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Research Article

First steps going down on algebraic frames

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Abstract

We extend the ring-theoretic concept of going down to algebraic frames and coherent maps. We then use the notion introduced to characterize algebraic frames of dimension 0 and frames of dimension at most 1. An application to rings yields a characterization of von Neumann regular rings that appears to have hitherto been overlooked. Namely, a commutative ring A with identity is von Neumann regular if and only if $\mathrm{Ann}(I) + P = A$, for every prime ideal P of A and any finitely generated ideal I of A contained in P.

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1. Introduction and motivation

The symbiosis that exists between ring theory and topology is epitomized by Melvin Hochster's epic theorem [9] that spectral spaces are, up to homeomorphism, exactly the prime spectra of unitary commutative rings. This result has been significantly sharpened by Bernhard Banaschewski [2], in that he showed, without choice, no less, that every coherent frame is isomorphic to the frame of radical ideals of a commutative ring with unit.

Rings and frames benefit a great deal from each other. In the words of Niefield and Rosenthal [16]:

> *There are some interesting insights to be gained by considering rings and lattices (in particular, locales) simultaneously.*

Indeed. A number of properties of rings have very lucid characterizations in terms of localic concepts. To give an example, we cite Banaschewski's result that a commutative ring with identity is a Gelfand ring (meaning that whenever $a + b = 1$, there are elements r and s in the ring such that $(1 - ar)(1 - bs) = 0$) if and only if the frame of radical ideals of the ring is normal [3].

In a series of papers (see, for instance, [14] and the references therein), Jorge Martínez and his former students have generalized a number of ring-theoretic results to algebraic frames. It is in that spirit that we extend the classical going-down property in rings to algebraic frames. Since we shall not put the condition that the algebraic frames in question be coherent, the results we present cannot be deduced from the corresponding

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ones in rings via the usual functors one encounters when dealing with rings and frames at the same time.

It is apposite to mention that the ring-theoretic notion of going-up has already been considered in algebraic frames by Martínez in [14]. He went up, we go down.

Apart from this introduction, the paper comprises four sections. In Section 2, we recall a few relevant facts regarding frames, and, in particular, algebraic frames. We are brief about it, counting on the reader who is not au fait with this subject to consult our main references, [10] and [17].

In Section 3, we define the going-down property for frame homomorphisms, and then give several characterizations of when a coherent map between algebraic frames with the finite intersection property on compact elements satisfies this property. These characterizations extend analogous ones for rings [8]. One of them is in terms of the localic version of the localization technique in rings. We end the section with some sufficient conditions for a coherent map to satisfy the going-down property. We show, in particular, that if the continuous function $\Sigma h\colon \Sigma M \to \Sigma L$ induced by a coherent map $h\colon L \to M$ is open, then the localic map $h_*\colon M \to L$ is open and h satisfies the going-down property. We also establish the converse to this result.

Section 4 uses the material in the preceding section to characterize algebraic frames of Krull dimension 0 and of Krull dimension at most 1. These results accord with similar ones for rings of these dimensions [7], with some notable differences here and there. For instance, Dobbs and Fontana [7] use flat ring homomorphisms, among other things, to characterize zero-dimensional rings. In our case, we introduce what we call "slightly open" coherent maps, defined to be those that preserve pseudocomplements of compact elements, and then prove that the dimension of L is zero if and only if the natural homomorphism $L \to {\uparrow}p$, given by $x \mapsto x \vee p$, is slightly open for every prime element $p \in L$. As a corollary, we obtain the characterization of von Neumann regular rings stated in the abstract.

2. Preliminaries

2.1. Frames

We refer to [10] and [17] for background on frames and frame homomorphisms. The right adjoint of a frame homomorphism h is denoted by h_*. If L is a frame and $a \in L$, we write $\kappa_a\colon L \to {\uparrow}a$ for the frame homomorphism $x \mapsto a \vee x$. Its right adjoint is the inclusion ${\uparrow}a \hookrightarrow L$.

We shall, from time to time, alternate between frames and locales. Our usage of terms such as "sublocale" and "localic map" will be as in [17]. The notation regarding these will also be of that text. Thus, for instance, the coframe of sublocales of L is written as $\mathcal{S}(L)$, and the frame obtained from this by "standing it on its head" (to quote Isbell) is written as $\mathcal{S}(L)^{\mathrm{op}}$.

An element $p \in L$ is *prime* if $p \neq 1$ and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. We use the word "prime" both as a noun and an adjective. The set of primes of L will be denoted by $\mathrm{Pr}(L)$, and the set of minimal primes by $\mathrm{Min}(L)$. A Zorn's Lemma argument shows that for any $p \in \mathrm{Pr}(L)$, there is some $q \in \mathrm{Min}(L)$ such that $q \leq p$. The *spectrum* of L, denoted ΣL, is the topological space whose underlying set is $\mathrm{Pr}(L)$, and whose topology consists of the sets

$$\Sigma_a = \{p \in \mathrm{Pr}(L) \mid a \nleq p\},$$

for $a \in L$. A frame homomorphism $h\colon L \to M$ gives rise to a continuous function $\Sigma h\colon \Sigma M \to \Sigma L$, which sends any $q \in \mathrm{Pr}(M)$ to $h_*(q)$. We shall write Σ'_a for the set-theoretic complement $\Sigma L \smallsetminus \Sigma_a$.

Any frame, L, is a Heyting algebra, with the Heyting implication explicitly given by

$$a \to b = \bigvee \{x \in L \mid a \wedge x \leq b\}.$$

The element $a \to 0$ is usually denoted by a^*, and called the *pseudocomplement* of a. Coming from ℓ-groups, Jorge Martínez calls a^* the *polar* of a, and denotes it by $a^\perp$. Since we use the asterisk for the right adjoint of a homomorphism, we shall adopt this notation, but not the terminology. An element $a \in L$ is *complemented* if $a \vee a^\perp = 1$, and *dense* if $a^\perp = 0$.

2.2. Algebraic frames

An element $a \in L$ is *compact* if, for any $S \subseteq L$, $a \leq \bigvee S$ implies that there is a finite $T \subseteq S$ with $a \leq \bigvee T$. We denote by $\mathfrak{k}(L)$ the set of all compact elements of L. If every element of L is the join of compact elements below it, then L is said to be *algebraic*. If $a \wedge b \in \mathfrak{k}(L)$ for every $a, b \in \mathfrak{k}(L)$, then L is said to have the *finite intersection property* on compact elements, throughout abbreviated as FIP. A compact algebraic frame with FIP is called *coherent*. A frame homomorphism between algebraic frames is called a *coherent map* if it maps compact elements to compact elements. The usage of the same adjective "coherent" for frames and homomorphisms is purely historical (see [10, Notes to Chapter II]).

If L is an algebraic frame, then, for any $a \in L$, $\uparrow a$ is an algebraic frame, and $\kappa_a \colon L \to \uparrow a$ is a coherent map. If, furthermore, L has FIP, then the same holds for $\uparrow a$. In general, if $h \colon L \to M$ is a surjective coherent map, then $\mathfrak{k}(M) = \{h(c) \mid c \in \mathfrak{k}(L)\}$.

In [15], Martínez and Zenk study inductive nuclei on algebraic frames with FIP. Let us briefly recall some special features. To start, a nucleus $\gamma \colon L \to L$ is called *inductive* if, for any $x \in L$,

$$\gamma(x) = \bigvee \{\gamma(c) \mid c \in \mathfrak{k}(L),\, c \leq x\}.$$

The *inductivization* of γ is the nucleus $\widehat{\gamma} \colon L \to L$ defined by

$$\widehat{\gamma}(x) = \bigvee \{\gamma(c) \mid c \in \mathfrak{k}(L),\, c \leq x\}.$$

It is an inductive nucleus, and $\mathrm{Fix}(\gamma) \subseteq \mathrm{Fix}(\widehat{\gamma})$. Furthermore, $\mathrm{Fix}(\widehat{\gamma})$ is an algebraic frame with FIP, and $\mathfrak{k}(\mathrm{Fix}(\widehat{\gamma})) = \{\gamma(c) \mid c \in \mathfrak{k}(L)\}$. We shall freely use the machinery in [15].

