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Research Article

On involutiveness of linear combinations of a quadratic matrix and an arbitrary matrix

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Abstract

We characterize the involutiveness of the linear combinations of the form $a\mathbf{A}+b\mathbf{B}$ when a, b are nonzero complex numbers, $\mathbf{A}$ is a quadratic $n\times n$ nonzero matrix and $\mathbf{B}$ is an arbitrary $n \times n$ nonzero matrix, under certain properties imposed on $\mathbf{A}$ and $\mathbf{B}$.

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1. Introduction

Let $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^{m\times n}$, and $\mathbb{C}^n$ denote the sets of complex numbers, nonzero complex numbers, all $m\times n$ complex matrices, and all $n\times n$ complex matrices, respectively. $\mathbf{0}$, $\mathbf{0}_n$, and $\mathbf{I}_n$ stand for a zero matrix of appropriate size, the zero matrix of order n, and the identity matrix of order n, respectively. The symbol $\oplus$ will denote the direct sum of matrices. Let $\alpha, \beta \in \mathbb{C}$, a matrix $\mathbf{A}\in\mathbb{C}^n$ is called an idempotent, an involutive, and an $\{\alpha,\beta\}$–quadratic matrix if $\mathbf{A}^2=\mathbf{A}$, $\mathbf{A}^2=\mathbf{I}_n$, and $(\mathbf{A}-\alpha\mathbf{I}_n)(\mathbf{A}-\beta\mathbf{I}_n)=\mathbf{0}$, respectively. It is noteworthy that a $\{0,1\}$–quadratic matrix is idempotent and a $\{-1,1\}$–quadratic matrix is involutive. Moreover, a matrix $\mathbf{A}\in\mathbb{C}^n$ is called a generalized $\{\alpha,\beta\}$–quadratic matrix with respect to an idempotent matrix $\mathbf{P}\in\mathbb{C}^n$ if $(\mathbf{A}-\alpha\mathbf{P})(\mathbf{A}-\beta\mathbf{P})=\mathbf{0}$ and $\mathbf{AP}=\mathbf{PA}=\mathbf{A}$ hold for $\alpha,\beta\in\mathbb{C}$.

In [1,2,4,7,13], it has been characterized the involutiveness of the form $a\mathbf{A}+b\mathbf{B}$ when $a,b\in\mathbb{C}$ and $\mathbf{A},\mathbf{B}$ are special types of matrices. Moreover, there are a lot of studies related to the linear combinations including involutive matrices [4,7,9,14] and quadratic, generalized quadratic matrices [2,3,5,6,8,10,11]. These special types of matrices have applications to digital image encryption (for example, [12]).

Consider a linear combination of the form

$$\mathbf{K}=a\mathbf{A}+b\mathbf{B},\ \ \mathbf{A},\mathbf{B}\in\mathbb{C}^n,\ \ a,b\in\mathbb{C}^*. \tag{1.1}$$

Liu et al. characterized the involutiveness of the linear combinations of the form (1.1) when $\mathbf{A}$ is a quadratic or a tripotent matrix and $\mathbf{B}$ is an arbitrary matrix [2]. Sarduvan and Kalaycı established necessary and sufficient conditions for the idempotency of linear

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combinations of the form (1.1) when $\mathbf{A}$ is a quadratic matrix and $\mathbf{B}$ is an arbitrary matrix [8].

This paper aims to give necessary and sufficient conditions in which a linear combination of the form (1.1) is an involutive matrix when $\mathbf{A}$ is a quadratic matrix and $\mathbf{B}$ is an arbitrary matrix with some certain conditions.

Now we can give the main results.

2. Main results

In this section, we will investigate the involutiveness of the linear combinations of the form (1.1), under some certain conditions.

Theorem 2.1. *Let $a, b, \alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$, and $\alpha \neq \beta$. Moreover, let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}^n \backslash \{\mathbf{0}\}$ be an $\{\alpha, \beta\}$–quadratic matrix and an arbitrary matrix, respectively. Furthermore, let $\mathbf{K}$ be their linear combination of the form $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$. Then $\mathbf{K}$ is an involutive matrix and $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$ if and only if there exists a nonsingular matrix $\mathbf{V} \in \mathbb{C}^n$ such that*

$$\mathbf{A} = \mathbf{V} \begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix} \mathbf{V}^{-1}$$

and $\mathbf{B}$ satisfies one of the following cases.

(a) $\alpha = 1$ *and* $\beta = 0$.

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{1-a}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1},$$

being $\mathbf{Z}_2 \in \mathbb{C}^{r\times(p-q)}$ and $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ arbitrary.

(b) $\alpha = 1$, $a\beta = 1$, *and* $a \neq 1$.

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{\beta-1}{\beta b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\beta-1}{\beta b}\mathbf{I}_{p-q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \mathbf{0}_{n-p} \end{pmatrix} \mathbf{V}^{-1},$$

being $\mathbf{Z}_2 \in \mathbb{C}^{(n-p)\times(p-q)}$ arbitrary.

(c) $\alpha = 1$, $a\beta = -1$, *and* $a \neq -1$.

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{\beta+1}{\beta b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\beta+1}{\beta b}\mathbf{I}_{p-q} & \mathbf{0} \\ \mathbf{Z}_1 & \mathbf{0} & \mathbf{0}_{n-p} \end{pmatrix} \mathbf{V}^{-1},$$

being $\mathbf{Z}_1 \in \mathbb{C}^{(n-p)\times q}$ arbitrary.

(d) $\beta = 0$, $a\alpha = 1$, *and* $a \neq 1$.

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_2 & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1},$$

being $\mathbf{Z}_2 \in \mathbb{C}^{(n-p-r)\times p}$ arbitrary.

(e) $\beta = 0$, $a\alpha = -1$, *and* $a \neq -1$.

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_1 & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1},$$

being $\mathbf{Z}_1 \in \mathbb{C}^{r\times p}$ arbitrary.

(f) *$\beta = 1$, $a\alpha = 1$, and $a \neq 1$.*

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-\alpha-1}{\alpha b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Y}_2 \in \mathbb{C}^{p\times(n-p-r)}$ arbitrary.

(g) *$\beta = 1$, $a\alpha = -1$, and $a \neq -1$.*

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \frac{\alpha+1}{\alpha b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-\alpha+1}{\alpha b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Y}_1 \in \mathbb{C}^{p\times r}$ arbitrary.

