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Research Article

A note on commuting graphs for general linear groups

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Abstract

Let G be a group and X a subset of G. Then $\mathcal{C}(G,X)$ is a graph with vertex set X in which two distinct elements x, $y \in X$ are joined by an edge if $xy = yx$. In this paper, we study the clique number, the domination number, the diameter, the planarity, the perfection and regularity of $\mathcal{C}(G,X)$ where $G = GL(n,q)$ and X is the set of transvections.

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1. Introduction and preliminaries

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph Γ, we denote the sets of the vertices and edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. A graph Γ is regular if all the vertices of Γ have the same degree. A subset X of $V(\Gamma)$ is called a clique if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and is denoted by $\omega(\Gamma)$. A subset X of $V(\Gamma)$ is called an independent set if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the independence number of Γ and is denoted by $\alpha(\Gamma)$. A k-vertex colouring of a graph Γ is an assignment of k colours to the vertices of Γ such that no two adjacent vertices have the same colour. The vertex chromatic number $\chi(\Gamma)$ of a graph Γ, is the minimum k for which Γ has a k-vertex colouring. For a graph Γ and a subset S of the vertex set $V(\Gamma)$, denote by $N_\Gamma[S]$ the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$, then S is said to be a dominating set of vertices in Γ. The domination number of a graph Γ, denoted by $\gamma(\Gamma)$, is the minimum size of a dominating set of vertices in Γ. The length of the shortest cycle in a graph Γ is called the girth of Γ and denoted by girth(Γ). If v and w are vertices in Γ, then $d(v,w)$ denotes the length of the shortest path between v and w. The largest distance between all pairs of the vertices of Γ is called the diameter of Γ, and is denoted by diam(Γ). A graph Γ is connected if there is a path between each pair of vertices of Γ. A planar graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which both are incident. A graph Γ is called perfect if for every induced subgraph H of Γ, $\omega(H) = \chi(H)$, and Γ is Berge if

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no induced subgraph of Γ is an odd cycle of length at least five or the complement of one. The following theorems and definitions will be used repeatedly.

Theorem 1.1. [2, Theorem 1.2] *A graph is perfect if and only if it is Berge.*

Definition 1.2. [4, Definition and Theorem 8.5] If V is an n-dimensional vector space over a field F, then the general linear group $GL(V)$ is the group of all nonsingular linear transformations on V with respect to the composition of mappings.
Choosing an ordered basis of V gives an isomorphism $GL(V) \longrightarrow GL(n, F)$, where $GL(n, F)$ is the group of all invertible $n \times n$ matrices over F. If F is finite, with q elements, this group is denoted by $GL(n, q)$. Also

$$|GL(n, q)| = (q^n - 1)(q^n - q) \ldots (q^n - q^{n-1}).$$

The determinant function $det : GL(n, F) \longrightarrow F^\star$ is a homomorphism which maps the identity matrix to 1, and it is multiplicative, as desired. The special linear group, $SL(n, F)$, is the kernel of this homomorphism. The center of $GL(V)$ is $Z(GL(V)) = \{\lambda I : \lambda \in F^\star\}$ and the center of $SL(n, F)$ is $Z(SL(n, F)) = \{\lambda I : \lambda \in F^\star, \lambda^n = 1\}$. Define the projective general linear group and the projective special linear group on V to be

$$PGL(V) = \frac{GL(V)}{Z(GL(V))} \quad , \quad PSL(V) = \frac{SL(V)}{Z(SL(V))}.$$

A hyperplane W in V is a subspace of dimension $n - 1$. Let W be a hyperplane of V. If $I \neq T \in GL(V)$ satisfies:

$$T(w) = w \quad \forall w \in W, \; T(v) - v \in W \quad \forall \, v \in V$$

then T is called a transvection with respect to W and W is called the axis of the transvection T. For each transvection T, $det(T) = 1$. So $T \in SL(V)$. The inverse of a transvection is a transvection. The set of transvections generates $SL(V)$. Given a nonzero linear functional f on V and a nonzero vector $a \in \ker f$, define $T_{a,f} : V \longrightarrow V$ by $T_{a,f} : v \longmapsto v - f(v)a$. It is clear that $T_{a,f}$ is a transvection. Moreover, for every transvection T there exist $f \neq 0$ and $a \neq 0$ with $T = T_{a,f}$.