2.3. Rings

Throughout, the term "ring" means a commutative ring with identity $1 \neq 0$. The *radical* of an ideal I of a ring A is the ideal

$$\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

If $I = \sqrt{I}$, then I is called a *radical ideal*. The lattice of radical ideals of A is denoted by $\mathrm{RId}(A)$. It is a coherent frame [3], whose compact elements are precisely the radicals of finitely generated ideals. If A is *reduced*, which is to say it has no nonzero nilpotent elements, then the bottom of $\mathrm{RId}(A)$ is the zero ideal.

We denote the annihilator of an ideal I by $\mathrm{Ann}(I)$, and the annihilator of an element a by $\mathrm{Ann}(a)$. If A is reduced, then the pseudocomplement of any $I \in \mathrm{RId}(A)$ is $\mathrm{Ann}(I)$. Every ring homomorphism $\phi \colon A \to B$ induces a coherent map $\mathrm{RId}(\phi) \colon \mathrm{RId}(A) \to \mathrm{RId}(B)$ that sends a radical ideal I of A to the radical ideal $\sqrt{\langle \phi[I] \rangle}$ of B.

3. The Going-Down property for frames

We shall adopt and adapt nomenclature from ring theory regarding the concept we wish to study. As mentioned in the introduction, Martínez [14] has already defined what it means to say a frame homomorphism has the going-up property. In this section we define the going-down property, and characterize in several ways frame homomorphisms that have this property.

Definition 3.1. Let $h\colon L \to M$ be a frame homomorphism, and let $p \in \mathrm{Pr}(L)$. We say h *goes down* to p if whenever $p \leq h_*(q)$ for some $q \in \mathrm{Pr}(M)$, then there is an $r \in \mathrm{Pr}(M)$ such that $r \leq q$ and $p = h_*(r)$. If h goes down to every prime of L, we say h satisfies the *going-down* (abbreviated GD) property. When we say a localic map satisfies the GD property, we mean that its left adjoint does.

This definition extends "conservatively" its ring-theoretic namesake, in the following sense. A ring homomorphism $\phi\colon A \to B$ satisfies the GD property if and only if the induced frame homomorphism $\mathrm{RId}(\phi)\colon \mathrm{RId}(A) \to \mathrm{RId}(B)$ satisfies the GD property. This is so because $(\mathrm{RId}(\phi))_*(J) = \phi^{-1}[J]$, for every radical ideal J of B, and, for any ring R, the primes of the frame $\mathrm{RId}(R)$ are precisely the prime ideals of R.

Before we proceed to characterizations, let us give two examples of such homomorphisms, neither of which is induced by a ring homomorphism. Recall from [17, Lemma III 10.1.1] that, in any frame L, for any $x \in L$ and $p \in \mathrm{Pr}(L)$,

$$x \to p = \begin{cases} 1 & \text{if} \quad x \leq p \\ p & \text{if} \quad x \nleq p. \end{cases}$$

Recall also from [17, III 10.2] that, for any $a \in L$, $\mathfrak{b}(a)$ denotes the sublocale

$$\mathfrak{b}(a) = \{x \to a \mid x \in L\},$$

and that a sublocale S of L is prime in the frame $\mathcal{S}(L)^{\mathrm{op}}$ if and only if it is of the form $S = \mathfrak{b}(p)$, for some $p \in \mathrm{Pr}(L)$, in which case $\mathfrak{b}(p) = \{p, 1\}$. The open sublocale associated with a is denoted by $\mathfrak{o}(a)$, and the closed one by $\mathfrak{c}(a)$.

Example 3.2. For any frame L and any $a \in L$, the frame homomorphism $\varphi\colon L \to \mathfrak{o}(a)$, given by $\varphi(x) = a \to x$, satisfies the GD property. Indeed, suppose for some $p \in \mathrm{Pr}(L)$ there exists $q \in \mathrm{Pr}(\mathfrak{o}(a))$ such that $p \leq \varphi_*(q)$. Since the right adjoint of φ is the inclusion $\mathfrak{o}(a) \hookrightarrow L$, this means $p \leq q$. Since $a \to q = q$, we have that $a \nleq q$, which, in turn, implies $a \nleq p$, so that $a \to p = p$. Thus, $p \in \mathfrak{o}(a)$, and, in fact, $p \in \mathrm{Pr}(\mathfrak{o}(a))$. Since $p \leq q$ and $p = \varphi_*(p)$, it follows that φ satisfies the GD property.

Example 3.3. For any frame L, the frame homomorphism $\gamma\colon L \to \mathcal{S}(L)^{\mathrm{op}}$, given by $\gamma(x) = \mathfrak{c}(x)$, satisfies the GD property if and only if $\mathrm{Pr}(L)$ is an antichain. For the proof, assume first that γ satisfies the GD property. Suppose, by way of contradiction, that there exists a chain $p < q$ in $\mathrm{Pr}(L)$. The right adjoint of γ sends a sublocale S to $\bigwedge S$. Thus, $\mathfrak{b}(q)$ is a prime element of $\mathcal{S}(L)^{\mathrm{op}}$ such that $p \leq q = \bigwedge \mathfrak{b}(q) = \gamma_*(\mathfrak{b}(q))$. By the GD property, there exists $r \in \mathrm{Pr}(L)$ such that $\mathfrak{b}(r) \leq \mathfrak{b}(q)$, and $p = \gamma_*(\mathfrak{b}(r)) = r$. Since we are in $\mathcal{S}(L)^{\mathrm{op}}$, this means $\mathfrak{b}(q) \subseteq \mathfrak{b}(p)$, so that $q \in \mathfrak{b}(p) = \{p, 1\}$. This is a contradiction since q is neither p nor 1. The other implication is verified along similar lines.

Remark 3.4. Let us reiterate that in these examples the frames are not assumed to be algebraic. We shall see further down that, in the case of algebraic frames with FIP, the first example is actually a special case of a more general result about coherent open localic injections satisfying the GD property.

We shall now give characterizations of when a homomorphism goes down to a prime element. Since our main interest is in algebraic frames, we shall restrict to coherent maps. We start by recording the frame version of Krull's result that if an ideal of a ring misses some multiplicative set, then it can be expanded to a prime ideal missing the set. We should point out that Martínez's [13, Lemma 2.3] is pretty much the result we shall state, except that his "multiplicative set" is somewhat restricted. Although a proof is indicated in [13], and is, in fact, merely an adaptation of the ring-theoretic one, we write it out in detail for the sake of completeness.

Lemma 3.5. *Let L be an algebraic frame with* FIP*, F be a nonempty subset of $\mathfrak{k}(L)$ such that $c \wedge d \in F$ for any $c, d \in F$, and $a \in L$ be above no element of F. Then there exists $p \in L$ such that $a \le p$, p is above no element of F, and p is maximal with this latter property. Furthermore, p is a prime element in L.*

Proof. Define the set $S \subseteq L$ by

$$S = \{x \in L \mid a \le x \text{ and } x \text{ is above no element of } F\}.$$

Then $S \neq \emptyset$ as $a \in S$. Let $T \subseteq S$ be a chain, and put $t_0 = \bigvee T$. Then $a \le t_0$, and if we suppose $c \le t_0$ for some $c \in F$, then the compactness of c yields a $t \in T$ (since T is a chain) such that $c \le t$, which is false. Therefore S has a maximal element, p, say. To show that p is prime, suppose, by way of contradiction, that there exist $u, v \in L$ such that $u \wedge v \le p$, but $u \nleq p$ and $u \nleq p$. Then $p < p \vee u$, and so $p \vee u \notin S$. Since $a \le p \vee u$, this implies that there exists $c \in F$ such that $c \le p \vee u$. Similarly, there exists $d \in F$ such that $d \le p \vee v$. Thus,

$$c \wedge d \le (p \vee u) \wedge (p \vee v) = p \vee (u \wedge v) = p;$$

contrary to the fact that p is above no element of F. □

The characterizations that follow parallel the ring-theoretic ones recorded in [8, 2.2 and 2.4]. From here to the end of this section, whenever we speak of a coherent map, the domain and codomain will be assumed to be algebraic frames with FIP.

