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Periodicity and solutions for some systems of nonlinear rational difference equations

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Abstract

In this paper, we investigated the periodic nature and the form of the solutions of nonlinear difference equations systems of order three

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(\pm 1 \pm x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(\pm 1 \pm y_{n-2}x_{n-1}\right)},$$

with initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are nonzero real numbers.

Keywords: difference equations, recursive sequences, stability, periodic solution, solution of difference equation, system of difference equations.

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1. Introduction

In this paper we deal with the existence of solutions and the periodicity character of the following systems of rational difference equations with order three

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(\pm 1 \pm x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(\pm 1 \pm y_{n-2}x_{n-1}\right)},$$

with initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are nonzero real numbers.

In recent years, rational difference equations have attracted the attention of many researchers for varied reasons. On the one hand, they provide examples of nonlinear equations which are, in some cases, treatable but whose dynamics present some new features with respect to the linear case. On the other hand, rational equations frequently appear in some biological models, and, hence, their study is of interest also due to their applications. A good example of both facts is Ricatti difference equations; the richness of the dynamics of Ricatti equations is very well-known (see, e.g., [10, 29]), and a particular case of these equations provides the classical Beverton-Holt model on the dynamics of

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exploited fish populations [5]. Obviously, higher-order rational difference equations and systems of rational equations have also been widely studied but still have many aspects to be investigated. The reader can find in the following books [1, 29, 30], and the works cited therein, many results, applications, and open problems on higher-order equations and rational systems.

There are many papers related to the difference equations systems for example, The global asymptotic behavior of the positive solutions of the rational difference system

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}},$$

has been studied by Camouzis et al. in [6].

The periodicity of the positive solutions of the rational difference equations systems

$$\begin{aligned} x_{n+1} &= \frac{m}{y_n}, \ y_{n+1} = \frac{p y_n}{x_{n-1} y_{n-1}}, \\ x_{n+1} &= \frac{1}{z_n}, \ y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}, \ z_{n+1} = \frac{1}{x_{n-1}}, \end{aligned}$$

has been obtained by Cinar in [7-8].

In [9] Clark and Kulenovic investigated the global asymptotic stability

$$x_{n+1} = \frac{x_n}{a + c y_n}, \quad y_{n+1} = \frac{y_n}{b + d x_n}.$$

Elsayed [14] has got the solutions of the following systems of the difference equations

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + x_{n-1} y_n}, \ y_{n+1} = \frac{y_{n-1}}{\mp 1 + y_{n-1} x_n}.$$

Grove et al. [23] has studied existence and behavior of solution of the rational system

$$x_{n+1} = \frac{a}{x_n} + \frac{b}{y_n}, \ y_{n+1} = \frac{c}{x_n} + \frac{d}{y_n}.$$

The behavior of positive solutions of the following system

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1} y_n}, \ y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1} x_n}.$$

has been studied by Kurbanli et al. [31].

Özban [32] has investigated the positive solution of the system of rational difference equations

$$x_{n+1} = \frac{a}{y_{n-3}}, \ y_{n+1} = \frac{b y_{n-3}}{x_{n-q} y_{n-q}}.$$

Also, Touafek et al. [36] studied the periodicity and gave the form of the solutions of the following systems

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)}, \ y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

In [37] Yalçınkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \ y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.$$

Similar to difference equations and nonlinear systems of rational difference equations were investigated see [1]-[43].

1.1. Definition. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2. The First System: $x_{n+1}=\frac{x_{n-2}y_{n-1}}{y_n(1+x_{n-2}y_{n-1})}$, $y_{n+1}=\frac{y_{n-2}x_{n-1}}{x_n(1+y_{n-2}x_{n-1})}$

In this section, we investigate the solutions of the system of two difference equations

$$x_{n+1}=\frac{x_{n-2}y_{n-1}}{y_n\left(1+x_{n-2}y_{n-1}\right)},\ y_{n+1}=\frac{y_{n-2}x_{n-1}}{x_n\left(1+y_{n-2}x_{n-1}\right)}, \tag{1}$$

where $n\in\mathbb{N}_0$ and the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers.

2.1. Theorem. *Assume that* $\{x_n,y_n\}$ *are solutions of system (1). Then for* $n=0,1,2,...,$ *we see that all solutions of system (1) are given by the following formula*

$$\begin{aligned}
x_{4n-2} &= \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},\\
x_{4n-1} &= \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},\\
x_{4n} &= \frac{x_0^{n+1}y_0^n}{x_{-2}^n y_{-2}^n}\prod_{i=0}^{n-1}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},\\
x_{4n+1} &= \frac{y_{-1}x_{-2}^{n+1}y_{-2}^n}{x_0^n y_0^{n+1}(1+x_{-2}y_{-1})}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},
\end{aligned}$$

and

$$\begin{aligned}
y_{4n-2} &= \frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)},\\
y_{4n-1} &= \frac{y_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n}\prod_{i=0}^{n-1}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})},\\
y_{4n} &= \frac{x_0^n y_0^{n+1}}{x_{-2}^n y_{-2}^n}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)},\\
y_{4n+1} &= \frac{x_{-1}x_{-2}^n y_{-2}^{n+1}}{x_0^{n+1}y_0^n(1+x_{-1}y_{-2})}\prod_{i=0}^{n-1}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}.
\end{aligned}$$

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. that is,

$$\begin{aligned}
x_{4n-6} &= \frac{x_0^{n-1}y_0^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-2}}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},\\
x_{4n-5} &= \frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},\\
x_{4n-4} &= \frac{x_0^n y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},\\
x_{4n-3} &= \frac{y_{-1}x_{-2}^n y_{-2}^{n-1}}{x_0^{n-1}y_0^n(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},
\end{aligned}$$

$$\begin{aligned}
y_{4n-6} &= \frac{x_0^{n-1}y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-2}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)},\\
y_{4n-5} &= \frac{y_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})},\\
y_{4n-4} &= \frac{x_0^{n-1}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)},\\
y_{4n-3} &= \frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}.
\end{aligned}$$

Now it follows from Eq.(1) that

$$\begin{aligned}
x_{4n-2} &= \frac{x_{4n-5}y_{4n-4}}{y_{4n-3}\left(1+x_{4n-5}y_{4n-4}\right)}\\
&= \frac{\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(\frac{x_0^{n-1}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}\right)}{\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)\left(1+\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(\frac{x_0^{n-1}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}\right)\right)}\\
&= \frac{\left(x_{-1}y_0\prod_{i=0}^{n-2}(1+(2i)x_{-1}y_0)\right)\left(\prod_{i=0}^{n-2}\frac{1}{(1+(2i+2)x_{-1}y_0)}\right)}{\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)\left(1+\left(x_{-1}y_0\prod_{i=0}^{n-2}(1+(2i)x_{-1}y_0)\right)\left(\prod_{i=0}^{n-2}\frac{1}{(1+(2i+2)x_{-1}y_0)}\right)\right)}\\
&= \frac{\left(\frac{x_{-1}y_0}{(1+(2n-2)x_{-1}y_0)}\right)}{\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^{n}}{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)\left(1+\left(\frac{x_{-1}y_0}{(1+(2n-2)x_{-1}y_0)}\right)\right)}\\
&= \frac{x_0^{n}y_0^{n-1}(1+x_{-1}y_{-2})\left(\frac{x_{-1}y_0}{1+(2n-2)x_{-1}y_0}\right)}{x_{-1}x_{-2}^{n-1}y_{-2}^{n}\left(1+\frac{x_{-1}y_0}{1+(2n-2)x_{-1}y_0}\right)}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\\
&= \frac{x_0^{n}y_0^{n}(1+x_{-1}y_{-2})}{x_{-2}^{n-1}y_{-2}^{n}\left(1+(2n-2)x_{-1}y_0+x_{-1}y_0\right)}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{x_0^{n}y_0^{n}(1+x_{-1}y_{-2})}{x_{-2}^{n-1}y_{-2}^{n}\left(1+(2n-1)x_{-1}y_0\right)}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\\
&= \frac{x_0^{n}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n}}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)},
\end{aligned}$$

$$
\begin{aligned}
y_{4n-2} &= \frac{y_{4n-5}x_{4n-4}}{x_{4n-3}\left(1+y_{4n-5}x_{4n-4}\right)} \\
&= \frac{\left(\frac{y_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}\right)\left(\frac{x_0^{n}y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)}{\left(\frac{y_{-1}x_{-2}^{n}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(1+\left(\frac{y_{-1}x_{-2}^{n-1}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}\right)\left(\frac{x_0^{n}y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\right)} \\
&= \frac{\left(x_0y_{-1}\prod_{i=0}^{n-2}(1+(2i)x_0y_{-1})\right)\left(\prod_{i=0}^{n-2}\frac{1}{(1+(2i+2)x_0y_{-1})}\right)}{\left(\frac{y_{-1}x_{-2}^{n}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(1+\left(x_0y_{-1}\prod_{i=0}^{n-2}(1+(2i)x_0y_{-1})\right)\left(\prod_{i=0}^{n-2}\frac{1}{(1+(2i+2)x_0y_{-1})}\right)\right)} \\
&= \frac{\left(\frac{x_0y_{-1}}{(1+(2n-2)x_0y_{-1})}\right)}{\left(\frac{y_{-1}x_{-2}^{n}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\left(1+\left(\frac{x_0y_{-1}}{(1+(2n-2)x_0y_{-1})}\right)\right)} \\
&= \frac{x_0y_{-1}x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})}{y_{-1}x_{-2}^{n}y_{-2}^{n-1}\left(1+(2n-1)x_0y_{-1}\right)}\prod_{i=0}^{n-2}\frac{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)} \\
&= \frac{x_0^{n}y_0^{n}}{x_{-2}^{n}y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}.
\end{aligned}
$$