Proof. From Theorem 2.1 in [5], we can write a quadratic matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\mathbf{U}^{-1},$$

where $p \in \{0, \ldots, n\}$ and $\mathbf{U} \in \mathbb{C}^n$ is a nonsingular matrix. We can represent $\mathbf{B}$ as $\mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{T} \end{pmatrix}\mathbf{U}^{-1}$, where $\mathbf{X} \in \mathbb{C}^p$. In view of the hypotheses $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$ and $\alpha \neq 0$ we can write

$$\alpha\mathbf{X} = \mathbf{X}, \quad \beta\mathbf{Y} = \mathbf{Y}, \quad \alpha\beta^2\mathbf{Z} = \beta^2\mathbf{Z}, \quad \beta^3\mathbf{T} = \beta^2\mathbf{T}. \tag{2.1}$$

Now let us assume that $\mathbf{K}$ is an involutive matrix then we can write

$$\begin{aligned} &\left(a\alpha\mathbf{I}_p + b\mathbf{X}\right)^2 + b^2\mathbf{Y}\mathbf{Z} = \mathbf{I}_p, \quad ab\left(\alpha+\beta\right)\mathbf{Y} + b^2\left(\mathbf{X}\mathbf{Y} + \mathbf{Y}\mathbf{T}\right) = \mathbf{0}, \\ &ab\left(\alpha+\beta\right)\mathbf{Z} + b^2\left(\mathbf{Z}\mathbf{X} + \mathbf{T}\mathbf{Z}\right) = \mathbf{0}, \quad b^2\mathbf{Z}\mathbf{Y} + \left(a\beta\mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = \mathbf{I}_{n-p}. \end{aligned} \tag{2.2}$$

Depending on the scalar β, we have the following cases.

(i) Let $\beta \neq 1$. From (2.1), it is seen that $\mathbf{Y} = \mathbf{0}$. We can split this case into four cases depending on the values of α and β.

(i-1) Let $\alpha = 1$ and $\beta = 0$. Reorganizing the equations of (2.2), it can be written

$$\left(a\mathbf{I}_p + b\mathbf{X}\right)^2 = \mathbf{I}_p, \quad \left(b\mathbf{T}\right)^2 = \mathbf{I}_{n-p}, \quad ab\mathbf{Z} + b^2\left(\mathbf{Z}\mathbf{X} + \mathbf{T}\mathbf{Z}\right) = \mathbf{0}. \tag{2.3}$$

It is clear that $a\mathbf{I}_p + b\mathbf{X}$ and $b\mathbf{T}$ are involutive matrices from the first and second equations in (2.3), respectively. Since an involutive matrix is a $\{-1, 1\}$–quadratic matrix, there exist $q \in \{0, \ldots, p\}$, $r \in \{0, \ldots, n-p\}$ and nonsingular matrices $\mathbf{S}_1 \in \mathbb{C}^p$, $\mathbf{S}_2 \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{X} = \mathbf{S}_1\left(\frac{1-a}{b}\mathbf{I}_q \oplus \frac{-1-a}{b}\mathbf{I}_{p-q}\right){\mathbf{S}_1}^{-1}, \mathbf{T} = \mathbf{S}_2\left(\frac{1}{b}\mathbf{I}_r \oplus \frac{-1}{b}\mathbf{I}_{n-p-r}\right){\mathbf{S}_2}^{-1}. \tag{2.4}$$

Let us write $\mathbf{Z}$ as

$$\mathbf{Z} = \mathbf{S}_2\begin{pmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{pmatrix}{\mathbf{S}_1}^{-1}, \tag{2.5}$$

where $\mathbf{Z}_1 \in \mathbb{C}^{r\times q}$. Substituting (2.4) and (2.5) into the third equation in (2.3) it is obtained that $2\left(\mathbf{Z}_1 \oplus -\mathbf{Z}_4\right) = \mathbf{0}$. Then $\mathbf{Z}$ reduces to

$$\mathbf{Z} = \mathbf{S}_2\begin{pmatrix} \mathbf{0} & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{0} \end{pmatrix}{\mathbf{S}_1}^{-1}, \tag{2.6}$$

where $\mathbf{Z}_2 \in \mathbb{C}^{r\times(p-q)}$ and $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ are arbitrary matrices.

Let us define $\mathbf{V} := \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right)$. Then we get $\mathbf{A}$ as

$$\begin{aligned} \mathbf{A} &= \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{0}_{n-p}\right)\mathbf{U}^{-1} = \mathbf{V}\left({\mathbf{S}_1}^{-1} \oplus {\mathbf{S}_2}^{-1}\right)\left(\mathbf{I}_p \oplus \mathbf{0}_{n-p}\right)\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right)\mathbf{V}^{-1} \\ &= \mathbf{V}\left(\mathbf{I}_p \oplus \mathbf{0}_{n-p}\right)\mathbf{V}^{-1}. \end{aligned}$$

In view of (2.4) and (2.6), $\mathbf{B}$ is obtained that

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{1-a}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1},$$

which establishes part (a).

(i-2) Let $\alpha = 1$ and $\beta \neq 0$. From (2.1), it is seen that $\mathbf{T} = \mathbf{0}$. Reorganizing the equations of (2.2), it can be written

$$(a\mathbf{I}_p + b\mathbf{X})^2 = \mathbf{I}_p, \ \ (a\beta\mathbf{I}_{n-p})^2 = \mathbf{I}_{n-p}, \ \ ab\,(1+\beta)\,\mathbf{Z} + b^2\mathbf{Z}\mathbf{X} = \mathbf{0}. \tag{2.7}$$

It is clear that $a\mathbf{I}_p + b\mathbf{X}$ is an involutive matrix from the first equation in (2.7), so there exist $q \in \{0, \dots, p\}$ and a nonsingular matrix $\mathbf{S}_3 \in \mathbb{C}^p$ such that

$$\mathbf{X} = \mathbf{S}_3 \left(\frac{1-a}{b}\mathbf{I}_q \oplus \frac{-1-a}{b}\mathbf{I}_{p-q} \right) \mathbf{S}_3{}^{-1}. \tag{2.8}$$

Let us write $\mathbf{Z}$ as

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \end{pmatrix} \mathbf{S}_3{}^{-1}, \tag{2.9}$$

where $\mathbf{Z}_1 \in \mathbb{C}^{(n-p)\times q}$. Substituting (2.8) and (2.9) into the third equation in (2.7) it is obtained that $\begin{pmatrix} (a\beta+1)\,\mathbf{Z}_1 & (a\beta-1)\,\mathbf{Z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix}$. Moreover, it is clear that $a\beta \in \{-1, 1\}$ from the second equation in (2.7). Then $\mathbf{Z}$ reduces to

$$\mathbf{Z} = \begin{pmatrix} \mathbf{0} & \mathbf{Z}_2 \end{pmatrix} \mathbf{S}_3{}^{-1} \tag{2.10}$$

when $a\beta = 1$ or

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 & \mathbf{0} \end{pmatrix} \mathbf{S}_3{}^{-1} \tag{2.11}$$

when $a\beta = -1$.

Let us define $\mathbf{V} := \mathbf{U}\,(\mathbf{S}_3 \oplus \mathbf{I}_{n-p})$. Then we get $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\,(\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p})\,\mathbf{U}^{-1} = \mathbf{V}\left(\mathbf{S}_3{}^{-1} \oplus \mathbf{I}_{n-p}\right)(\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p})\,(\mathbf{S}_3 \oplus \mathbf{I}_{n-p})\,\mathbf{V}^{-1}$$
$$= \mathbf{V}\,(\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p})\,\mathbf{V}^{-1}.$$

In view of (2.8), (2.10) and (2.8), (2.11) we obtain, respectively, that

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{\beta-1}{\beta b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\beta-1}{\beta b}\mathbf{I}_{p-q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \mathbf{0}_{n-p} \end{pmatrix} \mathbf{V}^{-1}$$

and

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{\beta+1}{\beta b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\beta+1}{\beta b}\mathbf{I}_{p-q} & \mathbf{0} \\ \mathbf{Z}_1 & \mathbf{0} & \mathbf{0}_{n-p} \end{pmatrix} \mathbf{V}^{-1},$$

which establish parts (b) and (c).