Theorem 3.6. *Let $h\colon L \to M$ be a coherent map and $p \in \mathrm{Pr}(L)$. The following statements are equivalent.*

(1) *h goes down to p.*
(2) *For any $q \in \mathrm{Pr}(M)$ that is minimal over $h(p)$, $h_*(q) = p$.*
(3) *For any $q \in \mathrm{Pr}(M)$ that is minimal over $h(p)$, if $h(c) \wedge d \le h(p)$ for some $c \in \mathfrak{k}(L)$ and $d \in \mathfrak{k}(M)$, then $c \le p$ or $d \le q$.*

Proof. (1) ⇔ (2): Assume that h goes down to p, and let $q \in \mathrm{Pr}(M)$ be minimal over $h(p)$. Then $h(p) \le q$, so that $p \le h_*(q)$. Since h goes down to p, there is an $r \in \mathrm{Pr}(M)$ such that $r \le q$ and $p = h_*(r)$. Then $h(p) = hh_*(r) \le r \le q$, which implies $r = q$ since q is minimal over $h(p)$. Thus, $h_*(q) = p$.

Conversely, assume the condition stated in (2) holds. To show that h goes down to p, consider any $r \in \mathrm{Pr}(M)$ with $p \le h_*(r)$. Then $h(p) \le r$. Select $q \in \mathrm{Pr}(M)$ such that $h(p) \le q \le r$, and q is minimal over $h(p)$. By the present hypothesis, $p = h_*(q)$, which shows that h goes down to p.

(2) ⇔ (3): Assume that (2) holds, and let $q \in \mathrm{Pr}(M)$ be minimal over $h(p)$. Consider any $c \in \mathfrak{k}(L)$ and $d \in \mathfrak{k}(M)$ with $h(c) \wedge d \le h(p)$. Then $h(c) \wedge d \le q$, which implies $h(c) \le q$ or $d \le q$ since q is prime. Since $p = h_*(q)$, by (2), this implies $c \le h_*(q) = p$ or $d \le q$, which shows that (2) implies (3).

Conversely, assume that (3) holds, and let $q \in \mathrm{Pr}(M)$ be minimal over $h(p)$. Define the set $F \subseteq \mathfrak{k}(M)$ by

$$F = \{h(c) \wedge d \mid c \in \mathfrak{k}(L),\, c \nleq p \;\text{ and }\; d \in \mathfrak{k}(M),\, d \nleq q\}.$$

This set is not empty since $p < 1$, $q < 1$, and the frames under discussion are algebraic. An easy calculation shows that F is closed under binary meets. Next, let $S \subseteq M$ be defined by

$$S = \{y \in M \mid y \text{ is above no element of } F\}.$$

By (3), $h(p)$ is not above any element of F, so, by Lemma 3.5, there is an $r \in \mathrm{Pr}(M)$ such that $h(p) \le r$ and r is above no element of F. We claim that $r \le q$. Indeed, let d be a compact element in M with $d \le r$. We cannot have $d \nleq q$, because if that were the case, then for any $c \in \mathrm{Pr}(L)$ with $c \nleq p$ (and there are such compact elements in L), we would have $h(c) \wedge d \le d \le r$, contradicting the fact that r is above no element of F. So the minimality of q over $h(p)$ implies $r = q$. We now show that $p = h_*(q)$. Since $h(p) \leq q$, we have $p \leq h_*(q)$. Consider any $c \in \mathfrak{k}(L)$ with $c \leq h_*(q)$. We cannot have $c \nleq p$, otherwise for any $d \in \mathfrak{k}(M)$ with $d \nleq q$ we would have

$$h(c) \wedge d \leq h(c) \leq hh_*(q) \leq q = r,$$

contrary to the fact that r is above no element of F. Since $h_*(q)$ is the join of the compact elements below it, it follows that $h_*(q) \leq p$, and hence $p = h_*(q)$. This proves that (3) implies (2). □

We shall now give a topological characterization that generalizes [8, 2.5]. Let us lay the foundation first. Recall that a topological space is *irreducible* if each of its nonempty open subsets is dense. A subspace is irreducible if it is irreducible as a topological space with the subspace topology. An *irreducible component* of a space is a maximal irreducible subspace. Irreducible components are closed sets.

Let L be a spatial frame. We wish to identify the irreducible components of ΣL. Recall our notation that, for $a \in L$,

$$\Sigma'_a = \{p \in \mathrm{Pr}(L) \mid a \leq p\},$$

and that these are precisely the closed subsets of ΣL. Observe that, for any $a, b \in L$, spatiality yields the following:

$$a \leq b \quad \Longleftrightarrow \quad \Sigma_a \subseteq \Sigma_b.$$

Lemma 3.7. *For a spatial frame L and $a \in L$, the subspace Σ'_a is irreducible iff $a \in \mathrm{Pr}(L)$.*

Proof. The open subsets of Σ'_a are the sets $\Sigma'_a \cap \Sigma_x$, for $x \in L$. Observe that

$$\Sigma'_a \cap \Sigma_x = \emptyset \quad \text{iff} \quad \Sigma_x \subseteq \Sigma_a \quad \text{iff} \quad x \leq a.$$

Now, for any $x, y \in L$, $(\Sigma'_a \cap \Sigma_x) \cap (\Sigma'_a \cap \Sigma_y) = \Sigma'_a \cap \Sigma_{x \wedge y}$. Consequently, if $a \in \mathrm{Pr}(L)$ and $(\Sigma'_a \cap \Sigma_x) \cap (\Sigma'_a \cap \Sigma_y) = \emptyset$, then $x \wedge y \leq a$, so that $x \leq a$ or $y \leq a$, whence $\Sigma'_a \cap \Sigma_x = \emptyset$ or $\Sigma'_a \cap \Sigma_y = \emptyset$, which shows that Σ'_a is irreducible. The converse is shown similarly. □

We deduce from this lemma that, for any $a \in L$ (with L spatial), the irreducible components of Σ'_a are precisely the closed sets Σ'_p, for p minimal prime over a. Call such a p the *generic prime* of Σ'_p. This terminology is standard.

Theorem 3.8. *Let $h\colon L \to M$ be a coherent map and $p \in \mathrm{Pr}(L)$. The following statements are equivalent.*

(1) *h goes down to p.*
(2) *$(\Sigma h)^{-1}[\Sigma'_p] = \emptyset$ or $(\Sigma h)(q) = p$, for every generic point q of every irreducible component of $(\Sigma h)^{-1}[\Sigma'_p]$.*

Proof. (1) ⇒ (2): Suppose $(\Sigma h)^{-1}[\Sigma'_p] \neq \emptyset$, and observe that

$$(\Sigma h)^{-1}[\Sigma'_p] = \{q \in \mathrm{Pr}(M) \mid (\Sigma h)(q) \in \Sigma'_p\} = \{q \in \mathrm{Pr}(M) \mid p \leq h_*(q)\} = \Sigma'_{h(p)}$$

since $h(p) \leq q$ if and only if $p \leq h_*(q)$. Now let $q \in \mathrm{Pr}(M)$ be the generic prime of some irreducible component of $(\Sigma h)^{-1}[\Sigma'_p]$. This means that q is minimal prime over $h(p)$. By Theorem 3.6, $h_*(q) = p$, which proves that (1) implies (2) since $(\Sigma h)(q) = h_*(q)$.

(2) ⇒ (1): If $(\Sigma h)^{-1}[\Sigma'_p] = \emptyset$, then $h(p) = 1$, and there is no $q \in \mathrm{Pr}(M)$ with $p \leq h_*(q)$, so h goes down to p vacuously. Suppose then that $(\Sigma h)^{-1}[\Sigma'_p] \neq \emptyset$, and let q be minimal over $h(p)$. By what we have observed above, and the fact that the irreducible components of $\Sigma'_{h(p)}$ are precisely the closed sets Σ'_r, for $r \in \mathrm{Pr}(M)$ minimal over $h(p)$, the hypothesis in (2) says $h_*(q) = p$. As in the first implication, we deduce from Theorem 3.6 that h goes down to p. □

We shall now characterize going down coherent maps in terms of "localizations". Let us first construct the frame analogue of the ring notion of localization at a prime ideal. For a different perspective, see [16]. Let L be an algebraic frame with FIP. Fix a prime $p \in \mathrm{Pr}(L)$, and define the set

$$S(p) = \{q \in \mathrm{Pr}(L) \mid q \leq p\}.$$

Let $j_p \colon L \to L$ be the nucleus defined by

$$j_p(x) = \bigwedge\{q \in S(p) \mid x \leq q\}.$$

Observe that, for any $x \in L$,

$$j_p(x) = 1 \quad \Longleftrightarrow \quad x \nleq q, \quad \text{for every } q \in S(p).$$

Let L_p denote $\mathrm{Fix}(\widehat{j_p})$.