Also, we see from Eq.(1) that

$$
\begin{aligned}
x_{4n-1} &= \frac{x_{4n-4}y_{4n-3}}{y_{4n-2}\left(1+x_{4n-4}y_{4n-3}\right)} \\
&= \frac{\left(\frac{x_0^n y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^n}{x_0^n y_0^{n-1}(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)}{\left(\frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}\right)\left(1+\left(\frac{x_0^n y_0^{n-1}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\left(\frac{x_{-1}x_{-2}^{n-1}y_{-2}^n}{x_0^n y_0^{n-1}(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{(1+(2i+2)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+3)x_{-1}y_{-2})}\right)\right)} \\
&= \frac{\left(\prod_{i=0}^{n-2}(1+(2i+1)x_{-1}y_{-2})\right)\left(\frac{x_{-1}y_{-2}}{(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{1}{(1+(2i+3)x_{-1}y_{-2})}\right)}{\left(\frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}\right)\left(1+\left(\substack{n-2\\ i=0}(1+(2i+1)x_{-1}y_{-2})\right)\left(\frac{x_{-1}y_{-2}}{(1+x_{-1}y_{-2})}\prod_{i=0}^{n-2}\frac{1}{(1+(2i+3)x_{-1}y_{-2})}\right)\right)} \\
&= \frac{\left(\frac{x_{-1}y_{-2}}{(1+(2n-1)x_{-1}y_{-2})}\right)}{\left(\frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}\right)\left(1+\left(\frac{x_{-1}y_{-2}}{(1+(2n-1)x_{-1}y_{-2})}\right)\right)} \\
&= \frac{x_{-1}y_{-2}}{\left(\frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}\right)\left(1+(2n-1)x_{-1}y_{-2}+x_{-1}y_{-2}\right)} \\
&= \frac{x_{-2}^n y_{-2}^{n-1}x_{-1}y_{-2}}{x_0^n y_0^n\left(1+(2n)x_{-1}y_{-2}\right)}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i)x_{-1}y_{-2})} \\
&= \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n}\prod_{i=0}^{n-1}\frac{(1+(2i+1)x_0y_{-1})(1+(2i)x_{-1}y_0)}{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})},
\end{aligned}
$$

and

$$
\begin{aligned}
y_{4n-1} &= \frac{y_{4n-4}x_{4n-3}}{x_{4n-2}\left(1+y_{4n-4}x_{4n-3}\right)} \\
&= \frac{\left(\frac{x_0^{n-1}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}\right)\left(\frac{y_{-1}x_{-2}^{n}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)}{\left(\frac{x_0^{n}y_0^{n}}{y_{-2}^{n}x_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\left(1+\left(\frac{x_0^{n-1}y_0^{n}}{x_{-2}^{n-1}y_{-2}^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}\right)\left(\frac{y_{-1}x_{-2}^{n}y_{-2}^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_0y_{-1})(1+(2i+2)x_{-1}y_0)}{(1+(2i+3)x_{-2}y_{-1})(1+(2i+2)x_{-1}y_{-2})}\right)\right)} \\
&= \frac{\left(\prod_{i=0}^{n-2}(1+(2i+1)x_{-2}y_{-1})\right)\left(\frac{y_{-1}x_{-2}}{(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{1}{(1+(2i+3)x_{-2}y_{-1})}\right)}{\left(\frac{x_0^{n}y_0^{n}}{y_{-2}^{n}x_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\left(1+\left(\prod_{i=0}^{n-2}(1+(2i+1)x_{-2}y_{-1})\right)\left(\frac{y_{-1}x_{-2}}{(1+x_{-2}y_{-1})}\prod_{i=0}^{n-2}\frac{1}{(1+(2i+3)x_{-2}y_{-1})}\right)\right)} \\
&= \frac{\left(\frac{y_{-1}x_{-2}}{(1+(2n-1)x_{-2}y_{-1})}\right)}{\left(\frac{x_0^{n}y_0^{n}}{y_{-2}^{n}x_{-2}^{n-1}}\prod_{i=0}^{n-1}\frac{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}\right)\left(1+\left(\frac{y_{-1}x_{-2}}{(1+(2n-1)x_{-2}y_{-1})}\right)\right)} \\
&= \frac{y_{-2}^{n}x_{-2}^{n-1}y_{-1}x_{-2}}{x_0^{n}y_0^{n}\left(1+(2n-1)x_{-2}y_{-1}+y_{-1}x_{-2}\right)}\prod_{i=0}^{n-1}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})} \\
&= \frac{y_{-1}y_{-2}^{n}x_{-2}^{n}}{x_0^{n}y_0^{n}}\prod_{i=0}^{n-1}\frac{(1+(2i)x_0y_{-1})(1+(2i+1)x_{-1}y_0)}{(1+(2i+2)x_{-2}y_{-1})(1+(2i+1)x_{-1}y_{-2})}.
\end{aligned}
$$

Also, we can prove the other relations. The proof is complete. ∎

The following Theorems can be proved similarly:

2.2. Theorem. *Assume that $\{x_n, y_n\}$ are solutions of the system*

$$x_{n+1}=\frac{x_{n-2}y_{n-1}}{y_n\left(1-x_{n-2}y_{n-1}\right)},\ y_{n+1}=\frac{y_{n-2}x_{n-1}}{x_n\left(1-y_{n-2}x_{n-1}\right)}.$$

Then for $n = 0, 1, 2, ...,$

$$
\begin{aligned}
x_{4n-2} &= \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(1-(2i)x_{-2}y_{-1})(1-(2i+1)x_{-1}y_{-2})}{(1-(2i)x_0y_{-1})(1-(2i+1)x_{-1}y_0)}, \\
x_{4n-1} &= \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(1-(2i+1)x_0y_{-1})(1-(2i)x_{-1}y_0)}{(1-(2i+1)x_{-2}y_{-1})(1-(2i+2)x_{-1}y_{-2})}, \\
x_{4n} &= \frac{x_0^{n+1} y_0^n}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(1-(2i+2)x_{-2}y_{-1})(1-(2i+1)x_{-1}y_{-2})}{(1-(2i+2)x_0y_{-1})(1-(2i+1)x_{-1}y_0)}, \\
x_{4n+1} &= \frac{y_{-1}x_{-2}^{n+1} y_{-2}^n}{x_0^n y_0^{n+1}(1-x_{-2}y_{-1})} \prod_{i=0}^{n-1} \frac{(1-(2i+1)x_0y_{-1})(1-(2i+2)x_{-1}y_0)}{(1-(2i+3)x_{-2}y_{-1})(1-(2i+2)x_{-1}y_{-2})},
\end{aligned}
$$

and

$$
\begin{aligned}
y_{4n-2} &= \frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(1-(2i+1)x_{-2}y_{-1})(1-(2i)x_{-1}y_{-2})}{(1-(2i+1)x_0y_{-1})(1-(2i)x_{-1}y_0)}, \\
y_{4n-1} &= \frac{y_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(1-(2i)x_0y_{-1})(1-(2i+1)x_{-1}y_0)}{(1-(2i+2)x_{-2}y_{-1})(1-(2i+1)x_{-1}y_{-2})}, \\
y_{4n} &= \frac{x_0^n y_0^{n+1}}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(1-(2i+1)x_{-2}y_{-1})(1-(2i+2)x_{-1}y_{-2})}{(1-(2i+1)x_0y_{-1})(1-(2i+2)x_{-1}y_0)}, \\
y_{4n+1} &= \frac{x_{-1}x_{-2}^n y_{-2}^{n+1}}{x_0^{n+1} y_0^n(1-x_{-1}y_{-2})} \prod_{i=0}^{n-1} \frac{(1-(2i+2)x_0y_{-1})(1-(2i+1)x_{-1}y_0)}{(1-(2i+2)x_{-2}y_{-1})(1-(2i+3)x_{-1}y_{-2})}.
\end{aligned}
$$

2.3. Theorem. *Let* $\{x_n, y_n\}$ *are solutions of the following system*

$$
x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n(-1+x_{n-2}y_{n-1})}, \; y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n(-1-y_{n-2}x_{n-1})}.
$$

Then for $n = 0, 1, 2, ...,$

$$
\begin{aligned}
x_{4n-2} &= \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(-1+(2i)x_{-2}y_{-1})(-1-(2i+1)x_{-1}y_{-2})}{(-1-(2i)x_0y_{-1})(-1+(2i+1)x_{-1}y_0)}, \\
x_{4n-1} &= \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(-1-(2i+1)x_0y_{-1})(-1+(2i)x_{-1}y_0)}{(-1+(2i+1)x_{-2}y_{-1})(-1-(2i+2)x_{-1}y_{-2})}, \\
x_{4n} &= \frac{x_0^{n+1} y_0^n}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(-1+(2i+2)x_{-2}y_{-1})(-1-(2i+1)x_{-1}y_{-2})}{(-1-(2i+2)x_0y_{-1})(-1+(2i+1)x_{-1}y_0)}, \\
x_{4n+1} &= \frac{y_{-1}x_{-2}^{n+1} y_{-2}^n}{x_0^n y_0^{n+1}(-1+x_{-2}y_{-1})} \prod_{i=0}^{n-1} \frac{(-1-(2i+1)x_0y_{-1})(-1+(2i+2)x_{-1}y_0)}{(-1+(2i+3)x_{-2}y_{-1})(-1-(2i+2)x_{-1}y_{-2})},
\end{aligned}
$$

$$y_{4n-2} = \frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(-1+(2i+1)x_{-2}y_{-1})(-1-(2i)x_{-1}y_{-2})}{(-1-(2i+1)x_0y_{-1})(-1+(2i)x_{-1}y_0)},$$

$$y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(-1-(2i)x_0y_{-1})(-1+(2i+1)x_{-1}y_0)}{(-1+(2i+2)x_{-2}y_{-1})(-1-(2i+1)x_{-1}y_{-2})},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(-1+(2i+1)x_{-2}y_{-1})(-1-(2i+2)x_{-1}y_{-2})}{(-1-(2i+1)x_0y_{-1})(-1+(2i+2)x_{-1}y_0)},$$

$$y_{4n+1} = \frac{x_{-1}x_{-2}^n y_{-2}^{n+1}}{x_0^{n+1} y_0^n(-1-x_{-1}y_{-2})} \prod_{i=0}^{n-1} \frac{(-1-(2i+2)x_0y_{-1})(-1+(2i+1)x_{-1}y_0)}{(-1+(2i+2)x_{-2}y_{-1})(-1-(2i+3)x_{-1}y_{-2})}.$$

2.4. Theorem. *The solutions of the following system of difference equations*

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n(-1-x_{n-2}y_{n-1})}, \quad y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n(-1+y_{n-2}x_{n-1})}.$$

are given by the following formula for $n = 0, 1, 2, ...,$

$$x_{4n-2} = \frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(-1-(2i)x_{-2}y_{-1})(-1+(2i+1)x_{-1}y_{-2})}{(-1+(2i)x_0y_{-1})(-1-(2i+1)x_{-1}y_0)},$$

$$x_{4n-1} = \frac{x_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(-1+(2i+1)x_0y_{-1})(-1-(2i)x_{-1}y_0)}{(-1-(2i+1)x_{-2}y_{-1})(-1+(2i+2)x_{-1}y_{-2})},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(-1-(2i+2)x_{-2}y_{-1})(-1+(2i+1)x_{-1}y_{-2})}{(-1+(2i+2)x_0y_{-1})(-1-(2i+1)x_{-1}y_0)},$$

$$x_{4n+1} = \frac{y_{-1}x_{-2}^{n+1} y_{-2}^n}{x_0^n y_0^{n+1}(-1-x_{-2}y_{-1})} \prod_{i=0}^{n-1} \frac{(-1+(2i+1)x_0y_{-1})(-1-(2i+2)x_{-1}y_0)}{(-1-(2i+3)x_{-2}y_{-1})(-1+(2i+2)x_{-1}y_{-2})},$$

$$y_{4n-2} = \frac{x_0^n y_0^n}{x_{-2}^n y_{-2}^{n-1}} \prod_{i=0}^{n-1} \frac{(-1-(2i+1)x_{-2}y_{-1})(-1+(2i)x_{-1}y_{-2})}{(-1+(2i+1)x_0y_{-1})(-1-(2i)x_{-1}y_0)},$$

$$y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n}{x_0^n y_0^n} \prod_{i=0}^{n-1} \frac{(-1+(2i)x_0y_{-1})(-1-(2i+1)x_{-1}y_0)}{(-1-(2i+2)x_{-2}y_{-1})(-1+(2i+1)x_{-1}y_{-2})},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}}{x_{-2}^n y_{-2}^n} \prod_{i=0}^{n-1} \frac{(-1-(2i+1)x_{-2}y_{-1})(-1+(2i+2)x_{-1}y_{-2})}{(-1+(2i+1)x_0y_{-1})(-1-(2i+2)x_{-1}y_0)},$$

$$y_{4n+1} = \frac{x_{-1}x_{-2}^n y_{-2}^{n+1}}{x_0^{n+1} y_0^n(-1+x_{-1}y_{-2})} \prod_{i=0}^{n-1} \frac{(-1+(2i+2)x_0y_{-1})(-1-(2i+1)x_{-1}y_0)}{(-1-(2i+2)x_{-2}y_{-1})(-1+(2i+3)x_{-1}y_{-2})}.$$

2.5. Example. For confirming the results of this section, we consider numerical example for the difference system (1) with the initial conditions $x_{-2} = 3,\ x_{-1} = 5, x_0 = -4,\ y_{-2} = 2,\ y_{-1} = 6$ *and* $y_0 = 7$. (See Fig. 1).