Theorem 1.3. *Let V be an n-dimensional vector space over a field F. Then intersection of null space of k independent linear functionals is an $(n - k)$-dimensional subspace of V.*

Proposition 1.4. [4, Corollary 8.18 and Theorem 8.21] *All transvections in $GL(n, F)$ are conjugate. If $n \geq 3$, then they are conjugate in $SL(n, F)$.*

Lemma 1.5. [4, Lemma 8.19] *Let V be a vector space over F. $T_{a,f} = T_{b,g}$ if and only if there is a scalar $\alpha \in F^\star$ with $g = \alpha f$ and $a = \alpha b$.*

For a hyperplane W in a vector space V, we set $\tau(W) = \{$ all transvections fixing W $\} \cup \{1_V\}$.

Lemma 1.6. [4, Lemma 8.22] *Let W be a hyperplane in an n-dimensional vector space V over F. $\tau(W)$ is an abelian subgroup of $SL(V)$, and $\tau(W) \cong W$.*

Definition 1.7. [1] A graph Γ is vertex-transitive if the automorphism group of Γ acts transitively on the vertex set of Γ.

Theorem 1.8. [1, Theorem 7.1] *Let Γ be a k-regular, connected, vertex-transitive graph of order n. Then*

(1) *If n is even, then Γ has a 1-factor.*
(2) *The product of the clique number and the independence number of Γ is at most n.*

2. Main result

The purpose of this note is to study certain properties of the commuting graph $\mathcal{C}(G, X) = \Gamma$ where $G = GL(n, q)$ and X is the set of transvections in G. Throughout this paper V is a vector space with $\dim(V) = n$ on a finite field F with $|F| = q$.

Lemma 2.1. *For a proper subspace U of V, set $S_U = \{T_{a,f} | a \in U,\ U \subseteq \ker f\}$. Then $|S_U| = \frac{(q^i-1)(q^{n-i}-1)}{q-1}$, where $i = \dim(U)$.*

Proof. We have $\{f : V \to F | U \subseteq \ker f\} \cong \{\bar{f} : \frac{V}{U} \to F\}$. So there are $q^{n-i} - 1$ candidates for f, that is the number of nonzero linear functionals from $\frac{V}{U}$ into F, and $q^i - 1$ candidates for a (the zero vector is not a candidate). By Lemma 1.5, $|S_W| = \frac{(q^i-1)(q^{n-i}-1)}{q-1}$. □

Lemma 2.2. $|V(\Gamma)| = \frac{(q^n-1)(q^{n-1}-1)}{q-1}$.

Proof. We consider a fixed hyperplane W. It is sufficient to calculate $|\{T_{a,f} | a \in W\ ,\ W = \ker f\}|$. Since the number of hyperplanes in V is equal to $\frac{q^n-1}{q-1}$, we have $|V(\Gamma)| = \frac{(q^n-1)(q^{n-1}-1)}{q-1}$ by Lemma 2.1. □

Lemma 2.3. [3, Lemma 1, part (iv)] *Let $T_{a,f}$ and $T_{b,g}$ be two transvections on V with fixed hyperplanes W_1 and W_2, respectively. Then $[T_{a,f}, T_{b,g}] = 1$ if and only if $a \in \ker g$ and $b \in \ker f$.*

Lemma 2.4. *Γ is k-regular with $k = \frac{(q-1)(q^{n-1}-1)+q(q^{n-2}-1)^2}{q-1} - 1$.*