Lemma 3.9. *For any algebraic frame L with* FIP *and $p \in \mathrm{Pr}(L)$, we have the following:*

(a) $\mathrm{Pr}(L_p) = S(p)$;
(b) *p is the unique maximal element of L_p; and*
(c) $L_p = \mathrm{Fix}(j_p)$.

Proof. (a) Observe that if $q \in \mathrm{Pr}(L)$ and $q \leq p$, then $j_p(q) = q$, which then yields $S(p) \subseteq \mathrm{Pr}(L_p)$. On the other hand, let $q \in \mathrm{Pr}(L_p)$, so that, among other things, $q = j_p(q)$. Consider any $c \in \mathfrak{k}(L)$ with $c \leq q$. Then $j_p(c) \leq q$, whence we deduce that $c \leq p$, otherwise, there is no prime r of L such that $c \leq r \leq p$, which would imply $j_p(c) = 1$, leading to $q = 1$, which is false. Since q is the join of compact elements of L below it, it follows that $q \leq p$. Since L_p is a sublocale of L, its prime elements are exactly those prime elements of L that belong to L_p. Consequently, $\mathrm{Pr}(L_p) \subseteq S(p)$, and we have the claimed equality.

(b) This follows from (a) together with the fact that, in any frame, maximal elements are prime.

(c) Since $\mathrm{Fix}(j_p)$ and L_p are both *spatial* sublocales of L, this follows from the easy observation that $\mathrm{Pr}(\mathrm{Fix}(j_p)) = S(p) = \mathrm{Pr}(L_p)$. □

Since, for any nucleus $\gamma \colon L \to L$ on an algebraic frame L with FIP, $\gamma(c) = \widehat{\gamma}(c)$ for every $c \in \mathfrak{k}(L)$, we have the following corollary, in light of the fact that $L_p = \mathrm{Fix}(j_p)$.

Corollary 3.10. *The homomorphism $L \to L_p$ induced by $\widehat{j_p}$ maps precisely as j_p.*

In light of this, we shall denote by j_p the homomorphism $L \to L_p$ induced by $\widehat{j_p}$. There will be no danger of confusion.

Remark 3.11. The discussion leading up to the frame L_p is modelled on Martínez's work in [13]. The difference is that he starts with what he calls a unit system, which, in ring terms, extends to algebraic frames localization at multiplicative sets consisting of non-divisors of zero.

The characterization of going down coherent maps we are aiming for will be a corollary of the following result, which is of interest in its own right. In the proof we use [11, Lemma 3.3], which, somewhat paraphrased, says if L_1 and L_2 are algebraic frames with FIP, any lattice homomorphism $\phi \colon \mathfrak{k}(L_1) \to \mathfrak{k}(L_2)$ has a unique extension to a coherent map $\bar{\phi} \colon L_1 \to L_2$.

Theorem 3.12. *Let $h\colon L \to M$ be a coherent map. For any $q \in \mathrm{Pr}(M)$, there is a coherent map $h_q\colon L_{h_*(q)} \to M_q$ making the diagram*

$$\begin{array}{ccc} L & \xrightarrow{\ h\ } & M \\ {\scriptstyle j_{h_*(q)}}\downarrow & & \downarrow{\scriptstyle j_q} \\ L_{h_*(q)} & \xrightarrow{\ h_q\ } & M_q \end{array}$$

commute.

Proof. For brevity, let us put $p = h_*(q)$. Recall that $\mathfrak{k}(L_p) = \{j_p(c) \mid c \in \mathfrak{k}(L)\}$. To avoid confusion, we shall write $\sqcup$ for binary joins in L_p and M_q; so that, for instance, $j_p(c) \sqcup j_p(d) = j_p(c \vee d)$, for any $c, d \in L$. Define

$$\phi\colon \mathfrak{k}(L_p) \to \mathfrak{k}(M_q) \qquad \text{by} \qquad \phi(j_p(c)) = j_q(h(c)).$$

Let us verify that ϕ is well defined. Suppose $j_p(c) = j_p(d)$, for some $c, d \in \mathfrak{k}(L)$. Consider any $t \in \mathrm{Pr}(M_q)$ with $j_q(h(c)) \leq t$. Then $h(c) \leq t \leq q$, which implies $c \leq h_*(t) \leq p$. Therefore $h_*(t)$ is a prime of L_p above $j_p(c)$, and therefore above $j_p(d)$. Thus, $d \leq j_p(d) \leq h_*(t)$, which implies $h(d) \leq hh_*(t) \leq t$, and hence $j_q(h(d)) \leq j_q(t) = t$, since t is fixed by j_q. Consequently, by a symmetrical argument, $j_q(h(c))$ and $j_q(h(d))$ are below exactly the same primes of M_q, which, by the spatiality of M_q, implies $j_p(c) = j_p(d)$. Therefore ϕ is well defined.

Let us check that ϕ preserves binary joins. Let $c, d \in \mathfrak{k}(L)$. Then

$$\begin{aligned} \phi(j_p(c) \sqcup j_p(d)) = \phi(j_p(c \vee d)) &= j_q(h(c \vee d)) \\ &= j_q(h(c) \vee h(d)) \\ &= j_q(h(c)) \sqcup j_q(h(d)) = \phi(c) \sqcup \phi(d), \end{aligned}$$

showing that ϕ preserves binary joins. Preservation of binary meets is seen similarly. Now let $h_q\colon L_{h_*(q)} \to M_q$ be the unique frame homomorphism extending ϕ. It is clearly coherent, by construction. To see that it makes the diagram above commute, we need only check that the composites $j_q \cdot h$ and $h_q \cdot j_{h_*(q)}$ agree on compact elements of L. That is straightforward. ☐

Now here is a characterization of the GD property in terms of frame-theoretic localization.

Corollary 3.13. *The following are equivalent for a coherent map $h\colon L \to M$.*

(1) *h satisfies* GD.
(2) *For any $q \in \mathrm{Pr}(M)$, the continuous map $\Sigma(h_q)\colon \Sigma(M_q) \to \Sigma(L_{h_*(q)})$ is surjective.*
(3) *For any $q \in \mathrm{Pr}(M)$, the homomorphism $h_q\colon L_{h_*(q)} \to M_q$ satisfies* GD.

Proof. (1) $\Rightarrow$ (2): Let $r \in \Sigma(L_{h_*(q)})$. This means that $r \in \mathrm{Pr}(L)$ and $r \leq h_*(q)$. Since h goes down to r, there exists $t \in \mathrm{Pr}(M)$ such that $t \leq q$ and $r = h_*(t)$. Since the diagram in Theorem 3.12 commutes, $h_q \cdot j_{h_*(q)} = j_q \cdot h$, which, on taking right adjoints, yields $(j_{h_*(q)})_* \cdot (h_q)_* = h_* \cdot (j_q)_*$. Now, calculating the image of t under these maps, and keeping in mind that the right adjoint of each $j_{(-)}$ is the inclusion map, we get $(h_q)_*(t) = h_*(t) = r$. Thus, t is an element of the space $\Sigma(M_q)$ mapped to r by $\Sigma(h_q)$; which proves the surjectivity of this function.

(2) $\Rightarrow$ (1): Let $p \in \mathrm{Pr}(L)$, and suppose $p \leq h_*(q)$ for some $q \in \mathrm{Pr}(M)$. Thus, $p \in \Sigma(L_{h_*(q)})$. Since $\Sigma(h_q)\colon \Sigma(M_q) \to \Sigma(L_{h_*(q)})$ is surjective, by the present hypothesis, there exists $r \in \Sigma(M_q)$ such that $\Sigma(h_q)(r) = p$. This says $r \in \mathrm{Pr}(M)$, $r \leq q$ and $(h_q)_*(r) = p$. As argued above, the equality $(h_q)_*(r) = p$ implies $h_*(r) = p$. Therefore h goes down to p, and hence h satisfies GD.