(i-3) Let $\alpha \neq 1$ and $\beta = 0$. From (2.1), it is seen that $\mathbf{X} = \mathbf{0}$. Reorganizing the equations of (2.2), it can be written

$$(a\alpha\mathbf{I}_p)^2 = \mathbf{I}_p, \ \ (b\mathbf{T})^2 = \mathbf{I}_{n-p}, \ \ ab\alpha\mathbf{Z} + b^2\mathbf{T}\mathbf{Z} = \mathbf{0}. \tag{2.12}$$

It is clear that $b\mathbf{T}$ is an involutive matrix from the second equation in (2.12), so there exist $r \in \{0, \dots, n-p\}$ and a nonsingular matrix $\mathbf{S}_4 \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{T} = \mathbf{S}_4 \left(\frac{1}{b}\mathbf{I}_r \oplus \frac{-1}{b}\mathbf{I}_{n-p-r} \right) \mathbf{S}_4{}^{-1}. \tag{2.13}$$

Let us write $\mathbf{Z}$ as

$$\mathbf{Z} = \mathbf{S}_4 \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix}, \tag{2.14}$$

where $\mathbf{Z}_1 \in \mathbb{C}^{r\times p}$. Substituting (2.13) and (2.14) into the third equation in (2.12) it is obtained that $\begin{pmatrix} (a\alpha+1)\,\mathbf{Z}_1 \\ (a\alpha-1)\,\mathbf{Z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$. Moreover, it is clear that $a\alpha \in \{-1,1\}$ from the first equation in (2.12). Then $\mathbf{Z}$ turns to

$$\mathbf{Z} = \mathbf{S}_4 \begin{pmatrix} \mathbf{0} \\ \mathbf{Z}_2 \end{pmatrix} \tag{2.15}$$

when $a\alpha = 1$ or

$$\mathbf{Z} = \mathbf{S}_4 \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{0} \end{pmatrix} \tag{2.16}$$

when $a\alpha = -1$.

Let us define $\mathbf{V} := \mathbf{U}\,(\mathbf{I}_p \oplus \mathbf{S}_4)$. Then we get $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\,(\alpha\mathbf{I}_p \oplus \mathbf{0}_{n-p})\,\mathbf{U}^{-1} = \mathbf{V}\left(\mathbf{I}_p \oplus {\mathbf{S}_4}^{-1}\right)(\alpha\mathbf{I}_p \oplus \mathbf{0}_{n-p})\,(\mathbf{I}_p \oplus \mathbf{S}_4)\,\mathbf{V}^{-1}$$

$$= \mathbf{V}\,(\alpha\mathbf{I}_p \oplus \mathbf{0}_{n-p})\,\mathbf{V}^{-1}.$$

In view of (2.13), (2.15) and (2.13), (2.16) we obtain, respectively, that

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_2 & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}$$

and

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_1 & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

which establish parts (d) and (e).

(i-4) Let $\alpha \neq 1$ and $\beta \neq 0$. From (2.1), it is seen that $\mathbf{B} = \mathbf{0}$ which contradicts the hypothesis $\mathbf{B} \neq \mathbf{0}$. So, in this case there is no matrix form of $\mathbf{B}$.

(ii) Let $\beta = 1$. From the first and third equations in (2.1), we obtain $\mathbf{X} = \mathbf{0}$ and $\mathbf{Z} = \mathbf{0}$, respectively. Reorganizing the equations of (2.2), it is obtained that

$$(a\alpha)^2\mathbf{I}_p = \mathbf{I}_p,\ \ (a\mathbf{I}_{n-p} + b\mathbf{T})^2 = \mathbf{I}_{n-p},\ \ ab\,(\alpha+1)\,\mathbf{Y} + b^2\mathbf{Y}\mathbf{T} = \mathbf{0}. \tag{2.17}$$

It is obvious that $a\alpha \in \{1,-1\}$ and $a\mathbf{I}_{n-p} + b\mathbf{T}$ is an involutive matrix from the first and second equations in (2.17), respectively. Hence, there exist $r \in \{0,\ldots,n-p\}$ and a nonsingular matrix $\mathbf{S} \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{T} = \mathbf{S}\left(\frac{1-a}{b}\mathbf{I}_r \oplus \frac{-1-a}{b}\mathbf{I}_{n-p-r}\right)\mathbf{S}^{-1}. \tag{2.18}$$

Let us write $\mathbf{Y}$ as

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \end{pmatrix}\mathbf{S}^{-1}, \tag{2.19}$$

where $\mathbf{Y}_1 \in \mathbb{C}^{p\times r}$. Substituting (2.18) and (2.19) into the third equation in (2.17) yields $\begin{pmatrix} b\,(a\alpha+1)\,\mathbf{Y}_1 & b\,(a\alpha-1)\,\mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix}$. Using $a\alpha \in \{1,-1\}$, $\mathbf{Y}$ obtain that

$$\mathbf{Y} = \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \end{pmatrix}\mathbf{S}^{-1} \tag{2.20}$$

when $a\alpha = 1$ or

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 & \mathbf{0} \end{pmatrix}\mathbf{S}^{-1} \tag{2.21}$$

when $a\alpha = -1$.

Hence, we can easily write

$$\mathbf{A} = \mathbf{U}\,(\alpha\mathbf{I}_p \oplus \mathbf{I}_{n-p})\,\mathbf{U}^{-1} = \mathbf{U}\,(\mathbf{I}_p \oplus \mathbf{S})\,(\alpha\mathbf{I}_p \oplus \mathbf{I}_{n-p})\left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}.$$

In view of (2.18), (2.20) and (2.18), (2.21) we obtain, respectively, that

$$\mathbf{B} = \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right) \begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-\alpha-1}{\alpha b}\mathbf{I}_{n-p-r} \end{pmatrix} \left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}$$

and

$$\mathbf{B} = \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right) \begin{pmatrix} \mathbf{0}_p & \mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \frac{\alpha+1}{\alpha b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-\alpha+1}{\alpha b}\mathbf{I}_{n-p-r} \end{pmatrix} \left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1},$$

which establish parts of (f) and (g) by defining $\mathbf{V}$ as $\mathbf{V} := \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)$. So, the necessity part of the proof is completed and the sufficiency is obvious. □