Proof. By proposition 1.4, Γ is a k-regular graph. Let $T_{a,f}$ be a transvection. It is sufficient to calculate $|\{(b, g) | b \in \ker f\ ,\ a \in \ker g\}|$. We have

$$\begin{aligned} |\{(b, g) | b \in \langle a \rangle\ ,\ a \in \ker g\}| &= (|\langle a \rangle| - 1)\ \left(\left|\left(\frac{V}{\langle a \rangle}\right)^*\right| - 1\right) \\ &= (q - 1)(q^{n-1} - 1) \end{aligned}$$

and

$$\begin{aligned} |\{(b, g) | b \in \ker f\ ,\ a \in \ker g\ , \langle a \rangle \neq \langle b \rangle\}| &= |\ker f - \langle a \rangle|\ \left(\left|\left(\frac{V}{\langle a, b \rangle}\right)^*\right| - 1\right) \\ &= (q^{n-1} - q)(q^{n-2} - 1), \end{aligned}$$

where $(\frac{V}{\langle a \rangle})^*$ is the vector space of all linear functionals from $\frac{V}{\langle a \rangle}$ to F and $(\frac{V}{\langle a,b \rangle})^*$ is defined similarly. It follows that

$$k = \frac{(q-1)(q^{n-1}-1) + q(q^{n-2}-1)^2}{q-1} - 1.$$

□

Theorem 2.5. (1) *For* $\dim(V) = 2$, Γ *is disconnected.*
(2) *For* $\dim(V) = 3$, $\operatorname{diam}(\Gamma) = 3$.
(3) *For* $\dim(V) > 3$, $\operatorname{diam}(\Gamma) = 2$.

Proof. Let $T_{a,f}$ and $T_{b,g}$ be two transvections on V with fixed hyperplanes $W_1 = \ker f$ and $W_2 = \ker g$, respectively. If $\dim(V) = 2$ then $\dim(\ker f) = 1$. Since $a \in \ker f$, $b \in \ker f$ if and only if $\langle b \rangle = \langle a \rangle$. It follows that Γ is disconnected.
Suppose that $\dim(V) = 3$. If $W_1 = W_2$ then $d(T_{a,f}, T_{b,g}) = 1$. If $W_1 \neq W_2$ then $W_1 \cap W_2 \neq 0$ and there exists a nonzero element $u \in W_1 \cap W_2$. Since $\dim(\langle u, b \rangle) \leq 2$, there is $\gamma : V \longrightarrow F$ such that $\langle u, b \rangle \subseteq \ker \gamma$. It follows that $[T_{b,g}, T_{u,\gamma}] = 1$. Also there exists $\delta : V \longrightarrow F$ such that $\langle u, a \rangle \subseteq \ker \delta$. Hence $[T_{a,f}, T_{u,\delta}] = 1$. Also $[T_{u,\gamma}, T_{u,\delta}] = 1$, thus $d(T_{a,f}, T_{b,g}) \leq 3$ and $\operatorname{diam}(\Gamma) \leq 3$. Now we show that $\operatorname{diam}(\Gamma) = 3$. There exist

transvections T_{v_1,f_1} and T_{v_2,f_2}, where $\ker f_1 = W_1$ and $\ker f_2 = W_2$, such that $v_1 \notin W_2$ and $v_2 \notin W_1$. If there exists a transvection $T_{u,h}$, where $\ker h = W$, such that $[T_{v_1,f_1}, T_{u,h}] = 1$ and $[T_{v_2,f_2}, T_{u,h}] = 1$ then $v_1, v_2 \in W$. Since $\langle v_1\rangle \neq \langle v_2\rangle$ and $\dim(W) = 2$, we obtain $W = \langle v_1, v_2\rangle$. It follows that $u = \lambda_1 v_1 + \lambda_2 v_2$ and hence $u - \lambda_1 v_1 \in W_2$. This is a contradiction. Hence $\mathrm{diam}(\Gamma) = 3$.
Now assume that $\dim(V) > 3$. If $W_1 = W_2$ then $d(T_{a,f}, T_{b,g}) = 1$. If $W_1 \neq W_2$ then $W_1 \cap W_2 \neq 0$ and there exists a nonzero element $u \in W_1 \cap W_2$. Since $\dim(\langle u, a, b\rangle) \leq 3$, there is $\mu : V \longrightarrow F$ such that $\langle u, a, b\rangle \subseteq \ker \mu$. Now $[T_{a,f}, T_{u,\mu}] = 1$ and $[T_{b,g}, T_{u,\mu}] = 1$. This implies that $d(T_{a,f}, T_{b,g}) \leq 2$. Hence $\mathrm{diam}(\Gamma) \leq 2$. Now we show that $\mathrm{diam}(\Gamma) = 2$. Let W_1, W_2 be two distinct hyperplanes, $v_1 \in W_1 - W_2$ and $v_2 \in W_2 - W_1$. Notice that $[T_{v_1,f_1}, T_{v_2,f_2}] \neq 1$, where $\ker f_1 = W_1$ and $\ker f_2 = W_2$. Thus $\mathrm{diam}(\Gamma) = 2$. □