(1) ⇒ (3): Assume that (1) holds, and let $q \in \mathrm{Pr}(M)$. To show that $h_q\colon L_{h_*(q)} \to M_q$ satisfies GD, suppose that $r \leq (h_q)_*(s)$ for some $r \in \mathrm{Pr}(L_{h_*(q)})$ and $s \in \mathrm{Pr}(M_q)$. Now, as observed above, $(h_q)_*(s) = h_*(s)$. So, since h goes down to r, there exists $t \in \mathrm{Pr}(M)$ such that $t \leq s$ and $r = h_*(t)$. Then t is a prime in M_q with $t \leq s$ and $r = (h_q)_*(t)$, which shows that h_q satisfies GD.

(3) ⇒ (1): Assume (3), and suppose that $p \leq h_*(q)$ for some $p \in \mathrm{Pr}(L)$ and $q \in \mathrm{Pr}(M)$. By the present hypothesis, $h_q\colon L_{h_*(q)} \to M_q$ satisfies GD. Now, p and q are, respectively, primes in $L_{h_*(q)}$ and M_q with $p \leq (h_q)_*(q)$. So there exists $r \in \mathrm{Pr}(M_q)$ such that $r \leq q$ and $p = (h_q)_*(r) = h_*(r)$. Therefore h satisfies GD. □

One may wonder if, in light of the equivalence (2) ⇔ (3) in this corollary, the surjectivity of $\Sigma h\colon \Sigma M \to \Sigma L$ is sufficient for h to satisfy GD. The following simple example shows that even if Σh is bijective, it does not follow that h satisfies GD.

Example 3.14. Let $\mathbf{3} = \{0, \ell, 1\}$ be the three-element chain and $\mathbf{4} = \{0, a, a', 1\}$ be the four-element Boolean algebra. Let $h\colon \mathbf{3} \to \mathbf{4}$ be the embedding sending ℓ to a. Now, $\mathrm{Pr}(\mathbf{3}) = \{0, \ell\}$ and $\mathrm{Pr}(\mathbf{4}) = \{a, a'\}$. We see from this that h does not go down to 0, even though $\Sigma h\colon \Sigma\mathbf{4} \to \Sigma\mathbf{3}$ is bijective since $h_*(a) = \ell$ and $h_*(a') = 0$.

In rings, it is known that the extension $A \subseteq B$ satisfies GD if the induced continuous function $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open map. We have a similar situation for algebraic frames, which we shall prove shortly.

Recall that a localic map $f\colon M \to L$ is called *open* if for any open sublocale U of M, the sublocale $f[U]$ of L is open. This is equivalent to saying the left adjoint of f preserves all meets and the Heyting implication. Traditionally, there is only one notion of openness associated with a ring homomorphism, and it is arrived at by going to the spectra. To wit, for a ring homomorphism $\phi\colon A \to B$, openness refers to openness of the induced continuous function $\phi^*\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. On the other hand, if $h\colon L \to M$ is a frame homomorphism, we can speak of openness of either the localic map $h_*\colon M \to L$ or the continuous function $\Sigma h\colon \Sigma M \to \Sigma L$. We shall show that the latter implies the former, and that it also implies h satisfies GD. Furthermore, we shall give a condition that makes openness of Σh and openness of h_* equivalent. Here is an example motivating the condition that we are alluding to.

Example 3.15. For any completely regular Hausdorff space X, let $C^*(X)$ denote the subring of $C(X)$ consisting of bounded functions. If I is an ideal of $C(X)$ such that the ideal $C^*(X) \cap I$ is prime in $C^*(X)$, then I is prime. Indeed, suppose $uv \in I$ for some $u, v \in C(X)$. Then $\frac{u}{1+|u|} \cdot \frac{u}{1+|u|} \in C^*(X) \cap I$, from which we may assume that $\frac{u}{1+|u|} \in C^*(X) \cap I$. Thus, $u \in I$, showing that I is prime. Therefore, the coherent map $\mathrm{RId}(C^*(X)) \to \mathrm{RId}(C(X))$, induced by the ring embedding $C^*(X) \hookrightarrow C(X)$, has the property that its right adjoint maps only primes to primes. Without going into details, we remark that this actually holds in any f-ring with *bounded inversion.* The latter means an f-ring in which every $a \geq 1$ is invertible.

Recall that every localic map sends primes to primes. Of course a localic map may map non-primes to primes. We shall be interested in those that map only primes to primes.

Definition 3.16. A localic map is *primal* if it maps only primes to primes. We extend the terminology to frame homomorphisms, and say a frame homomorphism is primal in case its right adjoint is primal.

Observation 3.17. Every surjective frame homomorphism is primal. Note though that the embedding of the two-element Boolean algebra into the four-element Boolean algebra is a non-surjective primal homomorphism.

In the proof that follows, we will use the (easy to verify) fact that, for any frame L and $a \in L$, $\Sigma_a = \mathfrak{o}(a) \cap \mathrm{Pr}(L)$.

Theorem 3.18. *Consider the conditions below regarding a coherent map $h\colon L \to M$.*

(1) *The continuous function $\Sigma h\colon \Sigma M \to \Sigma L$ is open.*
(2) *The localic map $h_*\colon M \to L$ is open.*
(3) *h satisfies* GD*.*

We have the following implications:

(a) *Condition* (1) *implies* (2) *and* (3)*.*
(b) *Conditions* (2) *and* (3) *together imply* (1)*.*
(c) *If h is primal, then* (1) *is equivalent to* (2)*.*

Proof. (a) Assuming that Σh is open, we show that h_* is open. Given $b \in M$, we must produce an $a \in L$ such that $h_*[\mathfrak{o}(b)] = \mathfrak{o}(a)$. Now, since Σh is open, there exists $a \in L$ such that $(\Sigma h)[\Sigma_b] = \Sigma_a$. Since complemented sublocales of spatial locales are spatial [17, Proposition VI 3.3], for any $y \in \mathfrak{o}(b)$, there are primes $\{q_\alpha\}$ in M such that each $q_\alpha \in \mathfrak{o}(b)$ and $y = \bigwedge q_\alpha$. Thus,

$$h_*(y) = h_*\Big(\bigwedge q_\alpha\Big) = \bigwedge h_*(q_\alpha) \in \mathfrak{o}(a);$$

that last part valid because each $h_*(q_\alpha) \in \Sigma_a \subseteq \mathfrak{o}(a)$, and $\mathfrak{o}(a)$ is closed under meets in L. This shows that $h_*[\mathfrak{o}(b)] \subseteq \mathfrak{o}(a)$. For the opposite inclusion, let $x \in \mathfrak{o}(a)$. Find primes $\{p_\alpha\}$ of L, each belonging to $\mathfrak{o}(a)$, such that $x = \bigwedge p_\alpha$. Since $(\Sigma h)[\Sigma_b] = \Sigma_a$, for each α there is a $q_\alpha \in \mathfrak{o}(b)$ such that $p_\alpha = h_*(q_\alpha)$. Therefore

$$x = \bigwedge h_*(q_\alpha) = h_*\Big(\bigwedge q_\alpha\Big) \in h_*[\mathfrak{o}(b)],$$

which yields the desired inclusion. Consequently, $h_*[\mathfrak{o}(b)] = \mathfrak{o}(a)$, and thus h_* is an open localic map.