Theorem 2.2. *Let a, b, $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$, and $\alpha \neq \beta$. Moreover, let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be an $\{\alpha, \beta\}$–quadratic matrix and an arbitrary matrix, respectively. Furthermore, let $\mathbf{K}$ be their linear combination of the form $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$. Then $\mathbf{K}$ is an involutive matrix and $\mathbf{A}^2\mathbf{B}^2 = (\mathbf{AB})^2$ if and only if there exists a nonsingular matrix $\mathbf{V} \in \mathbb{C}^n$ such that*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{V}^{-1} \tag{2.22}$$

and $\mathbf{B}$ satisfies one of the following cases.
(a) $\beta = 0$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & -\frac{1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}. \tag{2.23}$$

(b) $\beta \neq 0$, $a\alpha = 1$, *and* $a\beta = -1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & -\frac{2}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{2}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}. \tag{2.24}$$

(c) $\beta \neq 0$, $a\alpha \neq 1$, *and* $a\beta = -1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{2}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}. \tag{2.25}$$

(d) $\beta \neq 0$, $a\alpha = -1$, *and* $a\beta = 1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{2}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-2}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}. \tag{2.26}$$

(e) $\beta \neq 0$, $a\alpha \neq -1$, *and* $a\beta = 1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-2}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}. \tag{2.27}$$

(f) $\beta \neq 0$, $a\alpha = -1$, *and* $a\beta \neq 1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{2}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}. \tag{2.28}$$

(g) $\beta \neq 0$, $a\alpha = 1$, *and* $a\beta \neq -1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-2}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}. \tag{2.29}$$

(h) $\beta \neq 0$, $a\alpha \neq \pm 1$, *and* $a\beta \neq \pm 1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}. \tag{2.30}$$

Here $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$, $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$, $\mathbf{Z}_2 \in \mathbb{C}^{r\times(p-q)}$, $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ *arbitrary matrices and* $\mathbf{Z}_3\mathbf{Y}_2 = \mathbf{0}$, $\mathbf{Y}_2\mathbf{Z}_3 = \mathbf{0}$, $\mathbf{Z}_2\mathbf{Y}_3 = \mathbf{0}$, $\mathbf{Y}_3\mathbf{Z}_2 = \mathbf{0}$.

Proof. We can write a quadratic matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\mathbf{U}^{-1},$$

where $p \in \{0,\ldots,n\}$ and $\mathbf{U} \in \mathbb{C}^n$ is a nonsingular matrix. We can represent $\mathbf{B}$ as $\mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{T} \end{pmatrix}\mathbf{U}^{-1}$, where $\mathbf{X} \in \mathbb{C}^p$. Observe that under the hypotheses $\mathbf{A}^2\mathbf{B}^2 = (\mathbf{AB})^2$, $\alpha \neq 0$, and $\alpha \neq \beta$, one has

$$\mathbf{YZ} = \mathbf{0},\ \mathbf{YT} = \mathbf{0},\ \beta\mathbf{ZX} = \mathbf{0},\ \beta\mathbf{ZY} = \mathbf{0}. \tag{2.31}$$

Let us assume that $\mathbf{K}$ is an involutive matrix then

$$\begin{aligned} &(a\alpha\mathbf{I}_p + b\mathbf{X})^2 + b^2\mathbf{YZ} = \mathbf{I}_p, \quad ab\left(\alpha+\beta\right)\mathbf{Y} + b^2\left(\mathbf{XY} + \mathbf{YT}\right) = \mathbf{0}, \\ &ab\left(\alpha+\beta\right)\mathbf{Z} + b^2\left(\mathbf{ZX} + \mathbf{TZ}\right) = \mathbf{0}, \quad (a\beta\mathbf{I}_{n-p} + b\mathbf{T})^2 + b^2\mathbf{ZY} = \mathbf{I}_{n-p}. \end{aligned} \tag{2.32}$$

Now, let us separate the proof according to α and β. Firstly, we use the values of β.

(i) Let $\beta = 0$. Considering (2.31) and (2.32), we get

$$\begin{aligned} &(a\alpha\mathbf{I}_p + b\mathbf{X})^2 = \mathbf{I}_p, \quad ab\alpha\mathbf{Y} + b^2\mathbf{XY} = \mathbf{0}, \\ &(b\mathbf{T})^2 + b^2\mathbf{ZY} = \mathbf{I}_{n-p}, \quad ab\alpha\mathbf{Z} + b^2\left(\mathbf{ZX} + \mathbf{TZ}\right) = 0. \end{aligned} \tag{2.33}$$

It is clear that $a\alpha\mathbf{I}_p + b\mathbf{X}$ is an involutive matrix from the first equation in (2.33). So, there exist $q \in \{0,\ldots,p\}$ and a nonsingular matrix $\mathbf{S}_1 \in \mathbb{C}^p$ such that

$$\mathbf{X} = \mathbf{S}_1\left(\frac{1-a\alpha}{b}\mathbf{I}_q \oplus \frac{-1-a\alpha}{b}\mathbf{I}_{p-q}\right)\mathbf{S}_1{}^{-1}. \tag{2.34}$$

Let $\mathbf{Y}$ be written as

$$\mathbf{Y} = \mathbf{S}_1\begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \\ \mathbf{Y}_3 & \mathbf{Y}_4 \end{pmatrix}, \tag{2.35}$$

where $\mathbf{Y}_1 \in \mathbb{C}^{q\times r}$. Substituting (2.34) and (2.35) into the second equation in (2.33) it is obtained that $\mathbf{Y} = \mathbf{0}$. Considering the last result, the third equation of (2.33) turns to $(b\mathbf{T})^2 = \mathbf{I}_{n-p}$. Thus, it is clear that $b\mathbf{T}$ is an involutive matrix. So, there exist $r \in \{0,\ldots,n-p\}$ and a nonsingular matrix $\mathbf{S}_2 \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{T} = \mathbf{S}_2\left(\frac{1}{b}\mathbf{I}_r \oplus \frac{-1}{b}\mathbf{I}_{n-p-r}\right)\mathbf{S}_2{}^{-1}. \tag{2.36}$$

Let $\mathbf{Z}$ be written as

$$\mathbf{Z} = \mathbf{S}_2 \begin{pmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{pmatrix} \mathbf{S}_1{}^{-1}, \tag{2.37}$$

where $\mathbf{Z}_1 \in \mathbb{C}^{r \times q}$. Substituting (2.34), (2.36), and (2.37) into the fourth equation in (2.33) it is obtained that $2\left(\mathbf{Z}_1 \oplus -\mathbf{Z}_4\right) = \mathbf{0}$ in other words

$$\mathbf{Z} = \mathbf{S}_2 \begin{pmatrix} \mathbf{0} & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{0} \end{pmatrix} \mathbf{S}_1{}^{-1}. \tag{2.38}$$

Hence, defining $\mathbf{V} := \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right)$, we can write $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\left(\alpha \mathbf{I}_p \oplus \beta \mathbf{I}_{n-p}\right) \mathbf{U}^{-1} = \mathbf{V}\left(\mathbf{S}_1{}^{-1} \oplus \mathbf{S}_2{}^{-1}\right)\left(\alpha \mathbf{I}_p \oplus \beta \mathbf{I}_{n-p}\right)\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right) \mathbf{V}^{-1}$$

$$= \mathbf{V}\left(\alpha \mathbf{I}_p \oplus \beta \mathbf{I}_{n-p}\right) \mathbf{V}^{-1}.$$

In view of (2.34), (2.36), and (2.38), $\mathbf{B}$ is obtained that

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1},$$

which establishes part of (a).

