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Research Article

On localization of the star-Menger selection principle

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Abstract

In this paper we primarily introduce the local version of star-Menger property, namely locally star-Menger property (a space with this property is called locally star-Menger) and present some important topological observations. Certain interactions between the new notion and star-Menger property are also observed. Some observations on effectively locally star-Menger Pixley-Roy hyperspaces (introduced here) are obtained. Preservation like properties under several topological operations are also interpreted carefully. Besides, several results on decomposition and remainder of locally star-Menger spaces are also presented.

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1. Introduction

The systematic study of selection principles in topology was initiated by Scheepers [26] (see also [16]). Since then the study was enriched by several authors and it becomes one of the promising research areas in set-theoretic topology. One of the important selection principle is $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, nowadays called the Menger property (see [16, 26]). This property was introduced by K. Menger [20] and later reformulated by W. Hurewicz [15] in 1925.

Generalizing the idea of selection principles, Kočinac [17] introduced star selection principles in 1999. The seminal papers [16, 26] set up a framework for studying generalization of selection principles in many ways. Readers interested also in star-selection principles may consult the papers [17, 19] and references therein. The star version of the Menger property, called the star-Menger property, plays a central role in this article.

In this paper we primarily work on the localization of the star-Menger property. The paper is organised as follows. In Section 3, we introduce a more general class of topological spaces, namely locally star-Menger spaces. We present some equivalent formulations of locally star-Menger spaces under zero-dimensionality, as well as under metrizability conditions and consider certain situation when star-Menger property is identical with its local variation. Later in this section, we extend our study to locally star-Menger Pixley-Roy

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hyperspaces. Moreover, another local variation, namely effectively locally star-Menger in the context of Pixley-Roy hyperspaces is introduced and studied. In Section 4, we exhibit several preservation like properties under topological operations in locally star-Menger spaces. It is also pointed out that some of these properties do not hold in star-Menger settings. In the remaining part of this section, we turn our attention to study the decomposition [33] of locally star-Menger spaces. We further introduce another new class $\mathfrak{M}^\star$ of topological spaces as a more general approach to the class of locally star-Menger spaces. It is shown that $\mathfrak{M}^\star$ is identical with the class of all spaces which are obtained as a decomposition of locally star-Menger spaces. In the final section, we make an effort to present quite a few observations about the remainder of locally star-Menger spaces. Most of the results of this section are obtained as a subsequent observation in the realm of paracompact p-spaces [2, 14].

2. Preliminaries

Throughout the paper (X, τ) stands for a topological space. For undefined notions and terminologies, see [12]. We say that a subset N is a neighbourhood of x in X if there exists an open set U in X such that $x \in U \subseteq N$. For a subset A of a space X, $\overline{A}^X$ denotes the closure of A in X. If no confusion arises, $\overline{A}^X$ can be denoted by $\overline{A}$. For a subset A of a space X and a collection $\mathcal{C}$ of subsets of X, $St(A, \mathcal{C})$ denotes the star of A with respect to $\mathcal{C}$, that is the set $\cup\{B \in \mathcal{C} : A \cap B \neq \emptyset\}$. For $A = \{x\}$, $x \in X$, we write $St(x, \mathcal{C})$ instead of $St(\{x\}, \mathcal{C})$ [12]. A space X is said to be starcompact (resp. star-Lindelöf) if for every open cover $\mathcal{U}$ of X there exists a finite (resp. countable) $\mathcal{V} \subseteq \mathcal{U}$ such that $St(\cup\mathcal{V}, \mathcal{U}) = X$ [17].

We shall use the symbol $\mathcal{O}$ to denote the collection of all open covers of X. A space X is said to be Menger if it satisfies the selection principle $\mathsf{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$, i.e. for each sequence $(\mathcal{U}_n)$ of open covers of X there is a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is an open cover of X. Note that the Menger property is preserved under countable unions, F_σ subspaces and continuous mappings [26].

A space X is said to be star-Menger if it satisfies the selection hypothesis $\mathsf{S}^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$, i.e. for each sequence $(\mathcal{U}_n)$ of open covers of X there is a sequence $(\mathcal{V}_n)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup_{n\in\mathbb{N}}\{St(V, \mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an open cover of X. It is well known that every Menger space is star-Menger and every star-Menger paracompact Hausdorff space is Menger [17]. Also, note that the star-Menger property is preserved under clopen subspaces, as well as under continuous mappings [17]. For more information about star-Menger spaces, see also [9, 30].