Theorem 2.6. $\omega(\Gamma) = \frac{(q^k-1)(q^{n-k}-1)}{q-1}$ *where* $k = [\frac{n}{2}]$.

Proof. We have $[T_{a_i,f_i}, T_{a_j,f_j}] = 1$ for all $1 \leq i, j \leq t$ if and only if $a_1, a_2, \ldots, a_t \in \bigcap_{i=1}^{t} \ker f_i$. So $\{T_{a_1,f_1}, T_{a_2,f_2}, \ldots, T_{a_t,f_t}\}$ is a complete subgraph of Γ if and only if there exists a subspace W of V such that $a_1, a_2, \ldots, a_t \in W$ and $W \subseteq \ker f_i$ for all $1 \leq i \leq t$. It is sufficient to calculate $|S_W|$. Let U_1, U_2, …, U_{n-1} be subspaces of V with $\dim(U_i) = i$. By Lemma 2.1, $|S_{U_1}| = |S_{U_{n-1}}| < |S_{U_2}| = |S_{U_{n-2}}| < \cdots$. Therefore $\omega(\Gamma) = |S_{U_{[\frac{n}{2}]}}|$. □

Corollary 2.7. *If* $\dim(V) \geq 3$*, then the girth of* Γ *is equal to* 3.

Theorem 2.8. *For* $\dim(V) > 2$*,* Γ *is not planar.*

Proof. Let W be a hyperplane of V. By Lemma 1.6, $\tau(W) \cong W$ and we have a complete subgraph $K_{|W|-1}$. Hence if $q^{n-1} - 1 \geq 5$ then Γ is not planar. If $q^{n-1} - 1 < 5$ then $q = 2$, $n = 3$ and $V = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $W_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$, $W_2 = \mathbb{Z}_2 \times 0 \times \mathbb{Z}_2$, $W_3 = 0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $W_4 = \langle(1,0,0),(0,1,1)\rangle$, $W_5 = \langle(1,0,1),(0,1,0)\rangle$ be hyperplanes of V. Set $A_i = \{T_{a,f} | a \in \ker f = W_i\}$. For all $u \in W_i \cap W_j$, there exist $T_{u,f_i} \in A_i$, $T_{u,f_j} \in A_j$. Then, with contraction of A_i, where $1 \leq i \leq 5$, we obtain a complete graph K_5. This completes the proof. □

Theorem 2.9. Γ *is perfect if and only if* $\dim(V) = 3$.