Next, assuming that Σh is open, we show that h satisfies GD. If not, then, by the equivalence (1) $\Leftrightarrow$ (3) in Theorem 3.6, there exist $p \in \mathrm{Pr}(L)$, $q \in \mathrm{Pr}(M)$ minimal over $h(p)$, $c \in \mathfrak{k}(L)$, and $d \in \mathfrak{k}(M)$ such that $c \nleq p$, $d \nleq q$, but $h(c) \wedge d \leq h(p)$. Now observe that q belongs to the open set Σ_d. Since Σh is an open map, $(\Sigma h)[\Sigma_d]$ is an open set in ΣL. Therefore there exists $a \in L$ such that

$$(\Sigma h)[\Sigma_d] = \{h_*(r) \mid d \nleq r\} = \Sigma_a.$$

We claim that $p \in \Sigma_a$. If p were not in this set, we would have $a \leq p$, which would imply $h(a) \leq h(p) \leq q$, so that $a \leq h_*(q)$, which is false because the fact that $q \in \Sigma_d$ implies $h_*(q) = (\Sigma h)(q) \in \Sigma_a$, that is, $a \nleq h_*(q)$. So, $p \in \Sigma_a$, which then means $p = h_*(r)$ for some $r \in \mathrm{Pr}(M)$ with $d \nleq r$. This, in turn, implies $h(p) \leq r$. Since $h(c) \wedge d \leq h(p) \leq r$, we must have $h(c) \leq r$, since r is prime. Thus, $c \leq h_*(r) = p$, which is a contradiction because we took c such that $c \nleq p$.

(b) Assume that (2) and (3) hold. Let $b \in M$. Using condition (2), pick $a \in L$ such that $h_*[\mathfrak{o}(b)] = \mathfrak{o}(a)$. We will show that $(\Sigma h)[\Sigma_b] = \Sigma_a$. If $q \in \Sigma_b$, then $q \in \mathfrak{o}(b)$, and so

$$(\Sigma h)(q) = h_*(q) \in \mathfrak{o}(a) \cap \mathrm{Pr}(L) = \Sigma_a,$$

which shows that $(\Sigma h)[\Sigma_b] \subseteq \Sigma_a$. On the other hand, let $p \in \Sigma_a$. Since $\Sigma_a \subseteq \mathfrak{o}(a)$, there exists $y \in \mathfrak{o}(b)$ such that $p = h_*(y)$. Since $y < 1$ (else, $p = 1$) and $y = b \to y$, we have $b \nleq y$. By spatiality, there are primes $\{q_\alpha\} \subseteq \mathrm{Pr}(M)$ such that $y = \bigwedge q_\alpha$. Consequently, b cannot be below all the primes q_α. There is therefore a prime $q \in \mathrm{Pr}(M)$ such that $y \leq q$ and $b \nleq q$, that is, $y \leq q$ and $q \in \mathfrak{o}(b)$. Thus, $p = h_*(y) \leq h_*(q)$. Since (3) implies h goes down to p, there is an $r \in \mathrm{Pr}(M)$ with $r \leq q$ and $p = h_*(r)$. Since $b \nleq q$, we have $b \nleq r$, and so $r \in \mathfrak{o}(b)$. In all then,

$$p = (\Sigma h)(r) \in (\Sigma h)[\mathfrak{o}(b) \cap \mathrm{Pr}(M)] = (\Sigma h)[\Sigma_b],$$

which shows that $\Sigma_a \subseteq (\Sigma h)[\Sigma_b]$, and hence equality. Therefore Σh is an open map.

(c) The proof is similar to that of (b), but quicker since from $p = h_*(y)$ we deduce that $y \in \Sigma_b$ if h is primal. □

Remark 3.19. Another condition that ensures that an open coherent map satisfies GD is on the domain of the map. Recall that an element $p \in L$ is called a *covered prime* [5] in case, for any $S \subseteq L$, $p = \bigwedge S$ implies $p = s$ for some $s \in S$. Now if all the primes of L are covered primes and $h_*\colon M \to L$ is open, then $\Sigma h\colon \Sigma M \to \Sigma L$ is open, as can be deduced by an argument similar to that of the (b) part of the preceding theorem. Hence h satisfies GD.

Here is an example of a homomorphism which satisfies GD but is not open.

Example 3.20. Let $L = \{0, \ell, 1\}$, and let $\kappa\colon L \to {\uparrow}\ell$ be the mapping $x \mapsto \ell \vee x$. Evidently, κ satisfies GD. However, since ℓ is not complemented in L, κ is not open. Indeed, if it were, then, with pseudocomplement in ${\uparrow}\ell$ denoted by $(\ )^*$, we would have

$$\ell \vee \ell^\perp = \kappa(\ell^\perp) = \kappa(\ell \to 0) = \kappa(\ell) \to \kappa(0) = 0^*_{\uparrow \ell} = 1,$$

which is a contradiction.

4. Dimension and the GD-property

The study of the Krull-style dimension for distributive lattices is not a new industry. In [6], Coquand and Lombardi approach it from a constructive point of view, and in [12], Martínez takes up the subject for algebraic frames, with a definition lifted straight from ring theory.

To recall, let L be an algebraic frame. The *length* of a chain $p_0 < p_1 < \cdots < p_n$ of primes of L is the integer n. The *dimension* of L, denoted $\dim(L)$, is the supremum of the lengths of chains of primes of L, if such exists. Thus, for instance, $\dim(L) = 0$ if and only if every prime in L is minimal prime, and $\dim(L) = 1$ if and only if there is a non-minimal prime in L and there is no chain $p_0 < p_1 < p_2$ in $\mathrm{Pr}(L)$.

In [7], rings of dimension 0 and rings of dimension at most 1 are characterized in terms of the GD-property. We seek analogous characterizations for algebraic frames. In the process, we shall use frame-theoretic notions that are not analogues of any ring-theoretic notion used in [7]. For one such, recall that a frame homomorphism $h\colon L \to M$ is called *nearly open* [4] in case $h(a^\perp) = h(a)^\perp$ for very $a \in L$.

A number of the results below are characterized by the property that the mapping $\kappa_p\colon L \to {\uparrow}p$ satisfies GD for some various types of primes p. To facilitate the proofs, we record the following easy (but useful) characterization of when $\kappa_a\colon L \to {\uparrow}a$ satisfies GD.

Lemma 4.1. *For any frame L and $a \in L$, the following are equivalent.*

(1) *κ_a satisfies* GD.
(2) *For any $q \in \mathrm{Pr}(L)$, if $q \le p$ for some prime $p \ge a$ of L, then $q \ge a$.*
(3) *For any $q \in \mathrm{Pr}(L)$, if $q < p$ for some prime $p \ge a$ of L, then $q \ge a$.*

Proof. The proof follows easily from the fact that the right adjoint of $\kappa_a\colon L \to {\uparrow}a$ is the inclusion ${\uparrow}a \hookrightarrow L$, and that $\mathrm{Pr}({\uparrow}a) = \{p \in \mathrm{Pr}(L) \mid p \ge a\}$. □

The result that follows generalizes [7, Proposition 2.1]. In the previous section we agreed that whenever we mentioned a coherent map, its domain and codomain were to be assumed to be algebraic frames with FIP. In this section we do not assume the algebraic frames to have FIP, unless explicitly stated.

Proposition 4.2. *The following are equivalent for an algebraic frame L.*

(1) $\dim(L) = 0$.
(2) *Every coherent map $h\colon L \to M$ satisfies* GD.

(3) *For every* $p \in \mathrm{Pr}(L)$, $\kappa_p\colon L \to {\uparrow}p$ *satisfies* GD.
(4) *For every non-minimal* $p \in \mathrm{Pr}(L)$, κ_p *satisfies* GD.
(5) *For every non-minimal* $p \in \mathrm{Pr}(L)$, κ_p *is nearly open.*
(6) ΣL *is a* T_1*-space.*

Proof. (1) $\Rightarrow$ (2): Let $h\colon L \to M$ be a coherent map. Suppose $p \leq h_*(q)$ for some $p \in \mathrm{Pr}(L)$ and $q \in \mathrm{Pr}(M)$. By (1), this implies $p = h_*(q)$, which then shows that h satisfies GD.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): These implications are trivial.

(4) $\Rightarrow$ (1): If there were a chain $q < p$ in $\mathrm{Pr}(L)$, then p would be a non-minimal prime in L, and hence, by (4), κ_p would satisfy GD. Now, since $q < p$ and p is a prime of L above p, Lemma 4.1 would imply that $q \geq p$, which is false. Therefore there is no such chain $q < p$ in $\mathrm{Pr}(L)$, which says $\dim(L) = 0$.