For a space X, let $\mathsf{PR}(X)$ be the space of all non-empty finite subsets of X with the Pixley-Roy topology [11, 24]: for $A \in \mathsf{PR}(X)$ and U open in X, let $[A, U] = \{B \in \mathsf{PR}(X) : A \subseteq B \subseteq U\}$, then $\{[A, U] : A \in \mathsf{PR}(X),\ U \text{ open in } X\}$ is a base for the Pixley-Roy topology. If X is T_1, then $\mathsf{PR}(X)$ is zero-dimensional T_1, each basic open set $[A, U]$ is clopen in $\mathsf{PR}(X)$ and every subspace of $\mathsf{PR}(X)$ is metacompact. Also, for any subset Y of X, $\mathsf{PR}(Y)$ is a closed subset of $\mathsf{PR}(X)$. For each n let $PR_{\leq n}(X) = \{F \in \mathsf{PR}(X) : |F| \leq n\}$. If X is a T_1 space, then for each n $PR_{\leq n}(X)$ is a closed subspace of $\mathsf{PR}(X)$ and in particular, $PR_{\leq 1}(X)$ is a closed discrete subspace of $\mathsf{PR}(X)$.

For a space X, $e(X) = \sup\{|Y| : Y \text{ is a closed and discrete subspace of } X\}$ is called the extent of X. A collection $\mathcal{C}$ of subsets of X is said to be a network for X if for each $x \in X$ and any open set U containing x there exists a $A \in \mathcal{C}$ such that $x \in A \subseteq U$. A space X is said to be cosmic if it has a countable network. Also, for a collection $\mathcal{C}$ of subsets of X, let $\mathcal{C}_\delta$ denote the collection of all sets that can be expressed as the intersection of some nonempty subcollection of $\mathcal{C}$ and let $\mathcal{C}_{\delta,\sigma}$ denote the collection of all sets that can be expressed as the union of some subcollection of $\mathcal{C}_\delta$. We say that $\mathcal{C}$ is a source for a space Y in X if Y is a subspace of X such that $Y \in \mathcal{C}_{\delta,\sigma}$. A source $\mathcal{C}$ for Y in X is called open (closed) if every member of $\mathcal{C}$ is open (resp. closed) in X and a source $\mathcal{C}$ is countable if $\mathcal{C}$ is countable.

Recall that a collection $\mathcal{A}$ of subsets of $\mathbb{N}$ is said to be an almost disjoint family if each $A \in \mathcal{A}$ is infinite and for any two distinct elements $B, C \in \mathcal{A}$, $|B \cap C| < \aleph_0$. For an almost disjoint family $\mathcal{A}$, let $\Psi(\mathcal{A}) = \mathcal{A} \cup \mathbb{N}$ be the Isbell-Mrówka space. It is well known that $\Psi(\mathcal{A})$ is a locally compact zero-dimensional Hausdorff space (see [13, 22]).

Let $\mathbb{N}^{\mathbb{N}}$ be the Baire space. A natural pre-order $\leq^*$ on $\mathbb{N}^{\mathbb{N}}$ is defined by $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many n. A subset D of $\mathbb{N}^{\mathbb{N}}$ is said to be dominating if for each $g \in \mathbb{N}^{\mathbb{N}}$ there exists a $f \in D$ such that $g \leq^* f$. Let $\mathfrak{d}$ be the minimum cardinality of a dominating subset of $\mathbb{N}^{\mathbb{N}}$ and $\mathfrak{c}$ be the cardinality of the set of reals. It is well known that $\aleph_0 < \mathfrak{d} \leq \mathfrak{c}$. Throughout we use the terminology that if κ is any cardinal, then κ^+ denotes the smallest cardinal greater than κ.

A P-space is a space in which countable intersection of open sets is open. An equivalent condition is that countable union of closed sets is closed.

We start with the following two observations that will be useful in our context.

Lemma 2.1 (Folklore)**.** *Every starcompact paracompact Hausdorff space is compact.*

Lemma 2.2 (Folklore)**.** *If $X = \cup_{n\in\mathbb{N}} X_n$, where each X_n is a star-Menger subspace of X, then X is star-Menger.*

We also mention the following observation about the Isbell-Mrówka space $\Psi(\mathcal{A})$.

Lemma 2.3 (cf. [8, Corollary 11])**.** *If $|\mathcal{A}| = \mathfrak{c}$, then $\Psi(\mathcal{A})$ is not star-Menger.*

3. Certain investigations on the locally star-Menger spaces

3.1. Locally star-Menger property

We now introduce the main definition of the paper.

Definition 3.1. A space X is said to be locally star-Menger if for each $x \in X$ there exist an open set U and a star-Menger subspace M of X such that $x \in U \subseteq M$.

We also say that a space X has the locally star-Menger property if X is locally star-Menger. Clearly, the class of locally star-Menger spaces properly contains the class of star-Menger spaces. As for an example, if we consider the Isbell-Mrówka space $\Psi(\mathcal{A})$, where $\mathcal{A}$ is an almost disjoint family with $|\mathcal{A}| = \mathfrak{c}$, then $\Psi(\mathcal{A})$ is locally compact and locally star-Menger as well. But $\Psi(\mathcal{A})$ is not star-Menger by Lemma 2.3.