Proof. Let $\dim(V) = 3$. Suppose that Γ has an induced cycle of length $m \geq 4$. Also assume that T_{v_1,f_1}, T_{v_2,f_2}, T_{v_3,f_3} are three consecutive vertices of this cycle, where $\ker f_i = W_i$ for all $1 \leq i \leq 3$. We have $v_1 \in W_1 \cap W_2$, $v_2 \in W_1 \cap W_2 \cap W_3$, $v_3 \in W_2 \cap W_3$. If $W_i \neq W_j$ for all $1 \leq i, j \leq 3$, $1 \leq \dim(W_1 \cap W_2 \cap W_3) \leq \dim(W_1 \cap W_2) = 1$, then $W_1 \cap W_2 \cap W_3 = W_1 \cap W_2$. Similarly, $W_1 \cap W_2 \cap W_3 = W_2 \cap W_3$. Hence $W_1 \cap W_2 = W_2 \cap W_3$. Thus $v_1 \in W_3$ and $v_3 \in W_1$, a contradiction. Hence $W_1 = W_2$ or $W_2 = W_3$. Since $m \geq 4$, $W_1 \neq W_3$. Concequently we can assume $W_1 = W_2$, $W_3 = W_4$, $W_5 = W_6$, $\cdots$. If m is odd, then $T_{v_{m-1},f_{m-1}}$, T_{v_m,f_m}, T_{v_1,f_1} are three consecutive vertices and, $W_1 = W_2$, $W_{m-1} = W_{m-2}$. Then $W_m \neq W_1$ and $W_m \neq W_{m-1}$, a contradiction. Thus m is even and Γ has no odd induced cycle of length at least five. It follows that Γ is perfect.
Now let $\dim(V) = 4$ and $V = \langle v_1, v_2, v_3, v_4\rangle$. Assume that $W_1 = \langle v_1, v_2, v_3\rangle$, $W_2 = \langle v_1, v_2, v_4\rangle$ and $\overline{W} = \langle v_1, v_3, v_4\rangle$. Since $|V - (W_1 \cup W_2 \cup \overline{W})| = q^4 - 3q^3 + 3q^2 - q = q(q-1)^3 > 0$, there is $v_5 \in V - (W_1 \cup W_2 \cup \overline{W})$. Set $W_3 = \langle v_2, v_4, v_5\rangle$, $W_4 = \langle v_3, v_4, v_5\rangle$ and $W_5 = \langle v_1, v_3, v_5\rangle$. If $v_4 \in W_5$ then $v_4 = \lambda_1 v_1 + \lambda_3 v_3 + \lambda_5 v_5$. Since v_1, v_3, v_4 are independent, we have $\lambda_5 \neq 0$ and $v_5 \in \overline{W}$, which is a contradiction. Thus $v_4 \notin W_5$. Also $v_4 \notin W_1$. Hence $v_4 \notin W_1 \cap W_5$. Similarly, $v_3 \notin W_3 \cap W_2$, $v_1 \notin W_3 \cap W_4$, $v_2 \notin W_4 \cap W_5$, $v_5 \notin W_1 \cap W_2$. Now T_{v_1,f_1}, T_{v_2,f_2}, T_{v_4,f_3}, T_{v_5,f_4}, T_{v_3,f_5} forms an induced cycle of length 5, where $\ker f_i = W_i$ for all $1 \leq i \leq 5$. Since the complement of any induced cycle of length 5 in Γ is an induced cycle of length 5 in $\overline{\Gamma}$, Γ and $\overline{\Gamma}$ have induced cycles of length 5. Concequently Γ is not perfect.
Now assume that $\dim(V) \geq 5$ and $V = \langle v_1, \ldots, v_5\rangle \oplus W$. Suppose that $W_i =$

$\langle\{v_1, v_2, v_3, v_4, v_5\} - \{v_i\}\rangle \oplus W$ for $1 \leq i \leq 5$. Then T_{v_1,f_5}, T_{v_4,f_3}, T_{v_2,f_1}, T_{v_5,f_4}, T_{v_3,f_2} forms an induced cycle of length 5, where $\ker f_i = W_i$ for all $1 \leq i \leq 5$. Concequently, both Γ and $\overline{\Gamma}$ have induced cycles of length 5 and Γ is not perfect. □

Corollary 2.10. *If* $\dim(V) = 3$, *then* $\chi(\Gamma) = \omega(\Gamma) = q^2 - 1$.

Theorem 2.11. *For* $n \geq 3$, $\gamma(\Gamma) \leq min\{(q+1)^2, \frac{q^n-1}{q-1}\}$ *and for* $q = 2$ *or* $n \geq 5$, $\gamma(\Gamma) \geq q^2$.