3. The Second System: $x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n(-1+x_{n-2}y_{n-1})}$, $y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n(1+y_{n-2}x_{n-1})}$

In this section, we obtain the form of the solutions of the system of two difference equations

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n(-1+x_{n-2}y_{n-1})}, \quad y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n(1+y_{n-2}x_{n-1})}, \tag{2}$$

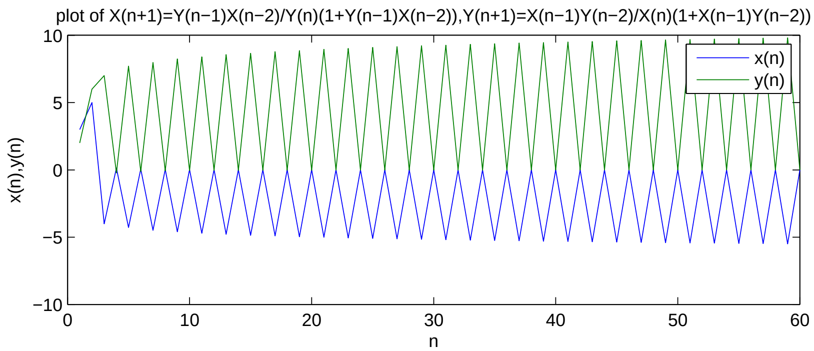

Figure 1

where $n \in \mathbb{N}_0$ and the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are arbitrary non zero real numbers with $x_{-1}y_0,\ x_{-2}y_{-1} \neq 1, \frac{1}{2}$, and $x_0y_{-1},\ x_{-1}y_{-2} \neq \pm 1$.

The following theorem is devoted to the expression of the form of the solutions of system (2).

3.1. Theorem. *Let* $\{x_n, y_n\}_{n=-2}^{+\infty}$ *be solutions of system (2). Then* $\{x_n\}_{n=-2}^{+\infty}$ *and* $\{y_n\}_{n=-2}^{+\infty}$ *are given by the formulae for* $n = 0, 1, 2, ...,$

$$x_{8n-2} = \frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}},$$

$$x_{8n-1} = \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}},$$

$$x_{8n} = \frac{x_0^{2n+1}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}},$$

$$x_{8n+1} = \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n+1}},$$

$$x_{8n+2} = \frac{x_0^{2n+1}y_0^{2n+1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n+1}},$$

$$x_{8n+3} = -\frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n+1}},$$

$$x_{8n+4} = -\frac{x_0^{2n+2}y_0^{2n+1}(-1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+x_{-1}y_0)^{2n+1}},$$

$$x_{8n+5} = \frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+2}(-1+2x_{-1}y_0)^{n+1}(-1+x_0y_{-1})^n(1+x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+2}(-1+x_{-2}y_{-1})^{2n+2}},$$

$$
\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1} x_{-2}^{2n} (-1+2x_{-1}y_0)^n (-1+x_0y_{-1})^n (1+x_0y_{-1})^n}, \\
y_{8n-1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1+x_{-1}y_0)^{2n}}{x_0^{2n} y_0^{2n} (-1+2x_{-2}y_{-1})^n (-1+x_{-1}y_{-2})^n (1+x_{-1}y_{-2})^n}, \\
y_{8n} &= \frac{x_0^{2n} y_0^{2n+1} (-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n} x_{-2}^{2n} (-1+2x_{-1}y_0)^n (-1+x_0y_{-1})^n (1+x_0y_{-1})^n}, \\
y_{8n+1} &= \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n} (-1+x_{-1}y_0)^{2n}}{x_0^{2n+1} y_0^{2n} (-1+2x_{-2}y_{-1})^n (-1+x_{-1}y_{-2})^n (1+x_{-1}y_{-2})^{n+1}}, \\
y_{8n+2} &= \frac{x_0^{2n+1} y_0^{2n+1} (-1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n} x_{-2}^{2n+1} (-1+2x_{-1}y_0)^n (-1+x_0y_{-1})^n (1+x_0y_{-1})^{n+1}}, \\
y_{8n+3} &= \frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (-1+x_{-1}y_0)^{2n+1}}{x_0^{2n+1} y_0^{2n+1} (-1+2x_{-2}y_{-1})^{n+1} (-1+x_{-1}y_{-2})^n (1+x_{-1}y_{-2})^{n+1}}, \\
y_{8n+4} &= -\frac{x_0^{2n+1} y_0^{2n+2} (-1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (-1+2x_{-1}y_0)^{n+1} (-1+x_0y_{-1})^n (1+x_0y_{-1})^{n+1}}, \\
y_{8n+5} &= -\frac{x_{-1} y_{-2}^{2n+2} x_{-2}^{2n+1} (-1+x_{-1}y_0)^{2n+1}}{x_0^{2n+2} y_0^{2n+1} (-1+2x_{-2}y_{-1})^{n+1} (-1+x_{-1}y_{-2})^{n+1} (1+x_{-1}y_{-2})^{n+1}}.
\end{aligned}
$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. that is,

$$
\begin{aligned}
x_{8n-10} &= \frac{x_0^{2n-2} y_0^{2n-2} (-1+2x_{-2}y_{-1})^{n-1} (-1+x_{-1}y_{-2})^{n-1} (1+x_{-1}y_{-2})^{n-1}}{y_{-2}^{2n-2} x_{-2}^{2n-3} (-1+x_{-1}y_0)^{2n-2}}, \\
x_{8n-9} &= \frac{x_{-1} y_{-2}^{2n-2} x_{-2}^{2n-2} (-1+2x_{-1}y_0)^{n-1} (-1+x_0y_{-1})^{n-1} (1+x_0y_{-1})^{n-1}}{x_0^{2n-2} y_0^{2n-2} (-1+x_{-2}y_{-1})^{2n-2}}, \\
x_{8n-8} &= \frac{x_0^{2n-1} y_0^{2n-2} (-1+2x_{-2}y_{-1})^{n-1} (-1+x_{-1}y_{-2})^{n-1} (1+x_{-1}y_{-2})^{n-1}}{y_{-2}^{2n-2} x_{-2}^{2n-2} (-1+x_{-1}y_0)^{2n-2}},
\end{aligned}
$$

$$
\begin{aligned}
x_{8n-7} &= \frac{y_{-1} y_{-2}^{2n-2} x_{-2}^{2n-1} (-1+2x_{-1}y_0)^{n-1} (-1+x_0y_{-1})^{n-1} (1+x_0y_{-1})^{n-1}}{x_0^{2n-2} y_0^{2n-1} (-1+x_{-2}y_{-1})^{2n-1}}, \\
x_{8n-6} &= \frac{x_0^{2n-1} y_0^{2n-1} (-1+2x_{-2}y_{-1})^{n-1} (-1+x_{-1}y_{-2})^{n-1} (1+x_{-1}y_{-2})^{n}}{y_{-2}^{2n-1} x_{-2}^{2n-2} (-1+x_{-1}y_0)^{2n-1}}, \\
x_{8n-5} &= -\frac{x_{-1} y_{-2}^{2n-1} x_{-2}^{2n-1} (-1+2x_{-1}y_0)^{n-1} (-1+x_0y_{-1})^{n-1} (1+x_0y_{-1})^{n}}{x_0^{2n-1} y_0^{2n-1} (-1+x_{-2}y_{-1})^{2n-1}}, \\
x_{8n-4} &= -\frac{x_0^{2n} y_0^{2n-1} (-1+2x_{-2}y_{-1})^{n} (-1+x_{-1}y_{-2})^{n-1} (1+x_{-1}y_{-2})^{n}}{y_{-2}^{2n-1} x_{-2}^{2n-1} (-1+x_{-1}y_0)^{2n-1}}, \\
x_{8n-3} &= \frac{y_{-1} y_{-2}^{2n-1} x_{-2}^{2n} (-1+2x_{-1}y_0)^{n} (-1+x_0y_{-1})^{n-1} (1+x_0y_{-1})^{n}}{x_0^{2n-1} y_0^{2n} (-1+x_{-2}y_{-1})^{2n}},
\end{aligned}
$$

and

$$\begin{aligned}
y_{8n-10} &= \frac{x_0^{2n-2}y_0^{2n-2}(-1+x_{-2}y_{-1})^{2n-2}}{y_{-2}^{2n-3}x_{-2}^{2n-2}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n-1}},\\
y_{8n-9} &= \frac{y_{-1}y_{-2}^{2n-2}x_{-2}^{2n-2}(-1+x_{-1}y_0)^{2n-2}}{x_0^{2n-2}y_0^{2n-2}(-1+2x_{-2}y_{-1})^{n-1}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n-1}},\\
y_{8n-8} &= \frac{x_0^{2n-2}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-2}}{y_{-2}^{2n-2}x_{-2}^{2n-2}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n-1}},\\
y_{8n-7} &= \frac{x_{-1}y_{-2}^{2n-1}x_{-2}^{2n-2}(-1+x_{-1}y_0)^{2n-2}}{x_0^{2n-1}y_0^{2n-2}(-1+2x_{-2}y_{-1})^{n-1}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n}},\\
y_{8n-6} &= \frac{x_0^{2n-1}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-2}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}},\\
y_{8n-5} &= \frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n-1}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^{n}},\\
y_{8n-4} &= -\frac{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}},\\
y_{8n-3} &= -\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}.
\end{aligned}$$

Now it follows from Eq.(2) that

$$x_{8n-2} = \frac{x_{8n-5}y_{8n-4}}{y_{8n-3}\left(-1+x_{8n-5}y_{8n-4}\right)}$$

$$\begin{aligned}
&= \frac{\left(-\frac{x_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}{x_0^{2n-1}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-1}}\right)\left(-\frac{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}\right)}{\left(-\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}\right)\left(-1+\left(-\frac{x_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n-1}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}{x_0^{2n-1}y_0^{2n-1}(-1+x_{-2}y_{-1})^{2n-1}}\right)\left(-\frac{x_0^{2n-1}y_0^{2n}(1-x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^{n}(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^{n}}\right)\right)}\\
&= \frac{\left(\frac{x_{-1}y_0}{(-1+2x_{-1}y_0)}\right)}{\left(\frac{-x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}\right)\left(-1+\frac{x_{-1}y_0}{-1+2x_{-1}y_0}\right)}\\
&= -\frac{x_{-1}y_0x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}\left(1-2x_{-1}y_0+x_{-1}y_0\right)}\\
&= \frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^{n}(-1+x_{-1}y_{-2})^{n}(1+x_{-1}y_{-2})^{n}}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}},
\end{aligned}$$

$$\begin{aligned}
y_{8n-2} &= \frac{y_{8n-5}x_{8n-4}}{x_{8n-3}\left(1+y_{8n-5}x_{8n-4}\right)} \\
&= \frac{\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n-1}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}\right)\left(-\frac{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}\right)}{\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)\left(1+\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n-1}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}\right)\left(-\frac{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}\right)\right)} \\
&= \frac{-x_0y_{-1}}{\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)(1-x_0y_{-1})} \\
&= \frac{x_0y_{-1}x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n\,(-1+x_0y_{-1})} \\
&= \frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}.
\end{aligned}$$