We say that a space X is locally starcompact if for each $x \in X$ there exist an open set U and a starcompact subspace K of X such that $x \in U \subseteq K$. Clearly, every locally starcompact space is locally star-Menger but not conversely. To see this, we first recall the definition of evenly spaced integer topology on the set of integers $\mathbb{Z}$ [28] which is generated by sets of the form $m + n\mathbb{Z} = \{m + n\lambda : \lambda \in \mathbb{Z}\}$, where $m,\ n \in \mathbb{Z}$ with $n \neq 0$. The set of integers with the evenly spaced integer topology is an example of a metrizable locally star-Menger space, which is not locally starcompact by Lemma 2.1 as it fails to be locally compact.

The following result presents a useful characterization for locally star-Menger spaces.

Theorem 3.2. *For a zero-dimensional (resp. metrizable) space X, the following assertions are equivalent.*

(1) *X is locally star-Menger.*
(2) *For each $x \in X$ and for each open set V with $x \in V$ there exist an open set U and a star-Menger subspace M of X such that $x \in U \subseteq M \subseteq V$.*
(3) *For each $x \in X$ there exists an open set U with $x \in U$ such that $\overline{U}$ is star-Menger.*
(4) *X has a base consisting of clopen (resp. closed) star-Menger neighbourhoods.*

Proof. We present the proof of (3) $\Rightarrow$ (4) when X is zero-dimensional. Let $x \in X$ and V be an open set containing x. Choose an open set U in X such that $x \in U$ and $\overline{U}$ is star-Menger. If $\mathcal{B}$ is a base for X consisting of clopen sets, then we can choose a $B \in \mathcal{B}$ such that $x \in B \subseteq U \cap V$. It follows that B is star-Menger. Clearly, such $\mathcal{B}$ is a base consisting of clopen star-Menger neighbourhoods.
Similarly we give a proof of (1) $\Rightarrow$ (2) for the metrizable case as other implications can be easily carried out. Let $x \in X$ and V be an open set containing x. Choose an open set W such that $x \in W \subseteq \overline{W} \subseteq V$. Also, choose an open set U and a star-Menger subspace M of X such that $x \in U \subseteq M$. Clearly, M is Menger and $W \cap U$ is open in X containing x. Since the Menger property is hereditary for closed subspaces, it follows that $\overline{W} \cap M$ is a star-Menger subspace contained in V. Thus (2) holds. $\square$

We now present another useful observation under the Lindelöf condition.

Proposition 3.3. *Let X be Lindelöf. Then X is locally star-Menger if and only if X is star-Menger.*

Proof. We only give proof of the forward implication. Let $x \in X$. Choose an open set U_x and a star-Menger subspace M_x such that $x \in U_x \subseteq M_x$. By Lindelöf condition, choose a countable subcollection $\{U_{x_n} : n \in \mathbb{N}\}$ of $\{U_x : x \in X\}$ that covers X. Clearly, $X = \cup_{n \in \mathbb{N}} M_{x_n}$ is star-Menger by Lemma 2.2. $\square$

Recall that if $\mathcal{P}$ is a property, then $\mathsf{non}(\mathcal{P})$ is the minimum cardinality of a set of reals that fails have the property $\mathcal{P}$. By a classical result of Hurewicz (see [16, Theorem 4.4]), a space X is Menger if and only if every continuous image of X in $\mathbb{N}^{\mathbb{N}}$ is not dominating and hence $\mathsf{non}(\text{Menger}) = \mathfrak{d}$. Clearly, $\mathsf{non}(\text{star-Menger}) = \mathfrak{d}$.

Corollary 3.4. non(*locally star-Menger*) $= \mathfrak{d}$.

Rephrasing [25, Proposition 1.7], we obtain the following.

Corollary 3.5. *Every star-Lindelöf space of cardinality less than $\mathfrak{d}$ is locally star-Menger.*

We now turn our attention to observe how large the extent $e(X)$ of a locally star-Menger space X can be.

Example 3.6.
(1) Let $X = \Psi(\mathcal{A})$ be the Isbell-Mrówka space with $|\mathcal{A}| = \mathfrak{c}$. Then X is a Tychonoff locally star-Menger space which is not star-Menger. Since $\mathcal{A}$ is a discrete and closed subspace of X, we have $e(X) \geq \mathfrak{c}$.