(1) $\Rightarrow$ (5): This holds vacuously because if $\dim(L) = 0$, then there are no non-minimal primes in L.

(5) $\Rightarrow$ (4): Let $p \in \mathrm{Pr}(L)$ be non-minimal. We apply Lemma 4.1 to prove that κ_p satisfies GD. So suppose $q \in \mathrm{Pr}(L)$ and $q \leq r$ for some prime r of L with $r \geq p$. We must show that $q \geq p$. We claim that p is complemented. To validate this claim, recall that, in any frame, $x \to y$ (the arrow signifying the Heyting implication) is the pseudocomplement of $x \vee y$ in the frame ${\uparrow}y$. Now, by (5), κ_p is nearly open, which implies $\kappa_p(p^\perp) = \kappa_p(p)^\perp$, that is, $p \vee p^\perp = p \to p = 1$. Since $p \wedge p^\perp = 0 \leq q$, and q is prime, we have $p \leq q$ or $p^\perp \leq q$. The latter cannot be true, lest we have $r \geq p \vee p^\perp = 1$. So $p \leq q$, as desired. Therefore κ_p satisfies GD.

(1) $\Leftrightarrow$ (6): Recall that, for any $p \in \Sigma L$, $\mathrm{cl}\{p\} = \{q \in \mathrm{Pr}(L) \mid p \leq q\}$. □

In [15, Theorem 2.4] it is shown that an algebraic frame L is regular if and only if it has FIP and $\mathrm{Pr}(L)$ is trivially ordered; that is, if and only if it has FIP and $\dim(L) = 0$. We therefore have the following corollary.

Corollary 4.3. *An algebraic frame* L *is regular iff it has* FIP *and* κ_p *satisfies* GD *for every prime* $p \in L$.

When dealing with algebraic frames, certain results that are defined (or characterized) in terms of arbitrary elements, can equally well be defined (or characterized) in terms of compact elements. An example is the property of being prime. As is well known, in an algebraic frame an element p is prime precisely when $c \wedge d \leq p$, with c and d compact, implies $c \leq p$ or $d \leq p$.

One of the necessary and sufficient conditions that $\dim(L)$ be equal to 0 is that every coherent map out of L be nearly open. Now, near openness requires the homomorphism to preserve pseudocomplements of all elements. One may wonder if requiring this only for compact elements is not already sufficient for the dimension of L to be zero. We shall see below that it actually is. An upshot of the result we will prove will be a characterization of von Neumann regular rings that is somewhat akin to [7, Corollary 2.2], but does not mention flatness of ring homomorphisms.

Let us formally introduce the following weaker version of near openness that concentrates only on compact elements.

Definition 4.4. A coherent map $h\colon L \to M$ is *slightly open* if $h(c^\perp) = h(c)^\perp$ for every $c \in \mathfrak{k}(L)$.

Since the inequality $h(c^\perp) \leq h(c)^\perp$ always holds, the thrust of this definition is that $h(c)^\perp \leq h(c^\perp)$ for every compact element c. In the event of the homomorphisms $\kappa_p\colon L \to {\uparrow}p$, for $p \in \mathrm{Pr}(L)$, we have the following rephrasing of slight openness. Recall that to say x is *rather below* y means that $x^\perp \vee y = 1$. We write $x \prec y$ to signify that x is rather below y.

Lemma 4.5. *For any $p \in \mathrm{Pr}(L)$, a necessary and sufficient condition that $\kappa_p \colon L \to {\uparrow}p$ be slightly open is that whenever a compact element of L is below p, it will be rather below p.*

Proof. Assume that κ_p is slightly open, and let c be a compact element of L with $c \le p$. The slight openness of κ_p says $\kappa_p(c)^{\perp} \le \kappa_p(c^{\perp})$, that is, $c \to p \le p \vee c^{\perp}$, which implies $1 \le c^{\perp} \vee p$, which says $c \prec p$.

Conversely, assume the condition holds, and let $c \in \mathfrak{k}(L)$. If $c \not\le p$, then $c^{\perp} \le p$, and so

$$\kappa_p(c^{\perp}) = p \vee c^{\perp} = p = c \to p = \kappa_p(c)^{\perp}.$$

On the other hand, if $c \le p$, then $c \prec p$, by hypothesis. Thus

$$\kappa_p(c)^{\perp} = c \to p = 1 = p \vee c^{\perp} = \kappa_p(c^{\perp}).$$

Therefore in either case we have $\kappa_p(c^{\perp}) = \kappa(c)^{\perp}$; so κ_p is slightly open. □

Here is the characterization of zero-dimensional algebraic frames in terms of slight openness. Recall from [11] that if L is an algebraic frame and $p \in \mathrm{Min}(L)$, then for any $c \in \mathfrak{k}(L)$, we cannot have both $c \le p$ and $c^{\perp} \le p$.

Proposition 4.6. *For any algebraic frame L, $\dim(L) = 0$ iff $\kappa_p \colon L \to {\uparrow}p$ is slightly open, for every $p \in \mathrm{Pr}(L)$.*

Proof. Assume that $\dim(L) = 0$. Let $p \in \mathrm{Pr}(L)$, and consider any $c \in \mathfrak{k}(L)$ with $c \le p$. By Lemma 4.5, it suffices to show that $p \vee c^{\perp} = 1$. Since $\dim(L) = 0$, p is minimal prime, and so $c^{\perp} \not\le p$, and hence $p < p \vee c^{\perp}$. Now, if $p \vee c^{\perp}$ were strictly below the top, then there would be a prime q such that $p \vee c^{\perp} \le q$, by spatiality, which would imply $p < q$, contradicting the fact that $\dim(L) = 0$. Therefore $p \vee c^{\perp} = 1$, showing that κ_p is slightly open.

Conversely, let $q \le p$ be a chain in $\mathrm{Pr}(L)$. Consider any compact $c \le p$. Then $c^{\perp} \vee p = 1$, by the present hypothesis. Now we cannot have $c^{\perp} \le q$, for that would imply $c^{\perp} \le p$, and hence $p = 1$; so then $c \le q$ since q is prime. Since p is the join of compact elements below it, we have $p \le q$, which then implies $q = p$, whence we deduce that $\dim(L) = 0$. □

This result has the following ring-theoretic application.

Corollary 4.7. *A ring A is von Neumann regular iff for any $P \in \mathrm{Spec}(A)$, $\mathrm{Ann}(I) + P = A$, for every finitely generated ideal I of A contained in P.*

Proof. Assume first that A is von Neumann regular. Then A is reduced, and hence in the frame $\mathrm{RId}(A)$, the pseudocomplement of any $J \in \mathrm{RId}(A)$ is $\mathrm{Ann}(J)$. Let $P \in \mathrm{Spec}(A)$, and let $I \subseteq P$ be a finitely generated ideal of A. Then $\sqrt{I}$ is a compact element in $\mathrm{RId}(A)$, still with $\sqrt{I} \subseteq P$. Since A is von Neumann regular, $\dim(A) = 0$, and hence $\dim(\mathrm{RId}(A)) = 0$ because, as sets, $\mathrm{Pr}(\mathrm{RId}(A)) = \mathrm{Spec}(A)$. Thus, Lemma 4.5 and Proposition 4.6 tell us that $\sqrt{I} \prec P$ in $\mathrm{RId}(A)$, that is, $\sqrt{I}^{\perp} \vee P = 1_{\mathrm{RId}(A)}$. Since A is reduced, $\mathrm{Ann}\left(\sqrt{I}\right) = \mathrm{Ann}(I)$, and so we have

$$A = \sqrt{I}^{\perp} \vee P = \sqrt{\mathrm{Ann}\left(\sqrt{I}\right) + P} = \sqrt{\mathrm{Ann}(I) + P},$$

whence we deduce that $\mathrm{Ann}(I) + P = A$.