Also, we see from Eq.(2) that

$$\begin{aligned}
x_{8n-1} &= \frac{x_{8n-4}y_{8n-3}}{y_{8n-2}\left(-1+x_{8n-4}y_{8n-3}\right)} \\
&= \frac{\left(-\frac{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}\right)\left(-\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)}{\left(\frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}\right)\left(-1+\left(-\frac{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^{n-1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}\right)\left(-\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\right)} \\
&= \frac{\left(\frac{x_{-1}y_{-2}}{(-1+x_{-1}y_{-2})}\right)}{\left(\frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}\right)\left(-1+\frac{x_{-1}y_{-2}}{(-1+x_{-1}y_{-2})}\right)} \\
&= \frac{x_{-1}y_{-2}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}\,(1-x_{-1}y_{-2}+x_{-1}y_{-2})} \\
&= \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}},
\end{aligned}$$

and

$$
\begin{aligned}
y_{8n-1} &= \frac{y_{8n-4}x_{8n-3}}{x_{8n-2}\left(1+y_{8n-4}x_{8n-3}\right)} \\
&= \frac{\left(-\frac{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}\right)\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)}{\left(\frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}}\right)\left(1+\left(-\frac{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n-1}}{y_{-2}^{2n-1}x_{-2}^{2n-1}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}\right)\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left(\frac{-x_{-2}y_{-1}}{(-1+x_{-2}y_{-1})}\right)}{\left(\frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}}\right)\left(1+\left(\frac{-x_{-2}y_{-1}}{(-1+x_{-2}y_{-1})}\right)\right)} \\
&= \frac{-x_{-2}y_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n\left(-1+x_{-2}y_{-1}-x_{-2}y_{-1}\right)} \\
&= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}.
\end{aligned}
$$

We get from Eq.(2) that

$$\begin{aligned}
x_{8n} &= \frac{x_{8n-3}y_{8n-2}}{y_{8n-1}\left(-1+x_{8n-3}y_{8n-2}\right)} \\
&= \frac{\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)\left(\frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}\right)}{\left(\frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\left(-1+\left(\frac{y_{-1}y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^{n-1}(1+x_0y_{-1})^n}{x_0^{2n-1}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)\left(\frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}\right)\right)} \\
&= \frac{\left(\frac{x_0y_{-1}}{(-1+x_0y_{-1})}\right)}{\left(\frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\left(-1+\frac{x_0y_{-1}}{-1+x_0y_{-1}}\right)} \\
&= \frac{x_0y_{-1}x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}\left(1-x_0y_{-1}+x_0y_{-1}\right)} \\
&= \frac{x_0^{2n+1}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}},
\end{aligned}$$

and

$$\begin{aligned}
y_{8n} &= \frac{y_{8n-3}x_{8n-2}}{x_{8n-1}\left(1+y_{8n-3}x_{8n-2}\right)} \\
&= \frac{\left(-\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\left(\frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}}\right)}{\left(\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}\right)\left(1+\left(-\frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n-1}}{x_0^{2n}y_0^{2n-1}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}\right)\left(\frac{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+x_{-1}y_0)^{2n}}\right)\right)} \\
&= \frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}\left(-\frac{x_{-1}y_0}{(-1+x_{-1}y_0)}\right)}{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n\left(1-\frac{x_{-1}y_0}{-1+x_{-1}y_0}\right)} \\
&= \frac{-x_{-1}y_0x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n\left(-1+x_{-1}y_0-x_{-1}y_0\right)} \\
&= \frac{x_0^{2n}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n}.
\end{aligned}$$

Also, we can prove the other relations. This completes the proof. ∎

Here, we consider the following systems and the proof of the theorems are similarly to above theorem and so, left to the reader.

$$(3.1)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1-x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(1-y_{n-2}x_{n-1}\right)}.3$$

$$(3.2)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(1+x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1-y_{n-2}x_{n-1}\right)}.4$$

$$(3.3)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(1-x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1+y_{n-2}x_{n-1}\right)}.5$$

The following theorems is devoted to the expressions of the form of the solutions of systems (3), (4), (5).

3.2. Theorem. *Let* $\{x_n, y_n\}_{n=-2}^{+\infty}$ *be solutions of system (3) and* $x_{-1}y_0,\ x_{-2}y_{-1} \neq -1, -\frac{1}{2}$*, and* $x_0y_{-1},\ x_{-1}y_{-2} \neq \pm 1$*. Then for* $n = 0, 1, 2, ...,$

$$x_{8n-2} = \frac{x_0^{2n}y_0^{2n}(1+2x_{-2}y_{-1})^n(1-x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(1+x_{-1}y_0)^{2n}},$$

$$x_{8n-1} = \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(1+2x_{-1}y_0)^n(1-x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(1+x_{-2}y_{-1})^{2n}},$$

$$x_{8n} = \frac{x_0^{2n+1}y_0^{2n}(1+2x_{-2}y_{-1})^n(1-x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}},$$

$$x_{8n+1} = -\frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(1-x_0y_{-1})^n(1+x_0y_{-1})^n}{x_0^{2n}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}},$$

$$x_{8n+2} = -\frac{x_0^{2n+1}y_0^{2n+1}(1+2x_{-2}y_{-1})^n(1-x_{-1}y_{-2})^{n+1}(1+x_{-1}y_{-2})^n}{y_{-2}^{2n+1}x_{-2}^{2n}(1+x_{-1}y_0)^{2n+1}},$$

$$x_{8n+3} = \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(1+x_0y_{-1})^n(1-x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}},$$

$$x_{8n+4} = -\frac{x_0^{2n+2}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^n(1-x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}},$$

$$x_{8n+5} = -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+2}(1+2x_{-1}y_0)^{n+1}(1+x_0y_{-1})^n(1-x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+2}(1+x_{-2}y_{-1})^{2n+2}},$$

and

$$
\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(1+2x_{-1}y_0)^n(1-x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n-1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(1+2x_{-2}y_{-1})^n(1-x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n},\\
y_{8n} &= \frac{x_0^{2n}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(1+2x_{-1}y_0)^n(1-x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n+1} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}}{x_0^{2n+1}y_0^{2n}(1+2x_{-2}y_{-1})^n(1+x_{-1}y_{-2})^n(1-x_{-1}y_{-2})^{n+1}},\\
y_{8n+2} &= -\frac{x_0^{2n+1}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(1+x_0y_{-1})^n(1-x_0y_{-1})^{n+1}},\\
y_{8n+3} &= \frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^n(1-x_{-1}y_{-2})^{n+1}},\\
y_{8n+4} &= -\frac{x_0^{2n+1}y_0^{2n+2}(1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(1+2x_{-1}y_0)^{n+1}(1+x_0y_{-1})^n(1-x_0y_{-1})^{n+1}},\\
y_{8n+5} &= \frac{x_{-1}y_{-2}^{2n+2}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}}{x_0^{2n+2}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^{n+1}(1-x_{-1}y_{-2})^{n+1}}.
\end{aligned}
$$

3.3. Theorem. *Assume that* $\{x_n, y_n\}$ *are solutions of system (4) with* $x_{-1}y_0,\ x_{-2}y_{-1} \neq -1, -\frac{1}{2}$*, and* $x_0y_{-1},\ x_{-1}y_{-2} \neq \pm 1$*. Then for* $n = 0, 1, 2, ...$,

$$
\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n-1}(1+x_{-1}y_0)^{2n}},\\
x_{8n-1} &= \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^n}{x_0^{2n}y_0^{2n}(1+x_{-2}y_{-1})^{2n}},\\
x_{8n} &= \frac{x_0^{2n+1}y_0^{2n}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^n}{y_{-2}^{2n}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}},\\
x_{8n+1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^n}{x_0^{2n}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}},\\
x_{8n+2} &= \frac{x_0^{2n+1}y_0^{2n+1}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n}(1+x_{-1}y_0)^{2n+1}},\\
x_{8n+3} &= -\frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}},\\
x_{8n+4} &= \frac{x_0^{2n+2}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}},\\
x_{8n+5} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+2}(1+2x_{-1}y_0)^{n+1}(-1+x_0y_{-1})^n(-1-x_0y_{-1})^{n+1}}{x_0^{2n+1}y_0^{2n+2}(1+x_{-2}y_{-1})^{2n+2}},
\end{aligned}
$$

and

$$\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^n},\\
y_{8n-1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^n},\\
y_{8n} &= \frac{x_0^{2n}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^n},\\
y_{8n+1} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n}(1+x_{-1}y_0)^{2n}}{x_0^{2n+1}y_0^{2n}(1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^{n+1}},\\
y_{8n+2} &= \frac{x_0^{2n+1}y_0^{2n+1}(1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n}x_{-2}^{2n+1}(1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(-1-x_0y_{-1})^{n+1}},\\
y_{8n+3} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^n(-1-x_{-1}y_{-2})^{n+1}},\\
y_{8n+4} &= \frac{x_0^{2n+1}y_0^{2n+2}(1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(1+2x_{-1}y_0)^{n+1}(-1+x_0y_{-1})^n(-1-x_0y_{-1})^{n+1}},\\
y_{8n+5} &= -\frac{x_{-1}y_{-2}^{2n+2}x_{-2}^{2n+1}(1+x_{-1}y_0)^{2n+1}}{x_0^{2n+2}y_0^{2n+1}(1+2x_{-2}y_{-1})^{n+1}(-1+x_{-1}y_{-2})^{n+1}(-1-x_{-1}y_{-2})^{n+1}}.
\end{aligned}$$

3.4. Theorem. *Suppose that $\{x_n, y_n\}$ are solutions of system (5) such that $x_{-1}y_0$, $x_{-2}y_{-1} \neq 1, \frac{1}{2}$, and x_0y_{-1}, $x_{-1}y_{-2} \neq \pm 1$. Then for $n = 0, 1, 2, ...$,*

$$\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(-1+x_{-1}y_{-2})^{2n}}{y_{-2}^{2n}x_{-2}^{2n-1}(-1+2x_0y_{-1})^n(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^n},\\
x_{8n-1} &= \frac{x_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_0y_{-1})^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-1}y_{-2})^n(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^n},\\
x_{8n} &= \frac{x_0^{2n+1}y_0^{2n}(-1+x_{-1}y_{-2})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(-1+2x_0y_{-1})^n(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^n},\\
x_{8n+1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n+1}(-1+x_0y_{-1})^{2n}}{x_0^{2n}y_0^{2n+1}(-1+2x_{-1}y_{-2})^n(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^{n+1}},\\
x_{8n+2} &= \frac{x_0^{2n+1}y_0^{2n+1}(-1+x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n}(-1+2x_0y_{-1})^n(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^{n+1}},\\
x_{8n+3} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+x_0y_{-1})^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(-1+2x_{-1}y_{-2})^{n+1}(-1+x_{-2}y_{-1})^n(1+x_{-2}y_{-1})^{n+1}},\\
x_{8n+4} &= -\frac{x_0^{2n+2}y_0^{2n+1}(-1+x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+2x_0y_{-1})^{n+1}(-1+x_{-1}y_0)^n(1+x_{-1}y_0)^{n+1}},\\
x_{8n+5} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+2}(-1+x_0y_{-1})^{2n+1}}{x_0^{2n+1}y_0^{2n+2}(-1+2x_{-1}y_{-2})^{n+1}(-1+x_{-2}y_{-1})^{n+1}(1+x_{-2}y_{-1})^{n+1}},
\end{aligned}$$