(2) Let κ be any infinite cardinal and D be a set of cardinality $\kappa \geq \aleph_0$. Take $X = [0, \kappa^+) \cup D$. We define a topology on X as follows. $[0, \kappa^+)$ has the usual order topology and $[0, \kappa^+)$ is an open subspace of X with a basic neighbourhood of a point $d \in D$ is of the form $O_\beta(d) = \{d\} \cup (\beta, \kappa^+)$, where $\beta < \kappa^+$. Clearly, X is T_1. We now show that X is locally star-Menger. Let $x \in X$. If $x \in [0, \kappa^+)$, then $[0, \kappa^+)$ is the required star-Menger open subspace as it is starcompact. Also, for $x \in D$, choose an open set $O_\beta(x) = \{x\} \cup (\beta, \kappa^+)$ for some $\beta < \kappa^+$. By Lemma 2.2, $\{x\} \cup [0, \kappa^+)$ is a star-Menger subspace of X with $O_\beta(x) \subseteq \{x\} \cup [0, \kappa^+)$. Thus X is locally star-Menger. Since D is a discrete closed subspace of X, we have $e(X) \geq \kappa$.

By [25, Proposition 2.12], if Y is a closed and discrete subspace of a normal star-Menger space X, then $|Y| < \mathfrak{d}$, i.e. $e(X) \leq \mathfrak{d}$ holds. This relation can not be outstretched to locally star-Menger cases.

Example 3.7. Let X be a discrete space with $|X| > \mathfrak{c}$. Then X is normal and locally star-Menger. If we take $Y = X$, then Y is a closed and discrete subspace of X with $|Y| > \mathfrak{c}$ and hence $e(X) > \mathfrak{c}$ holds.

3.2. Some observations on Pixley-Roy hyperspaces

The first study of star-Mengerness in Pixley-Roy spaces was appeared in [18] (see also [25]). For recent developments of the study of star selection principles in hyperspaces, see [10]. We begin with an illustrative example (Proposition 3.9) of a Pixley-Roy space which is locally star-Menger without being star-Menger. The following result is required.

Proposition 3.8 (cf. [25, Corollary 4.8]). *If* $\mathsf{PR}(X)$ *is star-Menger, then* $|X| < \mathfrak{c}$ *holds. Hence under CH,* $\mathsf{PR}(X)$ *is star-Menger if and only if* X *is countable.*

Recall that a collection $\mathcal{A}$ of sets is said to be a Δ-system if and only if there is a fixed set R, called the root of the Δ-system, such that for any two distinct members $A, B \in \mathcal{A}$, $A \cap B = R$ [32].

Proposition 3.9. *For each positive integer* n *and each cardinal* $\kappa \geq \mathfrak{c}$*,* $PR_{\leq n}(\kappa_{\mathsf{cof}})$ *is locally star-Menger but not star-Menger, where* κ_{cof} *is the cardinal* κ *with the cofinite topology.*

Proof. Choose a cardinal $\kappa \geq \mathfrak{c}$ and let $F \in PR_{\leq n}(\kappa_{\mathsf{cof}})$. Clearly, $[F, \kappa]$ is open in $\mathsf{PR}(\kappa_{\mathsf{cof}})$ containing F. We first show that $[F, \kappa]$ is countably compact. Let $\{F_m : m \in \mathbb{N}\}$ be a subset of $[F, \kappa]$. Without loss of generality assume that $|F_m| = k$ (finite) for each m and hence $\{F_m : m \in \mathbb{N}\}$ forms a Δ-system with root A. Clearly, A is a limit point of $\{F_m : m \in \mathbb{N}\}$ in $[F, \kappa]$ and hence $[F, \kappa]$ is countably compact. Now $[F, \kappa]$ is a compact subspace of $\mathsf{PR}(\kappa_{\mathsf{cof}})$ as $[F, \kappa]$ is metacompact. Clearly, $[F, \kappa] \cap PR_{\leq n}(\kappa_{\mathsf{cof}})$ is a compact subspace of $PR_{\leq n}(\kappa_{\mathsf{cof}})$ and hence $[F, \kappa] \cap PR_{\leq n}(\kappa_{\mathsf{cof}})$ is star-Menger. Since $[F, \kappa] \cap PR_{\leq n}(\kappa_{\mathsf{cof}})$ is open in $PR_{\leq n}(\kappa_{\mathsf{cof}})$, $PR_{\leq n}(\kappa_{\mathsf{cof}})$ is locally star-Menger.

To complete the proof we show that $PR_{\leq n}(\kappa_{\mathsf{cof}})$ is not star-Menger. Assume on the contrary that $PR_{\leq n}(\kappa_{\mathsf{cof}})$ is star-Menger. Since $\cup_{n \in \mathbb{N}} PR_{\leq n}(\kappa_{\mathsf{cof}}) = \mathsf{PR}(\kappa_{\mathsf{cof}})$, $\mathsf{PR}(\kappa_{\mathsf{cof}})$ is star-Menger by Lemma 2.2. It follows that $\kappa < \mathfrak{c}$ (see Proposition 3.8), a contradiction. $\square$

We now introduce another variety of the star-Menger property that will be useful in this context.