Conversely, suppose the stated condition holds. Let us show first that A is reduced. Consider any $a \in A$ with $a^2 = 0$. We must show that $a = 0$. If not, then $\mathrm{Ann}(a)$ is a proper ideal of A, and is therefore contained in some prime ideal, P, say. Now $a^2 = 0$ implies $a \in P$, by primeness of P. Thus, $\langle a \rangle$ is a finitely generated ideal of A contained in P. So, by hypothesis, $\mathrm{Ann}(a) + P = A$, which is not possible because $\mathrm{Ann}(a) \subseteq P$ and P is a proper ideal. Therefore $a = 0$, showing that A is reduced. Now we pass to the frame $\mathrm{RId}(A)$. The condition, in frame terms, says whenever a compact element of $\mathrm{RId}(A)$ is below some prime element, then it is rather below that prime element. So, by Proposition 4.6, $\dim(\mathrm{RId}(A)) = 0$, which implies $\dim(A) = 0$, and hence A is von Neumann regular because it is reduced. □

We now move to dimension at most 1, and obtain results that generalize [7, Proposition 2.4]. Recall from [1] that a topological space X is said to be a *T_{YS}-space* in case for any $x \neq y$ in X, $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$ or $\{x\}$ or $\{y\}$.

Proposition 4.8. *The following are equivalent for an algebraic frame L.*

(1) *For every non-maximal $p \in \mathrm{Pr}(L)$, the homomorphism κ_p satisfies* GD.
(2) $\dim(L) \leq 1$ *and each prime of L is above exactly one minimal prime.*
(3) ΣL *is a T_{YS}-space.*

Proof. (1) ⇔ (2): Assume that (1) holds. If there were a chain $p_1 < p < p_2$ in $\mathrm{Pr}(L)$, then p would be a non-maximal element in $\mathrm{Pr}(L)$, and so, by hypothesis, κ_p would satisfy GD. Now since $p_1 \leq p_2$, and p_2 is a prime of L above p, Lemma 4.1 would imply that $p_1 \geq p$, which is false. Therefore $\dim(L) \leq 1$. Next, let q be a non-minimal prime of L, and suppose q_1 and q_2 are minimal primes each below q. Now, q_1 is a non-maximal element in $\mathrm{Pr}(L)$, and so, by hypothesis, $\kappa_{q_1} : L \to {\uparrow} q_1$ satisfies GD. Since q_2 is a prime of L with $q_2 \leq p$, and p is a prime of L above q_1, it follows from Lemma 4.1 that $q_2 \geq q_1$, and hence $q_1 = q_2$ since q_2 is a minimal prime. Therefore q is above exactly one minimal prime. So, (1) implies (2).

Conversely, assume (2), and let p be a non-maximal element in $\mathrm{Pr}(L)$. Then there is a $w \in \mathrm{Pr}(L)$ such that $p < w$. To prove (using Lemma 4.1) that κ_p satisfies GD, consider any $q \in \mathrm{Pr}(L)$ with $q < r$, for some prime r of L with $p \leq r$. We must show that $q \geq p$. Now, we cannot have $p = r$, for that would yield the chain $q < r < w$, which is not possible since $\dim(L) \leq 1$. Therefore $p < r$. The strict inequalities $q < r$ and $p < r$ imply that p and q are minimal prime because there are no chains of length 2 in $\mathrm{Pr}(L)$. Since r is above both p and q, we must have $p = q$ by part of the hypothesis that every prime is above exactly one minimal prime. Therefore κ_p satisfies GD.

(2) ⇔ (3): Assume that (2) holds. Let $p \neq q$ in ΣL. If $r \in \overline{\{p\}} \cap \overline{\{q\}}$, then r is a prime of L such that $p \leq r$ and $q \leq r$. We cannot have $p < r$ and $q < r$, since the fact that $\dim(L) \leq 1$ would force p and q to be distinct minimal primes each below r, in violation of the hypothesis. So, any prime in $\overline{\{p\}} \cap \overline{\{q\}}$ is either p or q, which says $\overline{\{p\}} \cap \overline{\{q\}} = \{p\}$ or $\overline{\{p\}} \cap \overline{\{q\}} = \{q\}$. Therefore ΣL is a T_{YS}-space.

Conversely, assume that ΣL is a T_{YS}-space. We show first that every prime is above exactly one minimal prime. Let $p \in \mathrm{Pr}(L)$. If p is minimal prime, there is nothing to prove. So suppose that p is not minimal prime. Let q_1 and q_2 be minimal primes with $q_1 \leq p$ and $q_2 \leq p$. Since $p \in \overline{\{q_1\}} \cap \overline{\{q_2\}}$, and p is not equal to q_1 or q_2, we must have $q_1 = q_2$, otherwise the assumption that ΣL is a T_{YS}-space would be contradicted. That $\dim(L) \leq 1$ follows from the fact that a chain of the form $q < r < w$ yields $\overline{\{q\}} \cap \overline{\{r\}} \supseteq \{r, w\}$, which is proscribed by ΣL being a T_{YS}-space. Therefore (3) implies (2). □

We end this section with results that mirror the ones above, but characterized in terms of the going-up property of some appropriately chosen coherent map. In [14], Martínez says a frame homomorphism $h \colon L \to M$ satisfies the *going-up* property (abbreviated GU) if whenever $p \geq h_*(q)$ for some $p \in \mathrm{Pr}(L)$ and $q \in \mathrm{Pr}(M)$, there exists $r \in \mathrm{Pr}(M)$ such that $r \geq q$ and $p = h_*(r)$.

For a prime p in L, recall from Section 3 the homomorphism $j_p \colon L \to L_p$, associated with "localization". As in the case of κ_p, we have the following easy characterization of when j_p satisfies GU.

Lemma 4.9. *For any algebraic frame L and $p \in \mathrm{Pr}(L)$, the following are equivalent.*

(1) j_p *satisfies* GU.
(2) *For any $q \in \mathrm{Pr}(L)$, if $q \geq r$ for some prime $r \leq p$ of L, then $q \leq p$.*

(3) *For any* $q \in \mathrm{Pr}(L)$*, if* $q > r$ *for some prime* $r \leq p$ *of* L*, then* $q \leq p$*.*

Obvious modifications of the same argument as in 4.2 establish the following.

Proposition 4.10. *The following are equivalent for an algebraic frame* L*.*

(1) $\dim(L) = 0$.
(2) *Every frame homomorphism* $h\colon L \to M$ *satisfies* GU*.*
(3) $j_p\colon L \to L_p$ *satisfies* GU*, for every* $p \in \mathrm{Pr}(L)$*.*
(4) $j_p\colon L \to L_p$ *satisfies* GU*, for every non-maximal* $p \in \mathrm{Pr}(L)$*.*

If we think of the GU property as the up-side-down version of the GD property, then the result that follows is the up-side-down version of Proposition 4.8.

Proposition 4.11. *The following are equivalent for an algebraic frame* L *with* FIP*.*

(1) *For every non-minimal* $p \in \mathrm{Pr}(L)$*, the homomorphism* $j_p\colon L \to L_p$ *satisfies* GU*.*
(2) $\dim(L) \leq 1$ *and every prime in* L *is below at most one maximal element.*

Proof. Assume first that (1) holds. Of course every maximal element is below a unique maximal element. Now suppose $p \in \mathrm{Pr}(L)$ with $p < m$ and $p < n$, for some maximal elements m and n of L. Then m is a non-minimal prime element, and so, by (1), $j_m\colon L \to L_m$ satisfies GU. Since $n > p$ and $p \leq m$, we have $n \leq m$, by Lemma 4.9, and hence $n = m$, by maximality. Therefore above any prime element of L there is at most one maximal element. Next, if there were a chain $r < q < p$ in $\mathrm{Pr}(L)$, then q would be a non-minimal prime for which $j_q\colon L \to L_q$ does not satisfy GU. It follows therefore that $\dim(L) \leq 1$.

Conversely, assume that $\dim(L) \leq 1$ and every prime in L is below at most one maximal element. Let $p \in \mathrm{Pr}(L)$ be non-minimal. Since $\dim(L) \leq 1$, it follows that p is maximal. Consider any $q \in \mathrm{Pr}(L)$ such that $q > r$ for some $r \in \mathrm{Pr}(L)$ with $r \leq p$. Now, $\dim(L) \leq 1$ also ensures from the inequality $r < q$ that q is maximal. So p and q are both maximal elements both above r. Part of the hypothesis on L implies $q = p$. We deduce therefore from Lemma 4.9 that j_p satisfies GU. □

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