(ii) Now, let $\beta \neq 0$. From the third and fourth equations in (2.31), we can write $\mathbf{ZX} = \mathbf{0}$ and $\mathbf{ZY} = \mathbf{0}$. Then, reorganizing the equations in (2.32), we get

$$\begin{aligned} &\left(a\alpha \mathbf{I}_p + b\mathbf{X}\right)^2 = \mathbf{I}_p, \quad \left(a\beta \mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = \mathbf{I}_{n-p}, \\ &ab\left(\alpha+\beta\right)\mathbf{Y} + b^2\mathbf{XY} = \mathbf{0}, \quad ab\left(\alpha+\beta\right)\mathbf{Z} + b^2\mathbf{TZ} = \mathbf{0}. \end{aligned} \tag{2.39}$$

It is clear that $a\alpha \mathbf{I}_p + b\mathbf{X}$ and $a\beta \mathbf{I}_{n-p} + b\mathbf{T}$ are involutive matrices from the first and second equations in (2.39), respectively. So, there exist $q \in \{0, \ldots, p\}$, $r \in \{0, \ldots, n-p\}$ and nonsingular matrices $\mathbf{S}_3 \in \mathbb{C}^p$, $\mathbf{S}_4 \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{X} = \mathbf{S}_3\left(\frac{1-a\alpha}{b}\mathbf{I}_q \oplus \frac{-1-a\alpha}{b}\mathbf{I}_{p-q}\right)\mathbf{S}_3{}^{-1}, \mathbf{T} = \mathbf{S}_4\left(\frac{1-a\beta}{b}\mathbf{I}_r \oplus \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r}\right)\mathbf{S}_4{}^{-1}. \tag{2.40}$$

Defining $\mathbf{V} := \mathbf{U}\left(\mathbf{S}_3 \oplus \mathbf{S}_4\right)$, we can write $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\left(\alpha \mathbf{I}_p \oplus \beta \mathbf{I}_{n-p}\right) \mathbf{U}^{-1} = \mathbf{V}\left(\mathbf{S}_3{}^{-1} \oplus \mathbf{S}_4{}^{-1}\right)\left(\alpha \mathbf{I}_p \oplus \beta \mathbf{I}_{n-p}\right)\left(\mathbf{S}_3 \oplus \mathbf{S}_4\right) \mathbf{V}^{-1}$$

$$= \mathbf{V}\left(\alpha \mathbf{I}_p \oplus \beta \mathbf{I}_{n-p}\right) \mathbf{V}^{-1}.$$

Now, let $\mathbf{Y}$ and $\mathbf{Z}$ be written as

$$\mathbf{Y} = \mathbf{S}_3 \begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \\ \mathbf{Y}_3 & \mathbf{Y}_4 \end{pmatrix} \mathbf{S}_4{}^{-1} \text{ and } \mathbf{Z} = \mathbf{S}_4 \begin{pmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{pmatrix} \mathbf{S}_3{}^{-1}, \tag{2.41}$$

where $\mathbf{Y}_1 \in \mathbb{C}^{q \times r}$ and $\mathbf{Z}_1 \in \mathbb{C}^{r \times q}$. Substituting (2.40) and (2.41) into the third and fourth equations in (2.39), it is obtained that

$$\begin{pmatrix} (a\beta+1)\mathbf{Y}_1 & (a\beta+1)\mathbf{Y}_2 \\ (a\beta-1)\mathbf{Y}_3 & (a\beta-1)\mathbf{Y}_4 \end{pmatrix} = \mathbf{0}, \begin{pmatrix} (a\alpha+1)\mathbf{Z}_1 & (a\alpha+1)\mathbf{Z}_2 \\ (a\alpha-1)\mathbf{Z}_3 & (a\alpha-1)\mathbf{Z}_4 \end{pmatrix} = \mathbf{0}. \tag{2.42}$$

Depending on the values of $a\alpha$ and $a\beta$, we have the following cases.

(ii–1) Let $a\alpha = 1$ and $a\beta = -1$. It is clear that $\mathbf{Y}_3, \mathbf{Y}_4$ and $\mathbf{Z}_1, \mathbf{Z}_2$ are zero matrices from the equations in (2.42). Considering all of (2.31), (2.40), (2.41) and these facts, we obtain

$$\mathbf{Y} = \mathbf{S}_3 \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{S}_4{}^{-1} \text{ and } \mathbf{Z} = \mathbf{S}_4 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} \end{pmatrix} \mathbf{S}_3{}^{-1},$$

where $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$ and $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ are arbitrary matrices that satisfy the equalities $\mathbf{Y}_2\mathbf{Z}_3 = \mathbf{0}$ and $\mathbf{Z}_3\mathbf{Y}_2 = \mathbf{0}$. Therefore, we get $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-2}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{2}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

which establishes part (b).

(ii–2) Let $a\alpha \neq 1$ and $a\beta = -1$. From the equations in (2.42), it is clear that $\mathbf{Y}_3, \mathbf{Y}_4$, and $\mathbf{Z}$ are zero matrices. Thus, as in (ii–1), $\mathbf{Y}$ reduces to $\mathbf{Y} = \mathbf{S}_3\begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{S}_4{}^{-1}$ and then

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{2}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}.$$

So, it is completed part (c).

(ii–3) Let $a\alpha = -1$ and $a\beta = 1$. It is clear that $\mathbf{Y}_1,\ \mathbf{Y}_2$ and $\mathbf{Z}_3,\ \mathbf{Z}_4$ are zero matrices from the equations in (2.42). Considering all of (2.31), (2.40), (2.41) and these facts, we obtain

$$\mathbf{Y} = \mathbf{S}_3\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Y}_3 & \mathbf{0} \end{pmatrix}\mathbf{S}_4{}^{-1} \quad \text{and} \quad \mathbf{Z} = \mathbf{S}_4\begin{pmatrix} \mathbf{0} & \mathbf{Z}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{S}_3{}^{-1},$$

where $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$ and $\mathbf{Z}_2 \in \mathbb{C}^{r\times(p-q)}$ are arbitrary matrices that satisfy the equalities $\mathbf{Y}_3\mathbf{Z}_2 = \mathbf{0}$ and $\mathbf{Z}_2\mathbf{Y}_3 = \mathbf{0}$. Therefore, we get the matrix $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{2}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-2}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

which establishes part (d).

(ii–4) Let $a\alpha \neq -1$ and $a\beta = 1$. From the equations in (2.42), it is clear that $\mathbf{Y}_1, \mathbf{Y}_2$, and $\mathbf{Z}$ are zero matrices. Thus, as in (ii–3), $\mathbf{Y}$ reduces to $\mathbf{Y} = \mathbf{S}_3\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Y}_3 & \mathbf{0} \end{pmatrix}\mathbf{S}_4{}^{-1}$ and then

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-2}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}.$$

So, it is completed part (e).