Definition 3.10. Let X be a space. The Pixley-Roy space $\mathsf{PR}(X)$ is said to be effectively locally star-Menger if for each $F \in \mathsf{PR}(X)$ there exists an open set U in X such that $F \in \mathsf{PR}(U)$ and $\mathsf{PR}(U)$ is a star-Menger subspace of $\mathsf{PR}(X)$.

It is immediate that the effectively locally star-Menger property of $\mathsf{PR}(X)$ lies between the star-Menger and locally star-Menger property.

Proposition 3.11. *For a space* X *the following assertions are equivalent.*

(1) $\mathsf{PR}(X)$ *is effectively locally star-Menger.*
(2) *For each* $F \in \mathsf{PR}(X)$ *there exist an open set* U *in* X *and a set* $Y \subseteq X$ *such that* $\mathsf{PR}(Y)$ *is a star-Menger subspace of* $\mathsf{PR}(X)$ *and* $F \in [F, U] \subseteq \mathsf{PR}(Y)$.

Proof. (2) $\Rightarrow$ (1). Let $F \in \mathsf{PR}(X)$. Choose an open set U in X and a $Y \subseteq X$ such that $\mathsf{PR}(Y)$ is star-Menger and $F \in [F, U] \subseteq \mathsf{PR}(Y)$. Since U is open in X, $\mathsf{PR}(U)$ is clopen in $\mathsf{PR}(X)$. Thus $F \in \mathsf{PR}(U) \subseteq \mathsf{PR}(Y)$ and consequently $\mathsf{PR}(U)$ is the required star-Menger subspace of $\mathsf{PR}(X)$. $\square$

We now make few observations about this local variation of the star-Menger property. Since star-Menger implies both effectively locally star-Menger and locally star-Menger, we restate [25, Proposition 4.2] into our language in the following.

Proposition 3.12. *If* X *is a cosmic space of cardinality less than* $\mathfrak{d}$*, then every finite power of* $\mathsf{PR}(X)$ *is effectively locally star-Menger and also locally star-Menger.*

Corollary 3.13. *If* X *is a subspace of reals with cardinality less than* $\mathfrak{d}$*, then every finite power of* $\mathsf{PR}(X)$ *is effectively locally star-Menger and locally star-Menger as well.*

Also, the following result is useful.

Lemma 3.14 (cf. [25, Theorem 4.12])**.** *If* $\mathsf{PR}(X)$ *is star-Menger, then every finite power of* X *is Menger.*

In the following example we show that there exists a subspace X of reals with cardinality $\mathfrak{d}$ such that $\mathsf{PR}(X)$ is not effectively locally star-Menger.

Example 3.15. We first choose a subspace X of reals with cardinality $\mathfrak{d}$ which is not Menger. Now, if possible, suppose that $\mathsf{PR}(X)$ is effectively locally star-Menger. For each $x \in X$ choose an open set U_x containing x and a subset Y_x of X such that the subspace $\mathsf{PR}(Y_x)$ of $\mathsf{PR}(X)$ is star-Menger and $[\{x\}, U_x] \subseteq \mathsf{PR}(Y_x)$. By Lemma 3.14, each Y_x is Menger. Since X is Lindelöf, choose a countable collection $\{U_{x_n} : n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}} U_{x_n}$ and hence $X = \cup_{n\in\mathbb{N}} Y_{x_n}$. It now follows that X is Menger, a contradiction. Thus $\mathsf{PR}(X)$ is not effectively locally star-Menger.

Remark 3.16. In view of Example 3.15 and Corollary 3.13, we have the following equality.

$$\mathfrak{d} = \min\{|X| : X \subseteq \mathbb{R} \text{ and } \mathsf{PR}(X) \text{ is not effectively locally star-Menger}\}.$$

Theorem 3.17. *If* $\mathsf{PR}(X)$ *is effectively locally star-Menger, then every finite power of* X *is locally star-Menger.*

Proof. Let $m \in \mathbb{N}$ and $(x_1, x_2, \ldots, x_m) \in X^m$. Choose $F = \{x_1, x_2, \ldots, x_m\}$. Then $F \in \mathsf{PR}(X)$ and since $\mathsf{PR}(X)$ is effectively locally star-Menger, there exist an open set U in X and a set $Y \subseteq X$ such that $\mathsf{PR}(Y)$ is a star-Menger subspace of $\mathsf{PR}(X)$ and $F \in [F, U] \subseteq \mathsf{PR}(Y)$. Clearly, U^m is open in X^m and by Lemma 3.14, Y^m is a star-Menger subspace of X^m with $(x_1, x_2, \ldots, x_m) \in U^m \subseteq Y^m$. Thus X^m is locally star-Menger. $\square$