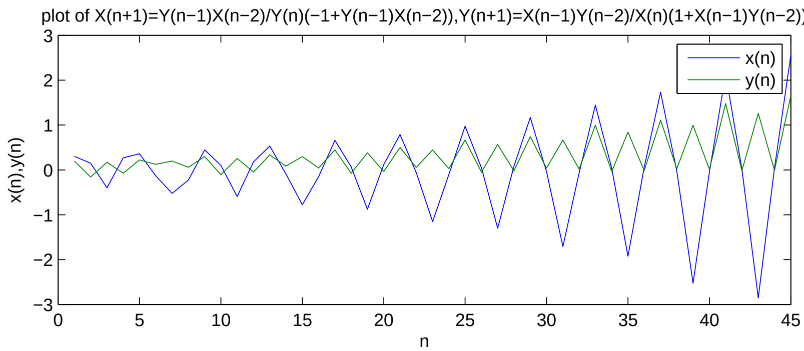

Figure 2

and

$$
\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n}y_0^{2n}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n-1}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n-1} &= \frac{y_{-1}y_{-2}^{2n}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n}y_0^{2n}(-1+2x_{-2}y_{-1})^n(-1+x_{-1}y_{-2})^n(1+x_{-1}y_{-2})^n},\\
y_{8n} &= \frac{x_0^{2n}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n}}{y_{-2}^{2n}x_{-2}^{2n}(-1+2x_{-1}y_0)^n(-1+x_0y_{-1})^n(1+x_0y_{-1})^n},\\
y_{8n+1} &= \frac{x_{-1}y_{-2}^{2n+1}x_{-2}^{2n}(-1+x_{-1}y_0)^{2n}}{x_0^{2n+1}y_0^{2n}(-1+2x_{-2}y_{-1})^n(1+x_{-1}y_{-2})^n(-1+x_{-1}y_{-2})^{n+1}},\\
y_{8n+2} &= -\frac{x_0^{2n+1}y_0^{2n+1}(-1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^n(1+x_0y_{-1})^n(-1+x_0y_{-1})^{n+1}},\\
y_{8n+3} &= -\frac{y_{-1}y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+x_{-1}y_0)^{2n+1}}{x_0^{2n+1}y_0^{2n+1}(-1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^n(-1+x_{-1}y_{-2})^{n+1}},\\
y_{8n+4} &= \frac{x_0^{2n+1}y_0^{2n+2}(-1+x_{-2}y_{-1})^{2n+1}}{y_{-2}^{2n+1}x_{-2}^{2n+1}(-1+2x_{-1}y_0)^{n+1}(1+x_0y_{-1})^n(-1+x_0y_{-1})^{n+1}},\\
y_{8n+5} &= \frac{x_{-1}y_{-2}^{2n+2}x_{-2}^{2n+1}(-1+x_{-1}y_0)^{2n+1}}{x_0^{2n+2}y_0^{2n+1}(-1+2x_{-2}y_{-1})^{n+1}(1+x_{-1}y_{-2})^{n+1}(-1+x_{-1}y_{-2})^{n+1}}.
\end{aligned}
$$

3.5. Example. We consider interesting numerical example for the difference system (2) with the initial conditions $x_{-2} = 0.3,\ x_{-1} = 0.15,\ x_0 = -0.4,\ y_{-2} = 0.2,\ y_{-1} = -0.16$ *and* $y_0 = 0.17$. (See Fig. 2).

4. The Third System: $x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n(1+x_{n-2}y_{n-1})}$, $y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n(-1+y_{n-2}x_{n-1})}$

In this section, we get the solutions of the system of the difference equations

$$(6)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\,(1+x_{n-2}y_{n-1})},\quad y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\,(-1+y_{n-2}x_{n-1})},$$

where $n = 0, 1, 2, ...$ and the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 are arbitrary nonzero real numbers with $x_{-1}y_0,\ x_{-2}y_{-1} \neq \pm 1$, and $x_0y_{-1},\ x_{-1}y_{-2} \neq 1, \frac{1}{2}$.

The following Theorems can be proved similarly as the previous section.

4.1. Theorem. *If* $\{x_n, y_n\}$ *are solutions of difference equation system (6). Then for* $n = 0, 1, 2, ...,$

$$x_{8n-2} = \frac{x_0^{2n} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n-1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0 y_{-1})^n},$$

$$x_{8n-1} = \frac{x_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1 + x_0 y_{-1})^{2n}}{x_0^{2n} y_0^{2n} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n},$$

$$x_{8n} = \frac{x_0^{2n+1} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0 y_{-1})^n},$$

$$x_{8n+1} = \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n+1} (-1 + x_0 y_{-1})^{2n}}{x_0^{2n} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^n},$$

$$x_{8n+2} = \frac{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 + 2x_0 y_{-1})^n},$$

$$x_{8n+3} = \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_0 y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^{n+1}},$$

$$x_{8n+4} = -\frac{x_0^{2n+2} y_0^{2n+1} (-1 + x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 + 2x_0 y_{-1})^{n+1}},$$

$$x_{8n+5} = -\frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+2} (-1 + x_0 y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+2} (-1 + x_{-2}y_{-1})^{n+1} (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^{n+1}},$$

and

$$y_{8n-2} = \frac{x_0^{2n} y_0^{2n} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n}{y_{-2}^{2n-1} x_{-2}^{2n} (-1 + x_0 y_{-1})^{2n}},$$

$$y_{8n-1} = \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0 y_{-1})^n}{x_0^{2n} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n}},$$

$$y_{8n} = \frac{x_0^{2n} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 + 2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n} (-1 + x_0 y_{-1})^{2n}},$$

$$y_{8n+1} = \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 + 2x_0 y_{-1})^n}{x_0^{2n+1} y_0^{2n} (-1 + x_{-1}y_{-2})^{2n+1}},$$

$$y_{8n+2} = \frac{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n+1} (-1 + x_0 y_{-1})^{2n+1}},$$

$$y_{8n+3} = -\frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 + 2x_0 y_{-1})^n}{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-1}y_{-2})^{2n+1}},$$

$$y_{8n+4} = -\frac{x_0^{2n+1} y_0^{2n+2} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 + 2x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_0 y_{-1})^{2n+1}},$$

$$y_{8n+5} = \frac{x_{-1} y_{-2}^{2n+2} x_{-2}^{2n+1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 + 2x_0 y_{-1})^{n+1}}{x_0^{2n+2} y_0^{2n+1} (-1 + x_{-1}y_{-2})^{2n+2}}.$$

4.2. Theorem. *If* $\{x_n, y_n\}$ *are solutions of the following difference equation system*

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1 - x_{n-2}y_{n-1}\right)}, \quad y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(1 + y_{n-2}x_{n-1}\right)},$$

where the initial conditions x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} *and* y_0 *are arbitrary nonzero real numbers with* $x_{-1}y_0$, $x_{-2}y_{-1} \neq \pm 1$, *and* $x_0 y_{-1}$, $x_{-1}y_{-2} \neq -1, -\frac{1}{2}$. *Then for* $n =$

$0, 1, 2, ...,$

$$\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (1 + x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n-1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 - 2x_0y_{-1})^n}, \\
x_{8n-1} &= \frac{x_{-1} y_{-2}^{2n} x_{-2}^{2n} (1 + x_0y_{-1})^{2n}}{x_0^{2n} y_0^{2n} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 - 2x_{-1}y_{-2})^n}, \\
x_{8n} &= \frac{x_0^{2n+1} y_0^{2n} (1 + x_{-1}y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 - 2x_0y_{-1})^n}, \\
x_{8n+1} &= -\frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n+1} (1 + x_0y_{-1})^{2n}}{x_0^{2n} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 - 2x_{-1}y_{-2})^n}, \\
x_{8n+2} &= -\frac{x_0^{2n+1} y_0^{2n+1} (1 + x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 - 2x_0y_{-1})^n}, \\
x_{8n+3} &= -\frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_0y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 - 2x_{-1}y_{-2})^{n+1}}, \\
x_{8n+4} &= \frac{x_0^{2n+2} y_0^{2n+1} (1 + x_{-1}y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 - 2x_0y_{-1})^{n+1}}, \\
x_{8n+5} &= -\frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+2} (1 + x_0y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+2} (-1 + x_{-2}y_{-1})^{n+1} (1 + x_{-2}y_{-1})^{n+1} (-1 - 2x_{-1}y_{-2})^{n+1}},
\end{aligned}$$

and

$$\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 - 2x_{-1}y_{-2})^n}{y_{-2}^{2n-1} x_{-2}^{2n} (1 + x_0y_{-1})^{2n}}, \\
y_{8n-1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 - 2x_0y_{-1})^n}{x_0^{2n} y_0^{2n} (1 + x_{-1}y_{-2})^{2n}}, \\
y_{8n} &= \frac{x_0^{2n} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^n (-1 - 2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n} (1 + x_0y_{-1})^{2n}}, \\
y_{8n+1} &= \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^n (-1 - 2x_0y_{-1})^n}{x_0^{2n+1} y_0^{2n} (1 + x_{-1}y_{-2})^{2n+1}}, \\
y_{8n+2} &= -\frac{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 - 2x_{-1}y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n+1} (1 + x_0y_{-1})^{2n+1}}, \\
y_{8n+3} &= \frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 - 2x_0y_{-1})^n}{x_0^{2n+1} y_0^{2n+1} (1 + x_{-1}y_{-2})^{2n+1}}, \\
y_{8n+4} &= \frac{x_0^{2n+1} y_0^{2n+2} (-1 + x_{-2}y_{-1})^n (1 + x_{-2}y_{-1})^{n+1} (-1 - 2x_{-1}y_{-2})^{n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_0y_{-1})^{2n+1}}, \\
y_{8n+5} &= -\frac{x_{-1} y_{-2}^{2n+2} x_{-2}^{2n+1} (-1 + x_{-1}y_0)^n (1 + x_{-1}y_0)^{n+1} (-1 - 2x_0y_{-1})^{n+1}}{x_0^{2n+2} y_0^{2n+1} (1 + x_{-1}y_{-2})^{2n+2}}.
\end{aligned}$$

4.3. Theorem. *If* $\{x_n, y_n\}$ *are solutions of the difference equations system*

$$x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n (1 - x_{n-2}y_{n-1})}, \; y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n (-1 - y_{n-2}x_{n-1})},$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ *and* y_0 *are arbitrary nonzero real numbers with* $x_{-1}y_0,\ x_{-2}y_{-1} \neq \pm 1,$ *and* $x_0y_{-1},\ x_{-1}y_{-2} \neq -1, -\frac{1}{2}$*. Then for* $n =$