(ii–5) Let $a\alpha = -1$ and $a\beta \neq 1$. It is obvious that $\mathbf{Z}_3, \mathbf{Z}_4$, and $\mathbf{Y}$ are zero matrices from the equations in (2.42). Thus, as in (ii–3), $\mathbf{Z}$ reduces to $\mathbf{Z} = \mathbf{S}_4\begin{pmatrix} \mathbf{0} & \mathbf{Z}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{S}_3{}^{-1}$ and then

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{2}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

where $\mathbf{Z}_2 \in \mathbb{C}^{r\times(p-q)}$ is an arbitrary matrix and which completes part (f).

(ii–6) Let $a\alpha = 1$ and $a\beta \neq -1$. It is obvious that $\mathbf{Z}_1, \mathbf{Z}_2$, and $\mathbf{Y}$ are zero matrices from the equations in (2.42). Thus, as in (ii–1), $\mathbf{Z}$ reduces to $\mathbf{Z} = \mathbf{S}_4 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} \end{pmatrix} \mathbf{S}_3{}^{-1}$ and then

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-2}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1},$$

where $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ is an arbitrary matrix. So, the part (g) of the proof is completed.

(ii–7) Let $a\beta \neq \pm 1$ and $a\alpha \neq \pm 1$. From the equations in (2.42), it is clear that $\mathbf{Y} = \mathbf{0}$ and $\mathbf{Z} = \mathbf{0}$. Hence,

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1},$$

which completes the part (h) of the proof. Therefore, the part of the necessity of the proof is completed.

On the other hand, it is evident that if $\mathbf{A}$ and $\mathbf{B}$ are represented as in (2.22) and (2.23)–(2.30) and if the scalars α, β satisfy the corresponding conditions, then $\mathbf{K}^2 = \mathbf{I}$. □

Theorem 2.3. *Let a, b, $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$, and $\alpha \neq \beta$. Moreover, let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be an $\{\alpha, \beta\}$–quadratic matrix and an arbitrary matrix, respectively. Furthermore, let $\mathbf{K}$ be their linear combination of the form $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$. Then $\mathbf{K}$ is an involutive matrix and $\mathbf{BAB} = \mathbf{AB}^2$ if and only if there exists a nonsingular matrix $\mathbf{V} \in \mathbb{C}^n$ such that*

$$\mathbf{A} = \mathbf{V} \begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix} \mathbf{V}^{-1}$$

and $\mathbf{B}$ satisfies one of the following cases.

(a) $a\alpha = 1$ *and* $a\beta = -1$,

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-2}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{2}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1}.$$

(b) $a\alpha \neq 1$ *and* $a\beta = -1$,

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{2}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1}.$$

(c) $a\alpha = -1$ *and* $a\beta = 1$,

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{2}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1}.$$

(d) $a\alpha \neq -1$ *and* $a\beta = 1$,

$$\mathbf{B} = \mathbf{V} \begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-2}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1}.$$

(e) $a\alpha = -1$ *and* $a\beta \neq 1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{2}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}.$$

(f) $a\alpha = 1$ *and* $a\beta \neq -1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-2}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}.$$

(g) $a\alpha \neq \pm 1$ *and* $a\beta \neq \pm 1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}.$$

Here $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$, $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$, $\mathbf{Z}_2 \in \mathbb{C}^{r\times(p-q)}$, $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ *are arbitrary matrices and* $\mathbf{Z}_3\mathbf{Y}_2 = \mathbf{0}$, $\mathbf{Y}_2\mathbf{Z}_3 = \mathbf{0}$, $\mathbf{Z}_2\mathbf{Y}_3 = \mathbf{0}$, $\mathbf{Y}_3\mathbf{Z}_2 = \mathbf{0}$.

Proof. This theorem is given under the condition $\mathbf{BAB} = \mathbf{AB}^2$. Premultiplying this condition by $\mathbf{A}$ leads to $\mathbf{A}^2\mathbf{B}^2 = (\mathbf{AB})^2$. Therefore, we get the proof if we apply Theorem 2.2. □

Lastly, let us give the following theorem.

Theorem 2.4. *Let* a, b, $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$, $\alpha \neq \beta$, *and* $(\alpha, \beta) \notin \{(-1, 1), (1, -1)\}$. *Moreover, let* $\mathbf{A}$ *and* $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *be an* $\{\alpha, \beta\}$*–quadratic matrix and an arbitrary matrix, respectively. Furthermore, let* $\mathbf{K}$ *be their linear combination of the form* $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$. *Then* $\mathbf{K}$ *is an involutive matrix and* $\mathbf{A}^2\mathbf{BA} = \mathbf{BA}$ *if and only if there exists a nonsingular matrix* $\mathbf{V} \in \mathbb{C}^n$ *such that*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{V}^{-1} \tag{2.43}$$

and $\mathbf{B}$ *satisfies one of the following cases.*

(a) $\beta^2 \neq 1$, $\alpha^2 = 1$, *and* $\beta = 0$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{2.44}$$

being $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$ *and* $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$ *arbitrary.*

(b) $\beta^2 \neq 1$, $\alpha^2 = 1$, $\beta \neq 0$, *and* $a\beta = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p} \end{pmatrix}\mathbf{V}^{-1}, \tag{2.45}$$

being $\mathbf{Y}_2 \in \mathbb{C}^{(p-q)\times(n-p)}$ *arbitrary.*

(c) $\beta^2 \neq 1$, $\alpha^2 = 1$, $\beta \neq 0$, *and* $a\beta = -1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{Y}_1 \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p} \end{pmatrix}\mathbf{V}^{-1}, \tag{2.46}$$

being $\mathbf{Y}_1 \in \mathbb{C}^{q\times(n-p)}$ *arbitrary.*

(d) $\beta^2 \neq 1,\ \alpha^2 \neq 1,\ \beta = 0,$ *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{2.47}$$

being $\mathbf{Y}_2 \in \mathbb{C}^{p\times(n-p-r)}$ *arbitrary.*

(e) $\beta^2 \neq 1,\ \alpha^2 \neq 1,\ \beta = 0,$ *and* $a\alpha = -1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{2.48}$$

being $\mathbf{Y}_1 \in \mathbb{C}^{p\times r}$ *arbitrary.*

(f) $\beta^2 = 1$ *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_2 & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{2.49}$$

being $\mathbf{Z}_2 \in \mathbb{C}^{(n-p-r)\times p}$ *arbitrary.*

(g) $\beta^2 = 1$ *and* $a\alpha = -1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_1 & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{2.50}$$

being $\mathbf{Z}_1 \in \mathbb{C}^{r\times p}$ *arbitrary.*

Proof. Let us write a quadratic matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\mathbf{U}^{-1},$$

where $p \in \{0,\ldots,n\}$ and $\mathbf{U} \in \mathbb{C}^n$ is a nonsingular matrix. We can represent $\mathbf{B}$ as $\mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{T} \end{pmatrix}\mathbf{U}^{-1}$ where $\mathbf{X} \in \mathbb{C}^p$. In view of the hypotheses $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{B}\mathbf{A}$ and $\alpha \neq 0$ we can write

$$\alpha^2\mathbf{X} = \mathbf{X},\ \alpha^2\beta\mathbf{Y} = \beta\mathbf{Y},\ \beta^2\mathbf{Z} = \mathbf{Z},\ \beta^3\mathbf{T} = \beta\mathbf{T}. \tag{2.51}$$