Proposition 3.18. *Let* X *be a space with the condition that* $\mathsf{PR}(Y)$ *is star-Menger for every star-Menger subspace* Y *of* X*. If* X *is locally star-Menger, then* $\mathsf{PR}(X)$ *is effectively locally star-Menger.*

Proof. Let $F \in \mathsf{PR}(X)$. For each $x \in F$ choose an open set U_x and a star-Menger subspace M_x of X such that $x \in U_x \subseteq M_x$. Define $U = \cup_{x\in F} U_x$ and $M = \cup_{x\in F} M_x$. Clearly, U is open and M is star-Menger with $F \subseteq U \subseteq M$. It follows that $F \in [F, U] \subseteq \mathsf{PR}(M)$, where $\mathsf{PR}(M)$ is a star-Menger subspace of $\mathsf{PR}(X)$. Thus $\mathsf{PR}(X)$ is effectively locally star-Menger. $\square$

Proposition 3.19. *If* Y *is open in* X *and* $\mathsf{PR}(X)$ *is effectively locally star-Menger, then also* $\mathsf{PR}(Y)$ *is effectively locally star-Menger.*

Proof. Let $F \in \mathsf{PR}(Y)$. Choose an open set U in X such that $F \in \mathsf{PR}(U)$ and $\mathsf{PR}(U)$ is a star-Menger subspace of $\mathsf{PR}(X)$. Also, $\mathsf{PR}(Y)$ is clopen in $\mathsf{PR}(X)$ as Y is open. Clearly, $\mathsf{PR}(U) \cap \mathsf{PR}(Y) = \mathsf{PR}(U \cap Y)$ is the required star-Menger subspace of $\mathsf{PR}(Y)$. $\square$

Proposition 3.20. *Let* $f : X \to Y$ *be an open continuous mapping from a space* X *onto a space* Y*. If* $\mathsf{PR}(X)$ *is effectively locally star-Menger, then* $\mathsf{PR}(Y)$ *is also effectively locally star-Menger.*

Proof. Define a mapping $\varphi : \mathsf{PR}(X) \to \mathsf{PR}(Y)$ by $\varphi(F) = f(F)$. Clearly, φ is surjective and continuous. Now let $F \in \mathsf{PR}(Y)$. Choose a $F' \in \mathsf{PR}(X)$ such that $\varphi(F') = F$. Since $\mathsf{PR}(X)$ is effectively locally star-Menger, there exists an open set U in X such that $F' \in \mathsf{PR}(U)$ and $\mathsf{PR}(U)$ is a star-Menger subspace of $\mathsf{PR}(X)$. Now $F \in \varphi(\mathsf{PR}(U))$ and $\varphi(\mathsf{PR}(U))$ is a star-Menger subspace of $\mathsf{PR}(Y)$. Also, $\varphi(\mathsf{PR}(U)) = \mathsf{PR}(f(U))$ and $f(U)$ is open in Y. Consequently, $\mathsf{PR}(Y)$ is effectively locally star-Menger. $\square$

We end this section with the following remark on the extent of effectively locally star-Menger Pixley-Roy spaces.

Remark 3.21. First observe that for any space X, we have $e(\mathsf{PR}(X)) = |X|$, since the one-point subsets of X form a closed discrete subspace of $PR(X)$. By Proposition 3.8, $e(\mathsf{PR}(X)) < \mathfrak{c}$ holds if $\mathsf{PR}(X)$ is star-Menger. The situation is quite different for effectively locally star-Menger Pixley-Roy spaces. If κ is any infinite cardinal and D is a discrete space with cardinality κ, then $\mathsf{PR}(D)$ is effectively locally star-Menger with $e(\mathsf{PR}(D)) = \kappa$, i.e. the extent can be sufficiently large in this case.

4. Preservation like properties

4.1. Preservation properties

We now exhibit some preservation properties in locally star-Menger spaces. Let κ be an infinite cardinal. A space X is said to be κ-concentrated (see [7, 29]) on a set $Y \subseteq X$ if $|X \setminus U| < \kappa$ for any open set U in X containing Y. Let $\mathrm{cf}(\mathfrak{d})$ be the cofinality of $\mathfrak{d}$.

Assume that $\mathrm{cf}(\mathfrak{d}) < \mathfrak{d}$. By [29, Theorem 2.10], there exist two sets of reals X and Y such that $|X| < \mathfrak{d}$ and Y is $\mathfrak{d}$-concentrated, but $X \times Y$ is not Menger. We use this fact to show that product of two locally star-Menger spaces need not be locally star-Menger.