$0, 1, 2, ...,$

$$
\begin{aligned}
x_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (1 + x_{-1} y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n-1} (1 - x_{-1} y_0)^n (1 + x_{-1} y_0)^n (1 + 2x_0 y_{-1})^n}, \\
x_{8n-1} &= \frac{x_{-1} y_{-2}^{2n} x_{-2}^{2n} (1 + x_0 y_{-1})^{2n}}{x_0^{2n} y_0^{2n} (1 - x_{-2} y_{-1})^n (1 + x_{-2} y_{-1})^n (1 + 2x_{-1} y_{-2})^n}, \\
x_{8n} &= \frac{x_0^{2n+1} y_0^{2n} (1 + x_{-1} y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n} (1 - x_{-1} y_0)^n (1 + x_{-1} y_0)^n (1 + 2x_0 y_{-1})^n}, \\
x_{8n+1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n+1} (1 + x_0 y_{-1})^{2n}}{x_0^{2n} y_0^{2n+1} (1 - x_{-2} y_{-1})^{n+1} (1 + x_{-2} y_{-1})^n (1 + 2x_{-1} y_{-2})^n}, \\
x_{8n+2} &= -\frac{x_0^{2n+1} y_0^{2n+1} (1 + x_{-1} y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n} (1 + x_{-1} y_0)^n (1 - x_{-1} y_0)^{n+1} (1 + 2x_0 y_{-1})^n}, \\
x_{8n+3} &= \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_0 y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+1} (1 + x_{-2} y_{-1})^n (1 - x_{-2} y_{-1})^{n+1} (1 + 2x_{-1} y_{-2})^{n+1}}, \\
x_{8n+4} &= -\frac{x_0^{2n+2} y_0^{2n+1} (1 + x_{-1} y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_{-1} y_0)^n (1 - x_{-1} y_0)^{n+1} (1 + 2x_0 y_{-1})^{n+1}}, \\
x_{8n+5} &= \frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+2} (1 + x_0 y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+2} (1 + x_{-2} y_{-1})^{n+1} (1 - x_{-2} y_{-1})^{n+1} (1 + 2x_{-1} y_{-2})^{n+1}},
\end{aligned}
$$

and

$$
\begin{aligned}
y_{8n-2} &= \frac{x_0^{2n} y_0^{2n} (1 + x_{-2} y_{-1})^n (1 - x_{-2} y_{-1})^n (1 + 2x_{-1} y_{-2})^n}{y_{-2}^{2n-1} x_{-2}^{2n} (1 + x_0 y_{-1})^{2n}}, \\
y_{8n-1} &= \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n} (1 + x_{-1} y_0)^n (1 - x_{-1} y_0)^n (1 + 2x_0 y_{-1})^n}{x_0^{2n} y_0^{2n} (1 + x_{-1} y_{-2})^{2n}}, \\
y_{8n} &= \frac{x_0^{2n} y_0^{2n+1} (1 + x_{-2} y_{-1})^n (1 - x_{-2} y_{-1})^n (1 + 2x_{-1} y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n} (1 + x_0 y_{-1})^{2n}}, \\
y_{8n+1} &= -\frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n} (1 + x_{-1} y_0)^n (1 - x_{-1} y_0)^n (1 + 2x_0 y_{-1})^n}{x_0^{2n+1} y_0^{2n} (1 + x_{-1} y_{-2})^{2n+1}}, \\
y_{8n+2} &= -\frac{x_0^{2n+1} y_0^{2n+1} (1 + x_{-2} y_{-1})^n (1 - x_{-2} y_{-1})^{n+1} (1 + 2x_{-1} y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n+1} (1 + x_0 y_{-1})^{2n+1}}, \\
y_{8n+3} &= \frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_{-1} y_0)^n (1 - x_{-1} y_0)^{n+1} (1 + 2x_0 y_{-1})^n}{x_0^{2n+1} y_0^{2n+1} (1 + x_{-1} y_{-2})^{2n+1}}, \\
y_{8n+4} &= -\frac{x_0^{2n+1} y_0^{2n+2} (1 + x_{-2} y_{-1})^n (1 - x_{-2} y_{-1})^{n+1} (1 + 2x_{-1} y_{-2})^{n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_0 y_{-1})^{2n+1}}, \\
y_{8n+5} &= -\frac{x_{-1} y_{-2}^{2n+2} x_{-2}^{2n+1} (1 + x_{-1} y_0)^n (1 - x_{-1} y_0)^{n+1} (1 + 2x_0 y_{-1})^{n+1}}{x_0^{2n+2} y_0^{2n+1} (1 + x_{-1} y_{-2})^{2n+2}}.
\end{aligned}
$$

4.4. Theorem. *Assume that* $\{x_n, y_n\}$ *are solutions of the following system with the initial conditions* x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} *and* y_0 *are arbitrary nonzero real numbers with* $x_{-1}y_0$, $x_{-2}y_{-1} \neq \pm 1$, *and* $x_0 y_{-1}$, $x_{-1}y_{-2} \neq 1, \frac{1}{2}$

$$
x_{n+1} = \frac{x_{n-2} y_{n-1}}{y_n \left(-1 + x_{n-2} y_{n-1}\right)}, \quad y_{n+1} = \frac{y_{n-2} x_{n-1}}{x_n \left(1 - y_{n-2} x_{n-1}\right)}.
$$

Then for $n = 0, 1, 2, ...,$

$$x_{8n-2} = \frac{x_0^{2n} y_0^{2n} (-1 + x_{-1} y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n-1} (-1 + x_{-1} y_0)^n (1 + x_{-1} y_0)^n (-1 + 2x_0 y_{-1})^n},$$

$$x_{8n-1} = \frac{x_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1 + x_0 y_{-1})^{2n}}{x_0^{2n} y_0^{2n} (-1 + x_{-2} y_{-1})^n (1 + x_{-2} y_{-1})^n (-1 + 2x_{-1} y_{-2})^n},$$

$$x_{8n} = \frac{x_0^{2n+1} y_0^{2n} (-1 + x_{-1} y_{-2})^{2n}}{y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1} y_0)^n (1 + x_{-1} y_0)^n (-1 + 2x_0 y_{-1})^n},$$

$$x_{8n+1} = \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n+1} (-1 + x_0 y_{-1})^{2n}}{x_0^{2n} y_0^{2n+1} (-1 + x_{-2} y_{-1})^{n+1} (1 + x_{-2} y_{-1})^n (-1 + 2x_{-1} y_{-2})^n},$$

$$x_{8n+2} = \frac{x_0^{2n+1} y_0^{2n+1} (1 - x_{-1} y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1} y_0)^{n+1} (1 + x_{-1} y_0)^n (-1 + 2x_0 y_{-1})^n},$$

$$x_{8n+3} = \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (1 - x_0 y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+1} (-1 + x_{-2} y_{-1})^{n+1} (1 + x_{-2} y_{-1})^n (-1 + 2x_{-1} y_{-2})^{n+1}},$$

$$x_{8n+4} = \frac{x_0^{2n+2} y_0^{2n+1} (1 - x_{-1} y_{-2})^{2n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_{-1} y_0)^n (-1 + x_{-1} y_0)^{n+1} (-1 + 2x_0 y_{-1})^{n+1}},$$

$$x_{8n+5} = -\frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+2} (1 - x_0 y_{-1})^{2n+1}}{x_0^{2n+1} y_0^{2n+2} (-1 + x_{-2} y_{-1})^{n+1} (1 + x_{-2} y_{-1})^{n+1} (-1 + 2x_{-1} y_{-2})^{n+1}},$$

and

$$y_{8n-2} = \frac{x_0^{2n} y_0^{2n} (-1 + x_{-2} y_{-1})^n (1 + x_{-2} y_{-1})^n (-1 + 2x_{-1} y_{-2})^n}{y_{-2}^{2n-1} x_{-2}^{2n} (1 - x_0 y_{-1})^{2n}},$$

$$y_{8n-1} = \frac{y_{-1} y_{-2}^{2n} x_{-2}^{2n} (-1 + x_{-1} y_0)^n (1 + x_{-1} y_0)^n (-1 + 2x_0 y_{-1})^n}{x_0^{2n} y_0^{2n} (1 - x_{-1} y_{-2})^{2n}},$$

$$y_{8n} = \frac{x_0^{2n} y_0^{2n+1} (-1 + x_{-2} y_{-1})^n (1 + x_{-2} y_{-1})^n (-1 + 2x_{-1} y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n} (1 - x_0 y_{-1})^{2n}},$$

$$y_{8n+1} = \frac{x_{-1} y_{-2}^{2n+1} x_{-2}^{2n} (-1 + x_{-1} y_0)^n (1 + x_{-1} y_0)^n (-1 + 2x_0 y_{-1})^n}{x_0^{2n+1} y_0^{2n} (1 - x_{-1} y_{-2})^{2n+1}},$$

$$y_{8n+2} = \frac{x_0^{2n+1} y_0^{2n+1} (1 + x_{-2} y_{-1})^n (-1 + x_{-2} y_{-1})^{n+1} (-1 + 2x_{-1} y_{-2})^n}{y_{-2}^{2n} x_{-2}^{2n+1} (1 - x_0 y_{-1})^{2n+1}},$$

$$y_{8n+3} = -\frac{y_{-1} y_{-2}^{2n+1} x_{-2}^{2n+1} (1 + x_{-1} y_0)^n (-1 + x_{-1} y_0)^{n+1} (-1 + 2x_0 y_{-1})^n}{x_0^{2n+1} y_0^{2n+1} (1 - x_{-1} y_{-2})^{2n+1}},$$

$$y_{8n+4} = -\frac{x_0^{2n+1} y_0^{2n+2} (1 + x_{-2} y_{-1})^n (-1 + x_{-2} y_{-1})^{n+1} (-1 + 2x_{-1} y_{-2})^{n+1}}{y_{-2}^{2n+1} x_{-2}^{2n+1} (1 - x_0 y_{-1})^{2n+1}},$$

$$y_{8n+5} = \frac{x_{-1} y_{-2}^{2n+2} x_{-2}^{2n+1} (1 + x_{-1} y_0)^n (-1 + x_{-1} y_0)^{n+1} (-1 + 2x_0 y_{-1})^{n+1}}{x_0^{2n+2} y_0^{2n+1} (1 - x_{-1} y_{-2})^{2n+2}}.$$

5. The Fourth System: $x_{n+1} = \frac{x_{n-2} y_{n-1}}{y_n (1 + x_{n-2} y_{n-1})}$, $y_{n+1} = \frac{y_{n-2} x_{n-1}}{x_n (1 - y_{n-2} x_{n-1})}$

In this section, we get the form of the solutions of the system of the difference equations

$$(7) \qquad x_{n+1} = \frac{x_{n-2} y_{n-1}}{y_n (1 + x_{n-2} y_{n-1})}, \; y_{n+1} = \frac{y_{n-2} x_{n-1}}{x_n (1 - y_{n-2} x_{n-1})},$$

where $n = 0, 1, 2, ...$ and the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$ *and* y_0 are arbitrary nonzero real numbers with $x_{-1} y_{-2}, x_0 y_{-1} \neq 1$, $x_{-2} y_{-1}, x_{-1} y_0 \neq -1$.