Let us assume that $\mathbf{K}$ is an involutive matrix then it follows that

$$\begin{aligned} &(a\alpha\mathbf{I}_p + b\mathbf{X})^2 + b^2\mathbf{Y}\mathbf{Z} = \mathbf{I}_p, \quad ab\,(\alpha+\beta)\,\mathbf{Y} + b^2\,(\mathbf{X}\mathbf{Y} + \mathbf{Y}\mathbf{T}) = \mathbf{0}, \\ &ab\,(\alpha+\beta)\,\mathbf{Z} + b^2\,(\mathbf{Z}\mathbf{X} + \mathbf{T}\mathbf{Z}) = \mathbf{0}, \quad (a\beta\mathbf{I}_{n-p} + b\mathbf{T})^2 + b^2\mathbf{Z}\mathbf{Y} = \mathbf{I}_{n-p}. \end{aligned} \tag{2.52}$$

The proof can be split into following cases depending on the scalar β.

(i) Let $\beta^2 \neq 1$. From (2.51), it is seen that $\mathbf{Z} = \mathbf{0}$. We can split this case into four cases depending on the values of α and β.

(i-1) Let $\alpha^2 = 1$ and $\beta = 0$. Reorganizing the equations of (2.52), it can be written

$$(a\alpha\mathbf{I}_p + b\mathbf{X})^2 = \mathbf{I}_p,\ \ (b\mathbf{T})^2 = \mathbf{I}_{n-p},\ \ ab\alpha\mathbf{Y} + b^2\,(\mathbf{X}\mathbf{Y} + \mathbf{Y}\mathbf{T}) = \mathbf{0}. \tag{2.53}$$

It is clear that $a\alpha\mathbf{I}_p + b\mathbf{X}$ and $b\mathbf{T}$ are involutive matrices from the first and second equations in (2.53), respectively. So, there exist $q \in \{0,\ldots,p\}$, $r \in \{0,\ldots,n-p\}$ and nonsingular matrices $\mathbf{S}_1 \in \mathbb{C}^p$, $\mathbf{S}_2 \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{X} = \mathbf{S}_1\left(\frac{1-a\alpha}{b}\mathbf{I}_q \oplus \frac{-1-a\alpha}{b}\mathbf{I}_{p-q}\right)\mathbf{S}_1{}^{-1} \text{ and } \mathbf{T} = \mathbf{S}_2\left(\frac{1}{b}\mathbf{I}_r \oplus \frac{-1}{b}\mathbf{I}_{n-p-r}\right)\mathbf{S}_2{}^{-1}. \tag{2.54}$$

Let us write $\mathbf{Y}$ as

$$\mathbf{Y} = \mathbf{S}_1\begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \\ \mathbf{Y}_3 & \mathbf{Y}_4 \end{pmatrix}\mathbf{S}_2{}^{-1}, \tag{2.55}$$

where $\mathbf{Y}_1 \in \mathbb{C}^{q\times r}$. Substituting (2.54) and (2.55) into the third equation in (2.53) yields $2(\mathbf{Y}_1 \oplus -\mathbf{Y}_4) = \mathbf{0}$. Then $\mathbf{Y}$ reduces to

$$\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{Y}_3 & \mathbf{0} \end{pmatrix} \mathbf{S}_2{}^{-1}, \tag{2.56}$$

where $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$ and $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$ are arbitrary matrices.

Let us define $\mathbf{V} := \mathbf{U}(\mathbf{S}_1 \oplus \mathbf{S}_2)$. Then we can write $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}(\alpha\mathbf{I}_p \oplus \mathbf{0}_{n-p})\mathbf{U}^{-1} = \mathbf{V}\left(\mathbf{S}_1{}^{-1} \oplus \mathbf{S}_2{}^{-1}\right)(\alpha\mathbf{I}_p \oplus \mathbf{0}_{n-p})(\mathbf{S}_1 \oplus \mathbf{S}_2)\mathbf{V}^{-1}$$

$$= \mathbf{V}(\alpha\mathbf{I}_p \oplus \mathbf{0}_{n-p})\mathbf{V}^{-1}.$$

In view of (2.54) and (2.56) we obtain that

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

which yields part (a).

(i-2) Let $\alpha^2 = 1$ and $\beta \neq 0$. From (2.51), it is seen that $\mathbf{T} = \mathbf{0}$. Reorganizing the equations of (2.52), it can be written

$$(a\alpha\mathbf{I}_p + b\mathbf{X})^2 = \mathbf{I}_p, \quad (a\beta\mathbf{I}_{n-p})^2 = \mathbf{I}_{n-p}, \quad ab(\alpha+\beta)\mathbf{Y} + b^2\mathbf{XY} = \mathbf{0}. \tag{2.57}$$

It is clear that $a\alpha\mathbf{I}_p + b\mathbf{X}$ is an involutive matrix from the first equation in (2.57), so there exist $q \in \{0,\ldots,p\}$ and a nonsingular matrix $\mathbf{S}_3 \in \mathbb{C}^p$ such that

$$\mathbf{X} = \mathbf{S}_3\left(\frac{1-a\alpha}{b}\mathbf{I}_q \oplus \frac{-1-a\alpha}{b}\mathbf{I}_{p-q}\right)\mathbf{S}_3{}^{-1}. \tag{2.58}$$

Let us write $\mathbf{Y}$ as

$$\mathbf{Y} = \mathbf{S}_3\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}, \tag{2.59}$$

where $\mathbf{Y}_1 \in \mathbb{C}^{q\times(n-p)}$. Substituting (2.58) and (2.59) into the third equation in (2.57) it is obtained that $\begin{pmatrix} (a\beta+1)\mathbf{Y}_1 \\ (a\beta-1)\mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$. Moreover, it is clear that $a\beta \in \{-1,1\}$ from the second equation in (2.57). Then $\mathbf{Y}$ reduces to

$$\mathbf{Y} = \mathbf{S}_3\begin{pmatrix} \mathbf{0} \\ \mathbf{Y}_2 \end{pmatrix} \tag{2.60}$$

when $a\beta = 1$ or

$$\mathbf{Y} = \mathbf{S}_3\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{0} \end{pmatrix} \tag{2.61}$$

when $a\beta = -1$.