Example 4.1. Assume that $\mathrm{cf}(\mathfrak{d}) < \mathfrak{d}$. First observe that any subset of reals with cardinality less than $\mathfrak{d}$ is Menger. Also, any $\mathfrak{d}$-concentrated set of reals is Menger (see [29]). We use Proposition 3.3 to conclude that there exist two locally star-Menger sets of reals X and Y such that $X \times Y$ is not locally star-Menger.

Recall that the product of a star-Menger space with a compact space is again star-Menger [17, Theorem 2.13]. If we replace 'compact' by 'locally compact', then the product need not be star-Menger, even if we consider a star-Menger Tychonoff space instead of a star-Menger space. Consider the Tychonoff star-Menger space $X = (aD \times [0, \mathfrak{c}^+)) \cup (D \times \{\mathfrak{c}^+\})$, where D is a discrete space of cardinality $\mathfrak{c}$ and aD is its one point compactification (see [27, Example 2.2]). The Isbell-Mrówka space $Y = \Psi(\mathcal{A})$ is locally compact. If we assume that $|\mathcal{A}| = \mathfrak{c}$, then Y is not star-Menger by Lemma 2.3. Clearly, $X \times Y$ is not star-Menger as star-Menger property is preserved under continuous mappings. Still we prove the following result.

Proposition 4.2. *If X is locally star-Menger and Y is locally compact, then $X \times Y$ is locally star-Menger.*

Proof. Let $(x, y) \in X \times Y$. Choose an open set U in X and a star-Menger subspace M of X such that $x \in U \subseteq M$. Also, we can choose an open set V and a compact subspace K of Y such that $y \in V \subseteq K$. Clearly, $X \times Y$ is locally star-Menger as $M \times K$ is star-Menger and $(x, y) \in U \times V \subseteq M \times K$. □

It is to be noted that locally star-Menger property is not hereditary, not even in regular spaces. The real line $\mathbb{R}$ with usual topology is regular and locally star-Menger. The subspace $\mathbb{S}$ of all irrational numbers is not star-Menger. By Proposition 3.3, $\mathbb{S}$ can not be locally star-Menger, as it is Lindelöf. However we prove the following.

Proposition 4.3. *Locally star-Menger property is hereditary for clopen subspaces.*

Proof. Let Y be a clopen subspace of a locally star-Menger space X. Let $y \in Y$ and choose an open set U and a star-Menger subspace M of X such that $y \in U \subseteq M$. Clearly, $Y \cap M$ is a clopen subset of M and hence $Y \cap M$ is star-Menger. It now follows that Y is locally star-Menger. □

We now observe preservation of locally star-Menger property under certain mappings for the next couple of results. We first recall the following definitions (from [21]). A surjective continuous mapping $f : X \to Y$ is said to be

(1) weakly perfect if f is closed and $f^{-1}(y)$ is Lindelöf for every $y \in Y$.

(2) bi-quotient if whenever $y \in Y$ and $\mathcal{U}$ is a cover of $f^{-1}(y)$ by open sets in X, then finitely many $f(U)$ with $U \in \mathcal{U}$ cover some open set containing y in Y.

It is immediate that surjective continuous open (and also perfect) mappings are bi-quotient.

Theorem 4.4. *Let X be locally star-Menger.*

(1) *If $f : X \to Y$ is weakly perfect, then Y is locally star-Menger.*
(2) *If $f : X \to Y$ is bi-quotient, then Y is locally star-Menger.*

Proof. (1). Let $y \in Y$. For each $x \in f^{-1}(y)$ choose an open set U_x and a star-Menger subspace M_x of X such that $x \in U_x \subseteq M_x$. Since $f^{-1}(y)$ is Lindelöf, there is a countable subcollection $\{U_{x_n} : n \in \mathbb{N}\}$ of $\{U_x : x \in f^{-1}(y)\}$ that covers $f^{-1}(y)$. Consequently, $f^{-1}(y) \subseteq \cup_{n\in\mathbb{N}} M_{x_n}$. Moreover $y \in Y \setminus f(X \setminus \cup_{n\in\mathbb{N}} U_{x_n}) \subseteq f(\cup_{n\in\mathbb{N}} M_{x_n})$. Since f is closed, $Y \setminus f(X \setminus \cup_{n\in\mathbb{N}} U_{x_n})$ is an open set in Y containing y. Also, since $\cup_{n\in\mathbb{N}} M_{x_n}$ is star-Menger, $f(\cup_{n\in\mathbb{N}} M_{x_n})$ is a star-Menger subspace of Y.
(2). For each $x \in X$ choose an open set U_x and a star-Menger subspace M_x of X such that $x \in U_x \subseteq M_x$. Let $y \in Y$ and consider $f^{-1}(y)$. Since f is a bi-quotient mapping, there exist a finite subset $\{U_{x_i} : 1 \le i \le k\}$ of $\{U_x : x \in X\}$ and an open set V containing y in Y such that $V \subseteq \cup_{i=1}^k f(U_{x_i})$. Clearly, $y \in \operatorname{Int} f(\cup_{i=1}^k U_{x_i})$ and $f(\cup_{i=1}^k M_{x_i})$ is a star-Menger subspace of Y with $\operatorname{Int} f(\cup_{i=1}^k U_{x_i}) \subseteq f(\cup_{i=1}^k M_{x_i})$. Thus Y is locally star-Menger. □