5.1. Theorem. *If $\{x_n, y_n\}$ are solutions of difference equation system (7). Then for $n = 0, 1, 2, ...,$*

$$x_{4n-2} = \frac{x_0^n y_0^n (1 - x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1} (1 + x_{-1}y_0)^n}, \quad x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (1 - x_0 y_{-1})^n}{x_0^n y_0^n (1 + x_{-2}y_{-1})^n},$$
$$x_{4n} = \frac{x_0^{n+1} y_0^n (1 - x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (1 + x_{-1}y_0)^n}, \quad x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1} (1 - x_0 y_{-1})^n}{x_0^n y_0^{n+1} (1 + x_{-2}y_{-1})^{n+1}},$$

and

$$y_{4n-2} = \frac{x_0^n y_0^n (1 + x_{-2}y_{-1})^n}{y_{-2}^{n-1} x_{-2}^n (1 - x_0 y_{-1})^n}, \quad y_{4n-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n (1 + x_{-1}y_0)^n}{x_0^n y_0^n (1 - x_{-1}y_{-2})^n},$$
$$y_{4n} = \frac{x_0^n y_0^{n+1} (1 + x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n (1 - x_0 y_{-1})^n}, \quad y_{4n+1} = \frac{x_{-1}y_{-2}^{n+1} x_{-2}^n (1 + x_{-1}y_0)^n}{x_0^{n+1} y_0^n (1 - x_{-1}y_{-2})^{n+1}}.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$. that is,

$$x_{4n-6} = \frac{x_0^{n-1} y_0^{n-1} (1 - x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1} x_{-2}^{n-2} (1 + x_{-1}y_0)^{n-1}}, \quad x_{4n-5} = \frac{x_{-1}y_{-2}^{n-1} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}{x_0^{n-1} y_0^{n-1} (1 + x_{-2}y_{-1})^{n-1}},$$
$$x_{4n-4} = \frac{x_0^n y_0^{n-1} (1 - x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1} x_{-2}^{n-1} (1 + x_{-1}y_0)^{n-1}}, \quad x_{4n-3} = \frac{y_{-1}y_{-2}^{n-1} x_{-2}^n (1 - x_0 y_{-1})^{n-1}}{x_0^{n-1} y_0^n (1 + x_{-2}y_{-1})^n},$$

$$y_{4n-6} = \frac{x_0^{n-1} y_0^{n-1} (1 + x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-2} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}, \quad y_{4n-5} = \frac{y_{-1}y_{-2}^{n-1} x_{-2}^{n-1} (1 + x_{-1}y_0)^{n-1}}{x_0^{n-1} y_0^{n-1} (1 - x_{-1}y_{-2})^{n-1}},$$
$$y_{4n-4} = \frac{x_0^{n-1} y_0^n (1 + x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}, \quad y_{4n-3} = \frac{x_{-1}y_{-2}^n x_{-2}^{n-1} (1 + x_{-1}y_0)^{n-1}}{x_0^n y_0^{n-1} (1 - x_{-1}y_{-2})^n}.$$

It follows from Eq.(7) that

$$\begin{aligned}
x_{4n-2} &= \frac{x_{4n-5}y_{4n-4}}{y_{4n-3}\left(+x_{4n-5}y_{4n-4}\right)} \\
&= \frac{\left(\frac{x_{-1}y_{-2}^{n-1} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}{x_0^{n-1} y_0^{n-1} (1 + x_{-2}y_{-1})^{n-1}}\right)\left(\frac{x_0^{n-1} y_0^n (1 + x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}\right)}{\left(\frac{x_{-1}y_{-2}^n x_{-2}^{n-1} (1 + x_{-1}y_0)^{n-1}}{x_0^n y_0^{n-1} (1 - x_{-1}y_{-2})^n}\right)\left(1 + \left(\frac{x_{-1}y_{-2}^{n-1} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}{x_0^{n-1} y_0^{n-1} (1 + x_{-2}y_{-1})^{n-1}}\right)\left(\frac{x_0^{n-1} y_0^n (1 + x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1} x_{-2}^{n-1} (1 - x_0 y_{-1})^{n-1}}\right)\right)} \\
&= \frac{x_{-1}y_0}{\left(\frac{x_{-1}y_{-2}^n x_{-2}^{n-1} (1 + x_{-1}y_0)^{n-1}}{x_0^n y_0^{n-1} (1 - x_{-1}y_{-2})^n}\right)(1 + x_{-1}y_0)} \\
&= \frac{x_{-1}y_0 x_0^n y_0^{n-1} (1 - x_{-1}y_{-2})^n}{x_{-1}y_{-2}^n x_{-2}^{n-1} (1 + x_{-1}y_0)^{n-1} (1 + x_{-1}y_0)} = \frac{x_0^n y_0^n (1 - x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1} (1 + x_{-1}y_0)^n},
\end{aligned}$$

$$\begin{aligned}
y_{4n-2} &= \frac{y_{4n-5}x_{4n-4}}{x_{4n-3}\left(1-y_{4n-5}x_{4n-4}\right)} \\
&= \frac{\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}{x_0^{n-1}y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}\right)\left(\frac{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}\right)}{\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})^{n}}\right)\left(1-\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}{x_0^{n-1}y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}\right)\left(\frac{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}\right)\right)} \\
&= \frac{x_0y_{-1}x_0^{n-1}y_0^n(1+x_{-2}y_{-1})^n}{y_{-1}y_{-2}^{n-1}x_{-2}^n(1-x_0y_{-1})^{n-1}\left(1-x_0y_{-1}\right)} = \frac{x_0^n y_0^n(1+x_{-2}y_{-1})^n}{y_{-2}^{n-1}x_{-2}^n(1-x_0y_{-1})^n}.
\end{aligned}$$

Also, we see from Eq.(7) that

$$\begin{aligned}
x_{4n-1} &= \frac{x_{4n-4}y_{4n-3}}{y_{4n-2}\left(1+x_{4n-4}y_{4n-3}\right)} \\
&= \frac{\left(\frac{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}\right)\left(\frac{x_{-1}y_{-2}^{n}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n}}\right)}{\left(\frac{x_0^n y_0^n(1+x_{-2}y_{-1})^n}{y_{-2}^{n-1}x_{-2}^n(1-x_0y_{-1})^n}\right)\left(1+\left(\frac{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}\right)\left(\frac{x_{-1}y_{-2}^{n}x_{-2}^{n-1}(1+x_{-1}y_0)^{n-1}}{x_0^n y_0^{n-1}(1-x_{-1}y_{-2})^{n}}\right)\right)} \\
&= \frac{\left(\frac{x_{-1}y_{-2}}{(1-x_{-1}y_{-2})}\right)}{\left(\frac{x_0^n y_0^n(1+x_{-2}y_{-1})^n}{y_{-2}^{n-1}x_{-2}^n(1-x_0y_{-1})^n}\right)\left(1+\frac{x_{-1}y_{-2}}{(1-x_{-1}y_{-2})}\right)} \\
&= \frac{y_{-2}^{n-1}x_{-2}^n(1-x_0y_{-1})^n x_{-1}y_{-2}}{x_0^n y_0^n(1+x_{-2}y_{-1})^n\left(1-x_{-1}y_{-2}+x_{-1}y_{-2}\right)} = \frac{x_{-1}y_{-2}^n x_{-2}^n(1-x_0y_{-1})^n}{x_0^n y_0^n(1+x_{-2}y_{-1})^n},
\end{aligned}$$

and

$$\begin{aligned}
y_{4n-1} &= \frac{y_{4n-4}x_{4n-3}}{x_{4n-2}\left(1-y_{4n-4}x_{4n-3}\right)} \\
&= \frac{\left(\frac{x_0^{n-1}y_0^n(1+x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1-x_0y_{-1})^{n-1}}\right)\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})^{n}}\right)}{\left(\frac{x_0^n y_0^n(1-x_{-1}y_{-2})^n}{y_{-2}^{n}x_{-2}^{n-1}(1+x_{-1}y_0)^n}\right)\left(1-\left(\frac{x_0^{n-1}y_0^n(1+x_{-2}y_{-1})^{n-1}}{y_{-2}^{n-1}x_{-2}^{n-1}(1-x_0y_{-1})^{n-1}}\right)\left(\frac{y_{-1}y_{-2}^{n-1}x_{-2}^{n}(1-x_0y_{-1})^{n-1}}{x_0^{n-1}y_0^{n}(1+x_{-2}y_{-1})^{n}}\right)\right)} \\
&= \frac{\left(\frac{y_{-1}x_{-2}}{(1+x_{-2}y_{-1})}\right)}{\left(\frac{x_0^n y_0^n(1-x_{-1}y_{-2})^n}{y_{-2}^{n}x_{-2}^{n-1}(1+x_{-1}y_0)^n}\right)\left(1-\left(\frac{y_{-1}x_{-2}}{(1+x_{-2}y_{-1})}\right)\right)} \\
&= \frac{y_{-2}^n x_{-2}^{n-1}(1+x_{-1}y_0)^n y_{-1}x_{-2}}{x_0^n y_0^n(1-x_{-1}y_{-2})^n\left(1+x_{-2}y_{-1}-x_{-2}y_{-1}\right)} = \frac{y_{-1}y_{-2}^n x_{-2}^n(1+x_{-1}y_0)^n}{x_0^n y_0^n(1-x_{-1}y_{-2})^n}.
\end{aligned}$$

Also, the other relations can be proved similarly. This completes the proof. ∎

We consider the following systems and the proof of the theorems are similarly to above theorem and so, left to the reader.

$$(5.1)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(1-x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(1+y_{n-2}x_{n-1}\right)}.8$$

$$(5.2)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1+x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1+y_{n-2}x_{n-1}\right)}.9$$

$$(5.3)\qquad x_{n+1} = \frac{x_{n-2}y_{n-1}}{y_n\left(-1-x_{n-2}y_{n-1}\right)},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{x_n\left(-1-y_{n-2}x_{n-1}\right)}.10$$

The following theorems is devoted to the expressions of the form of the solutions of systems (8), (9), (10).

