb{R}$ be a bivariate function that satisfies Sincov's functional equation. The following statements are equivalent:*

i) *For every $a \in X$, the trace function $F_a : X \to \mathbb{R}$ defined by $F_a(t) = F(a, t)$ for every $t \in X$, is bounded by below (i.e.: there exists a constant $A \in \mathbb{R}$ such that $F(a, t) > A$ for every $t \in X$).*
ii) *There exists an element $a \in X$ such that F_a is bounded by below.*
iii) *There exists a positively weightable quasi-metric $d : X \times X \to [0, +\infty)$ whose disymmetry function is F.*

Proof. The implication $i) \Rightarrow ii)$ is trivial.

To prove the fact $ii) \Rightarrow iii)$, let $A \in \mathbb{R}$ be such that $F(a, t) > A$ for every $t \in X$. Given $x, y \in X$, we define the function $w : X \to \mathbb{R}$ as $w(t) = F(a, t) + |A|$ for every $t \in X$. Notice that $w(t) > 0$ holds for every $t \in X$, since $A + |A| \geq 0$. Moreover, for every $x, y \in X$ we have that $F(x, y) = F(x, a) + F(a, y) = F(a, y) - F(a, x) = (F(a, y) + |A|) - (F(a, x) + |A|) = w(y) - w(x)$, so that by Lemma 5.1 there exists a positively weightable quasi-metric d whose disymmetry function is F.

To conclude, we prove the implication $iii) \Rightarrow i)$. To do so, suppose that $d : X \times X \to [0, +\infty)$ is a positively weightable quasi-metric whose disymmetry function is F. By Remark 2.7, we have that $F(x, y) = w(y) - w(x)$ for every $x, y \in X$, where w stands for the weighting function associated to d. Fix any element $a \in X$. Now, given $t \in X$, we have that $F(a, t) = w(t) - w(a) > -w(a)$. In other words: F_a is bounded by below. □

5.3. Example. Accordingly to Theorem 5.2, it is now easy to find an example of a nonempty set X and a function $F : X \times X \to \mathbb{R}$ such that F satisfies Sincov's functional equation, but there is *no* positively weightable quasi-metric on X whose disymmetry function is F. Consider for instance $X = \mathbb{R}$ and $F(x, y) = x - y$ for every $(x, y) \in \mathbb{R}^2$. As a matter of fact, we may notice that no trace of F is bounded by below.

5.4. Corollary. *Let X be a finite nonempty set. Let $F : X \times X \to \mathbb{R}$ be a bivariate function that satisfies Sincov's functional equation. Then there exists a positively weightable quasi-metric $d : X \times X \to \mathbb{R}$ whose disymmetry function is F.*

Proof. This is an immediate consequence of Theorem 5.2, because F is bounded since X is finite. $\square$

5.5. Corollary. *Let X be a nonempty set, endowed with a topology τ for which X is a compact set. Let $F : X \times X \to \mathbb{R}$ be a continuous*[††] *bivariate function that satisfies Sincov's functional equation. Then there exists a positively weightable quasi-metric $d : X \times X \to \mathbb{R}$ whose disymmetry function is F.*

Proof. Again, this is an immediate consequence of Theorem 5.2, because F is a continuous real-valued function defined on a compact set, so it is bounded (see e.g. [41], p. 20). $\square$

To finish this Section 5 we analyze the posibility of retrieving a weightable quasi-metric (in this case, *not necessarily a positively weightable one*) from a bivariate function satisfying Sincov's functional equation. Unlike Theorem 5.2 and Example 5.3, the answer is always positive, as next Theorem 5.6 proves.

5.6. Theorem. *Let X be a nonempty set. Let $F : X \times X \to \mathbb{R}$ be a bivariate function that satisfies Sincov's functional equation. Then there exists a weightable (T_1) quasi-metric $d : X \times X \to [0, +\infty)$ whose disymmetry function is F.*

Proof. Since F satisfies Sincov's functional equation, there exists a function $G : X \to \mathbb{R}$ such that F can be decomposed as $F(x, y) = G(y) - G(x)$, for every $x, y \in X$.

Define $d : X \times X \to \mathbb{R}$ as follows:

i) $d(x, y) = 0$ if $x = y \in X$.
ii) $d(x, y) = 1 + G(y) - G(x)$ if $x \neq y \in X$ are such that $G(x) \leq G(y)$.
iii) $d(x, y) = 1$ if $x \neq y \in X$ are such that $G(x) > G(y)$.

By definition, $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$, for every $x, y \in X$.

Let us see now that d satisfies the triangle inequality. To see this, given $x, y, z \in X$ we consider the following cases:

Case 1: If there is at least a coincidence between x, y, z then $d(x, z) \leq d(x, y) + d(y, z)$ trivially holds.

Case 2: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(x) \leq G(y) \leq G(z)$ then we have that $d(x, z) = 1 + G(z) - G(x) < 2 + G(z) - G(x) = (1 + G(y) - G(x)) + (1 + G(z) - G(y)) = d(x, y) + d(y, z)$.

Case 3: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(x) \leq G(z) < G(y)$ then we have that $d(x, z) = 1 + G(z) - G(x) < 1 + G(y) - G(x) < (1 + G(y) - G(x)) + 1 = d(x, y) + d(y, z)$.

Case 4: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(y) < G(x) \leq G(z)$ then we have that $d(x, z) = 1 + G(z) - G(x) < 1 + G(z) - G(y) < 1 + (1 + G(z) - G(y)) = d(x, y) + d(y, z)$.

Case 5: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(y) \leq G(z) < G(x)$ then we have that $d(x, z) = 1 < 1 + d(y, z) = d(x, y) + d(y, z)$.

Case 6: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(z) < G(x) \leq G(y)$ then we have that $d(x, z) = 1 < d(x, y) + 1 = d(x, y) + d(y, z)$.

Case 7: If $x \neq y$; $y \neq z$; $x \neq z$ and $G(z) < G(y) < G(x)$ then we have that $d(x, z) = 1 < 2 = 1 + 1 = d(x, y) + d(y, z)$.

Therefore d is a quasi-metric.

A final checking shows that $d(x, y) - d(y, x) = G(y) - G(x)$ for every $x, y \in X$, so that d is indeed weightable (T_1).

$\square$

[††] Here we consider that $X \times X$ is endowed with the product topology $\tau \times \tau$, whereas the real line $\mathbb{R}$ is given the usual Euclidean topology.

6. Final comments and some suggestions for further research

Weightable quasi-metrics are closely related to several functional equations stated for real-valued bivariate functions on a nonempty set.

As shown in the main results stated in Section 4 and Section 5, there is a close relationship between the concepts of weightable quasi-metrics, representable total preorders and solutions of Sincov's functional equation. Each of these concepts gives rise to any of the two other ones.

We leave as an open question the study of similar functional equations in the framework of generalized metric spaces of any kind (see e.g. [39]), as, in particular, cone metric spaces (see e.g. [1]), pseudo-metrics, quasi-pseudo metrics (see e.g. [19]), probabilistic and statistical metric and quasi-metric spaces (see e.g. [27, 42, 36, 14]), and/or partial metrics, as well as to extend some results arising in the classical crisp context to the fuzzy setting (see e.g. [30]).

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