Proof. For each hyperplane W, let a_W be a nonzero element of W and f_W be a linear functional with $\ker f_W = W$. Set $S = \{T_{a_W,f_W} |$ W is a hyperplane of $V\}$. Let $T_{b,g} \in V(\Gamma)$ and $\ker g = W_1$. Then $T_{a_{W_1},f_{W_1}} \in S$ and $[T_{b,g}, T_{a_{W_1},f_{W_1}}] = 1$. Hence S is a dominating set for Γ, and so $\gamma(\Gamma) \leq \frac{q^n-1}{q-1}$. Now assume that $\dim(V) \geq 4$. Let W be a subspace of V with $\dim(W) = n-2$. Observe that V has $q+1$ hyperplanes W_1, W_2, …, W_{q+1} containing W. Let $f_1, f_2, \ldots, f_{q+1}$ be linear functionals with $\ker f_j = W_j$ for $j = 1, 2, \ldots, q+1$. Clearly, we have $V = W_1 \cup W_2 \cup \ldots \cup W_{q+1}$. Let U be a subspace of W with $\dim(U) = 2$ and let $\langle a_1\rangle, \langle a_2\rangle, \ldots, \langle a_{q+1}\rangle$ be all distinct one dimensional subspaces of U. We claim now that $S = \{T_{a_i,f_j} | i, j \in \{1, 2, \ldots, q+1\}\}$ is a dominating set for Γ. For an ordinary transvection $T_{b,g}$, there exists W_j such that $b \in W_j$. Since $\dim(U \cap \ker g) = 1$, there exists $a_i \in \ker g$. It then follows that $[T_{a_i,f_j}, T_{b,g}] = 1$, which proves the claim. Thus, we get $\gamma(\Gamma) \leq |S| = (q+1)^2$ as desired. Now suppose $\gamma(\Gamma) = t$. Since Γ is k-regular, we have

$$\begin{aligned} t \geq \frac{|V(\Gamma)|}{k} &= \frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^{n-1}-1) + (q^{n-1}-q)(q^{n-2}-1) - (q-1)} \\ &\geq \frac{q^{2n-1} - q^n - q^{n-1} + 1}{q^{2n-3} + q^n - 3q^{n-1} - q + 2} \\ &> \frac{q^{2n-1} - q^n - q^{n-1}}{q^{2n-3} + q^n - 3q^{n-1}} \\ &> \frac{(q^n - q - 1)}{(q^{n-2} + q - 3)}. \end{aligned}$$

But for $q = 2$ or $n \geq 5$, $\frac{(q^n-q-1)}{(q^{n-2}+q-3)} > q^2 - 1$. Hence $q^2 \leq \gamma(\Gamma)$. This completes the proof. □

Theorem 2.12. *If* q *is odd, then* Γ *has a* 1*-factor and* $\alpha(\Gamma) \leq \frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^{[\frac{n}{2}]}-1)(q^{n-[\frac{n}{2}]}-1)}$.

Proof. It follows from Theorem 1.8. □

Example 2.13. Let $G = GL(3, 2)$. Then the commuting graph $\Gamma = \mathcal{C}(G, X)$ satisfies the following conditions:

(1) $|V(\Gamma)| = 21$.
(2) Γ is 4-regular.
(3) $|E(\Gamma)| = 42$.
(4) $\mathrm{diam}(\Gamma) = 3$.
(5) Γ is perfect.
(6) $\chi(\Gamma) = \omega(\Gamma) = 3$.
(7) $\gamma(\Gamma) = 5$.

Proof. Parts (1) – (6) are clear. For part (7), by Theorem 2.11, we have $4 \leq \gamma(\Gamma) \leq 7$. Since
$4 \times 5 \leq |V(\Gamma)| = 21$, $\gamma(\Gamma) \geq 5$. Observe that $\{T_{v_1,f_1}, T_{v_2,f_2}, T_{v_3,f_3}, T_{v_5,f_5}, T_{v_6,f_6}\}$ is a dominating set, where $v_1 = (1, 0, 0)$, $v_2 = (1, 0, 1)$, $v_3 = (0, 1, 1)$, $v_5 = (1, 1, 1)$, $v_6 = (1, 1, 0)$ and $\ker f_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$, $\ker f_2 = \mathbb{Z}_2 \times 0 \times \mathbb{Z}_2$, $\ker f_3 = 0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\ker f_5 = \langle(1, 0, 1), (0, 1, 0)\rangle$, $\ker f_6 = \langle(1, 1, 0), (0, 0, 1)\rangle$. Thus $\gamma(\Gamma) = 5$. □

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