5.2. Theorem. *Let* $\{x_n, y_n\}_{n=-2}^{+\infty}$ *be solutions of system (8) and* $x_{-1}y_{-2}$, $x_0y_{-1} \neq -1$, $x_{-2}y_{-1}$, $x_{-1}y_0 \neq 1$. *Then for* $n = 0, 1, 2, \ldots$,

$$x_{4n-2} = \frac{x_0^n y_0^n (1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(1-x_{-1}y_0)^n},\quad x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (1+x_0y_{-1})^n}{x_0^n y_0^n (1-x_{-2}y_{-1})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n (1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (1-x_{-1}y_0)^n},\quad x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1}(1+x_0y_{-1})^n}{x_0^n y_0^{n+1}(1-x_{-2}y_{-1})^{n+1}},$$

$$y_{4n-2} = \frac{x_0^n y_0^n (1-x_{-2}y_{-1})^n}{y_{-2}^{n-1} x_{-2}^n (1+x_0y_{-1})^n},\quad y_{4n-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n (1-x_{-1}y_0)^n}{x_0^n y_0^n (1+x_{-1}y_{-2})^n},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1} (1-x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n (1+x_0y_{-1})^n},\quad y_{4n+1} = \frac{x_{-1}y_{-2}^{n+1} x_{-2}^n (1-x_{-1}y_0)^n}{x_0^{n+1} y_0^n (1+x_{-1}y_{-2})^{n+1}}.$$

5.3. Theorem. *Assume that* $\{x_n, y_n\}$ *are solutions of system (9) with* $x_{-1}y_{-2}$, x_0y_{-1}, $x_{-2}y_{-1}$, $x_{-1}y_0 \neq 1$. *Then for* $n = 0, 1, 2, \ldots$,

$$x_{4n-2} = \frac{x_0^n y_0^n (-1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(-1+x_{-1}y_0)^n},\quad x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (-1+x_0y_{-1})^n}{x_0^n y_0^n (-1+x_{-2}y_{-1})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n (-1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (-1+x_{-1}y_0)^n},\quad x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1}(-1+x_0y_{-1})^n}{x_0^n y_0^{n+1}(-1+x_{-2}y_{-1})^{n+1}},$$

$$y_{4n-2} = \frac{x_0^n y_0^n (-1+x_{-2}y_{-1})^n}{y_{-2}^{n-1} x_{-2}^n (-1+x_0y_{-1})^n},\quad y_{4n-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n (-1+x_{-1}y_0)^n}{x_0^n y_0^n (-1+x_{-1}y_{-2})^n},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1} (-1+x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n (-1+x_0y_{-1})^n},\quad y_{4n+1} = \frac{x_{-1}y_{-2}^{n+1} x_{-2}^n (-1+x_{-1}y_0)^n}{x_0^{n+1} y_0^n (-1+x_{-1}y_{-2})^{n+1}}.$$

5.4. Theorem. *Suppose that* $\{x_n, y_n\}$ *are solutions of system (10) such that* $x_{-1}y_{-2}$, x_0y_{-1}, $x_{-2}y_{-1}$, $x_{-1}y_0 \neq -1$. *Then for* $n = 0, 1, 2, \ldots$,

$$x_{4n-2} = \frac{x_0^n y_0^n (-1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(-1-x_{-1}y_0)^n},\quad x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (-1-x_0y_{-1})^n}{x_0^n y_0^n (-1-x_{-2}y_{-1})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n (-1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (-1-x_{-1}y_0)^n},\quad x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1}(-1-x_0y_{-1})^n}{x_0^n y_0^{n+1}(-1-x_{-2}y_{-1})^{n+1}},$$

$$y_{4n-2} = \frac{x_0^n y_0^n (-1-x_{-2}y_{-1})^n}{y_{-2}^{n-1} x_{-2}^n (-1-x_0y_{-1})^n},\quad y_{4n-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n (-1-x_{-1}y_0)^n}{x_0^n y_0^n (-1-x_{-1}y_{-2})^n},$$

$$y_{4n} = \frac{x_0^n y_0^{n+1} (-1-x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n (-1-x_0y_{-1})^n},\quad y_{4n+1} = \frac{x_{-1}y_{-2}^{n+1} x_{-2}^n (-1-x_{-1}y_0)^n}{x_0^{n+1} y_0^n (-1-x_{-1}y_{-2})^{n+1}}.$$

5.5. Lemma. *The solution of system (7) is unbounded except in the following case.*

5.6. Theorem. *System (7) has a periodic solution of period four iff* $y_{-2} = -y_0,\ x_{-2} = -x_0$ *and it will be taken the following form* $\{x_n\} = \left\{x_{-2}, x_{-1}, x_0, \frac{y_{-1}x_{-2}}{y_0(1+x_{-2}y_{-1})}, x_{-2}, x_{-1}, x_0, ...\right\}$, $\{y_n\} = \left\{y_{-2}, y_{-1}, y_0, \frac{x_{-1}y_{-2}}{x_0(1+y_{-2}x_{-1})}, y_{-2}, y_{-1}, y_0, , ...\right\}$.

Proof. First suppose that there exists a prime period four solution

$$\begin{aligned}\{x_n\} &= \left\{x_{-2}, x_{-1}, x_0, \tfrac{y_{-1}x_{-2}}{y_0(1+x_{-2}y_{-1})}, x_{-2}, x_{-1}, x_0, ...\right\},\\ \{y_n\} &= \left\{y_{-2}, y_{-1}, y_0, \tfrac{x_{-1}y_{-2}}{x_0(1+y_{-2}x_{-1})}, y_{-2}, y_{-1}, y_0, , ...\right\},\end{aligned}$$

of system (7), we see from the form of the solution of system (7) that

$$\begin{aligned}x_{4n-2} &= x_{-2} = \frac{x_0^n y_0^n (1 - x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1}(1 + x_{-1}y_0)^n},\ x_{4n-1} = x_{-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (1 - x_0 y_{-1})^n}{x_0^n y_0^n (1 + x_{-2}y_{-1})^n},\\ x_{4n} &= x_0 = \frac{x_0^{n+1} y_0^n (1 - x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (1 + x_{-1}y_0)^n},\ x_{4n+1} = \tfrac{y_{-1}x_{-2}}{y_0(1+x_{-2}y_{-1})} = \tfrac{y_{-1}y_{-2}^n x_{-2}^{n+1}(1-x_0y_{-1})^n}{x_0^n y_0^{n+1}(1+x_{-2}y_{-1})^{n+1}},\\ y_{4n-2} &= y_{-2} = \frac{x_0^n y_0^n (1 + x_{-2}y_{-1})^n}{y_{-2}^{n-1} x_{-2}^n (1 - x_0 y_{-1})^n},\ y_{4n-1} = y_{-1} = \frac{y_{-1}y_{-2}^n x_{-2}^n (1 + x_{-1}y_0)^n}{x_0^n y_0^n (1 - x_{-1}y_{-2})^n},\\ y_{4n} &= y_0 = \frac{x_0^n y_0^{n+1} (1 + x_{-2}y_{-1})^n}{y_{-2}^n x_{-2}^n (1 - x_0 y_{-1})^n},\ y_{4n+1} = \tfrac{x_{-1}y_{-2}}{x_0(1+y_{-2}x_{-1})} = \tfrac{x_{-1}y_{-2}^{n+1}x_{-2}^n(1+x_{-1}y_0)^n}{x_0^{n+1}y_0^n(1-x_{-1}y_{-2})^{n+1}}.\end{aligned}$$

Then we get

$$y_{-2} = -y_0,\ x_{-2} = -x_0.$$

Second assume that $y_{-2} = -y_0,\ x_{-2} = -x_0$. Then we see from the form of the solution of system (7) that

$$\begin{aligned}x_{4n-2} &= x_{-2},\ x_{4n-1} = x_{-1},\ \ x_{4n} = x_0,\ x_{4n+1} = \tfrac{y_{-1}x_{-2}}{y_0(1+x_{-2}y_{-1})},\\ y_{4n-2} &= y_{-2},\ \ \ y_{4n-1} = y_{-1},\ \ y_{4n} = y_0,\ \ \ y_{4n+1} = \tfrac{x_{-1}y_{-2}}{x_0(1+y_{-2}x_{-1})}.\end{aligned}$$

Thus we have a periodic solution of period four and the proof is complete. ∎

Also, we can prove the following Theorems:

5.7. Lemma. *The solutions of all systems (8), (9) and (10) are unbounded except in the following cases.*

5.8. Theorem. *System (8) has a periodic solution of period four iff* $y_{-2} = -y_0,\ x_{-2} = -x_0$ *and it will be taken the following form* $\{x_n\} = \left\{x_{-2}, x_{-1}, x_0, \frac{y_{-1}x_{-2}}{y_0(1-x_{-2}y_{-1})}, x_{-2}, x_{-1}, x_0, ...\right\}$, $\{y_n\} = \left\{y_{-2}, y_{-1}, y_0, \frac{x_{-1}y_{-2}}{x_0(1-y_{-2}x_{-1})}, y_{-2}, y_{-1}, y_0, , ...\right\}$.

5.9. Theorem. *All Solutions of the difference equations system (9) are periodic solution with period four iff* $y_{-2} = y_0,\ x_{-2} = x_0$ *and it will be taken the following form* $\{x_n\} = \left\{x_{-2}, x_{-1}, x_0, \frac{y_{-1}x_{-2}}{y_0(-1+x_{-2}y_{-1})}, x_{-2}, ...\right\}$, $\{y_n\} = \left\{y_{-2}, y_{-1}, y_0, \frac{x_{-1}y_{-2}}{x_0(-1+y_{-2}x_{-1})}, y_{-2}, , ...\right\}$.

5.10. Theorem. *If* $\{x_n\}$, $\{y_n\}$ *are solutions of system (10), then* $\{x_n\}$, $\{y_n\}$ *are periodic solutions of period four iff* $y_{-2} = y_0,\ x_{-2} = x_0$ *and it will be in the following form* $\{x_n\} = \left\{x_{-2}, x_{-1}, x_0, \frac{y_{-1}x_{-2}}{y_0(-1-x_{-2}y_{-1})}, x_{-2}, ...\right\}$, $\{y_n\} = \left\{y_{-2}, y_{-1}, y_0, \frac{x_{-1}y_{-2}}{x_0(-1-y_{-2}x_{-1})}, y_{-2}, , ...\right\}$.

5.11. Example. We consider interesting numerical example for the difference system (7) with the initial conditions $x_{-2} = 0.3,\ x_{-1} = 0.15,\ x_0 = -0.4,\ y_{-2} = 0.2,\ y_{-1} = -0.16$ *and* $y_0 = 0.17$. See Figure (3).

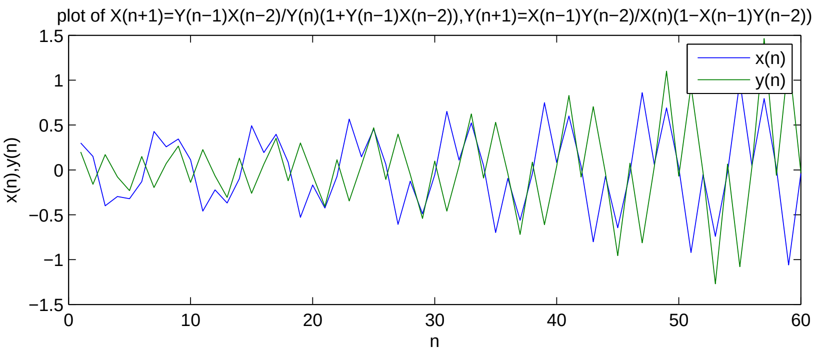

Figure 3

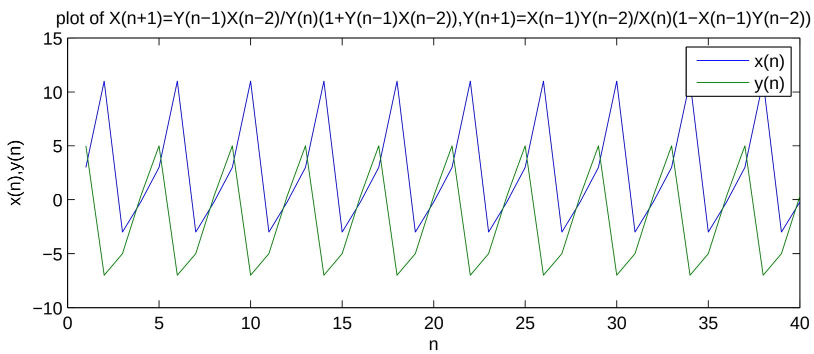

Figure 4

5.12. Example. See Figure (4) when we take system (7) with the initial conditions $x_{-2} = 3,\ x_{-} = 11,\ x_0 = -3,\ y_{-2} = 5,\ y_{-1} = -7$ *and* $y_0 = -5$.

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