Let us define $\mathbf{V} := \mathbf{U}(\mathbf{S}_3 \oplus \mathbf{I}_{n-p})$. Then we get $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p})\mathbf{U}^{-1} = \mathbf{V}\left(\mathbf{S}_3{}^{-1} \oplus \mathbf{I}_{n-p}\right)(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p})(\mathbf{S}_3 \oplus \mathbf{I}_{n-p})\mathbf{V}^{-1}$$

$$= \mathbf{V}(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p})\mathbf{V}^{-1}.$$

In view of (2.58), (2.60) and (2.58), (2.61) we obtain, respectively, that

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p} \end{pmatrix}\mathbf{V}^{-1},$$

and

$$\mathbf{B}=\mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{Y}_1 \\ \mathbf{0} & \frac{-1-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p} \end{pmatrix}\mathbf{V}^{-1},$$

which establish parts (b) and (c).

(i-3) Let $\alpha^2 \neq 1$ and $\beta = 0$. From (2.51), it is seen that $\mathbf{X}=\mathbf{0}$. Reorganizing the equations of (2.52), it can be written

$$(a\alpha\mathbf{I}_p)^2=\mathbf{I}_p, \quad (b\mathbf{T})^2=\mathbf{I}_{n-p}, \quad ab\alpha\mathbf{Y}+b^2\mathbf{Y}\mathbf{T}=\mathbf{0}. \tag{2.62}$$

It is clear that $b\mathbf{T}$ is an involutive matrix from the second equation in (2.62), so there exist $r\in\{0,\ldots,n-p\}$ and a nonsingular matrix $\mathbf{S}_4\in\mathbb{C}^{(n-p)}$ such that

$$\mathbf{T}=\mathbf{S}_4\left(\frac{1}{b}\mathbf{I}_r\oplus\frac{-1}{b}\mathbf{I}_{n-p-r}\right)\mathbf{S}_4{}^{-1}. \tag{2.63}$$

Let us write $\mathbf{Y}$ as

$$\mathbf{Y}=\begin{pmatrix}\mathbf{Y}_1 & \mathbf{Y}_2\end{pmatrix}\mathbf{S}_4{}^{-1}, \tag{2.64}$$

where $\mathbf{Y}_1\in\mathbb{C}^{p\times r}$. Substituting (2.63) and (2.64) into the third equation in (2.62) it is obtained that $\begin{pmatrix}(a\alpha+1)\,\mathbf{Y}_1 & (a\alpha-1)\,\mathbf{Y}_2\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{0}\end{pmatrix}$. Moreover, it is clear that $a\alpha\in\{-1,1\}$ from the first equation in (2.62). Then $\mathbf{Y}$ turns to

$$\mathbf{Y}=\begin{pmatrix}\mathbf{0} & \mathbf{Y}_2\end{pmatrix}\mathbf{S}_4{}^{-1} \tag{2.65}$$

when $a\alpha=1$ or

$$\mathbf{Y}=\begin{pmatrix}\mathbf{Y}_1 & \mathbf{0}\end{pmatrix}\mathbf{S}_4{}^{-1} \tag{2.66}$$

when $a\alpha=-1$.

Let us define $\mathbf{V}:=\mathbf{U}\left(\mathbf{I}_p\oplus\mathbf{S}_4\right)$. Then we get $\mathbf{A}$ as

$$\mathbf{A}=\mathbf{U}\left(\alpha\mathbf{I}_p\oplus\mathbf{0}_{n-p}\right)\mathbf{U}^{-1}=\mathbf{V}\left(\mathbf{I}_p\oplus\mathbf{S}_4{}^{-1}\right)\left(\alpha\mathbf{I}_p\oplus\mathbf{0}_{n-p}\right)\left(\mathbf{I}_p\oplus\mathbf{S}_4\right)\mathbf{V}^{-1}$$

$$=\mathbf{V}\left(\alpha\mathbf{I}_p\oplus\mathbf{0}_{n-p}\right)\mathbf{V}^{-1}.$$

In view of (2.63), (2.65) and (2.63), (2.66) we obtain, respectively, that

$$\mathbf{B}=\mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

and

$$\mathbf{B}=\mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

which establish parts (d) and (e).

(i-4) Let $\alpha^2\neq 1$ and $\beta\neq 0$. From (2.51), it is seen that $\mathbf{B}=\mathbf{0}$ which contradicts the hypothesis $\mathbf{B}\neq\mathbf{0}$. So, in this case there is no matrix form of $\mathbf{B}$.

(ii) Let $\beta^2=1$. From the first and second equations in (2.51) and considering hypotheses $(\alpha,\beta)\notin\{(-1,1),(1,-1)\}$ and $\alpha\neq\beta$, it is obvious that $\mathbf{X}=\mathbf{0}$ and $\mathbf{Y}=\mathbf{0}$. Reorganizing the equations of (2.52), it can be written

$$(a\alpha)^2\mathbf{I}_p=\mathbf{I}_p, \quad (a\beta\mathbf{I}_{n-p}+b\mathbf{T})^2=\mathbf{I}_{n-p}, \quad ab\left(\alpha+\beta\right)\mathbf{Z}+b^2\mathbf{T}\mathbf{Z}=\mathbf{0}. \tag{2.67}$$

It is explicit that $a\alpha\in\{-1,1\}$ and $a\beta\mathbf{I}_{n-p}+b\mathbf{T}$ is an involutive matrix from the first and second equations in (2.67). So, there exist $r\in\{0,\ldots,n-p\}$ and a nonsingular matrix $\mathbf{S}\in\mathbb{C}^{(n-p)}$ such that

$$\mathbf{T}=\mathbf{S}\left(\frac{1-a\beta}{b}\mathbf{I}_r\oplus\frac{-1-a\beta}{b}\mathbf{I}_{n-p-r}\right)\mathbf{S}^{-1}. \tag{2.68}$$

Let us write $\mathbf{Z}$ as

$$\mathbf{Z} = \mathbf{S}\begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix}, \tag{2.69}$$

where $\mathbf{Z}_1 \in \mathbb{C}^{r\times p}$. Substituting (2.68) and (2.69) into the third equation in (2.67), it is obtained that $\begin{pmatrix} (a\alpha+1)\,\mathbf{Z}_1 \\ (a\alpha-1)\,\mathbf{Z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$. Using $a\alpha \in \{-1,1\}$, $\mathbf{Z}$ obtained that

$$\mathbf{Z} = \mathbf{S}\begin{pmatrix} \mathbf{0} \\ \mathbf{Z}_2 \end{pmatrix} \tag{2.70}$$

when $a\alpha = 1$ or

$$\mathbf{Z} = \mathbf{S}\begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{0} \end{pmatrix} \tag{2.71}$$

when $a\alpha = -1$.

Hence, we can easily write

$$\mathbf{A} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\mathbf{U}^{-1} = \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}.$$

In view of (2.68), (2.70) and (2.68), (2.71) we obtain, respectively, that

$$\mathbf{B} = \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_2 & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}$$

and

$$\mathbf{B} = \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_1 & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}.$$

The necessity part of the proof is completed by defining $\mathbf{V}$ as $\mathbf{V} := \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)$.

Now, it is evident that if $\mathbf{A}$ is represented as in (2.43), $\mathbf{B}$ is represented as in (2.44)–(2.50) and the scalars α, β satisfy the corresponding conditions, then $\mathbf{K}^2 = \mathbf{I}$. □

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