Corollary 4.5. *If $f : X \to Y$ is a perfect (or an open continuous) mapping from a locally star-Menger space X onto a space Y, then Y is locally star-Menger.*

Next we define star-Menger covering mapping.

Definition 4.6. A continuous mapping $f : X \to Y$ is said to be a star-Menger covering mapping if for each star-Menger subspace N of Y, there is a star-Menger subspace M of X such that $f(M) = N$.

For example, the projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are star-Menger covering mappings.

Proposition 4.7. *Locally star-Menger property is an inverse invariant under injective star-Menger covering mappings (as well as under injective closed open continuous mappings).*

Proof. We only give proof for the case of star-Menger covering mappings. Let $f : X \to Y$ be an injective star-Menger covering mapping from a space X onto a locally star-Menger space Y. Let $x \in X$. Choose an open set U in Y and a star-Menger subspace $M \subseteq Y$ such that $f(x) \in U \subseteq M$. We can also choose a star-Menger subspace N of X such that $f(N) = M$, i.e. $N = f^{-1}(M)$. Clearly, $x \in f^{-1}(U) \subseteq N$ and hence X is locally star-Menger. □

Observe that the locally star-Menger property is not an inverse invariant under open, as well as under star-Menger covering mappings.

Example 4.8. By Corollary 3.4, there is a set of reals X with cardinality $\mathfrak{d}$ which is not locally star-Menger. The Isbell-Mrówka space $Y = \Psi(\mathcal{A})$ is locally star-Menger. By considering the projection onto the first component (and also by Corollary 4.5), we obtain $X \times Y$ is not locally star-Menger. The projection onto second component witnesses that the locally star-Menger property is not an inverse invariant under open, as well as under star-Menger covering mappings.

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. Let $X = \oplus_{\alpha\in\Lambda} X_\alpha$ be the topological sum which is defined as $\oplus_{\alpha\in\Lambda} X_\alpha = \cup_{\alpha\in\Lambda}\{(x, \alpha) : x \in X_\alpha\}$. For each $\alpha \in \Lambda$, let $\varphi_\alpha : X_\alpha \to X$ be defined by $\varphi_\alpha(x) = (x, \alpha)$. The topology on X is defined as follows.

A subset U of X is open in X if and only if $\varphi_\alpha^{-1}(U)$ is open in X_α for each $\alpha \in \Lambda$. Also, if for each $\alpha \in \Lambda$ the space X_α is homeomorphic to a fixed space Y, then the topological sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is homeomorphic to $Y \times \Lambda$, where Λ has the discrete topology.

Proposition 4.9. *Let Λ be an arbitrary index set.*

(1) *If $X = \cup_{\alpha\in\Lambda} Y_\alpha$, where each Y_α is an open locally star-Menger subspace of X, then X is locally star-Menger.*

(2) *The topological sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is locally star-Menger if and only if each X_α is locally star-Menger.*

Proof. We only present proof of (2). Let $X = \oplus_{\alpha\in\Lambda} X_\alpha$. Since each $X_\alpha \times \{\alpha\}$ is clopen in X, it follows that $X_\alpha \times \{\alpha\}$ is locally star-Menger. Clearly, each X_α is locally star-Menger. For the other direction let $y \in X$. Choose $x \in X_\beta$ for some $\beta \in \Lambda$ such that $y = (x, \beta)$. Also, choose an open set U and a star-Menger subspace M of X_β such that $x \in U \subseteq M$. Now $U' = U \times \{\beta\}$ and $M' = M \times \{\beta\}$ are respectively open and star-Menger subspace of X such that $y \in U' \subseteq M'$. Consequently, X is locally star-Menger. □

In contrast to the local version, the above result fails in star-Menger settings.

Example 4.10.

(1) Let $X = \Psi(\mathcal{A})$ be the Isbell-Mrówka space with $|\mathcal{A}| = \mathfrak{c}$. Since X is locally compact zero-dimensional and Hausdorff, there exists a cover $\{K_x : x \in X\}$ for X such that each K_x is an open compact subspace of X. Clearly, X is a union of open star-Menger subspaces, but X is not star-Menger by Lemma 2.3.

(2) Let X be the topological sum of ω_1 copies of $[0, 1]$. Then X is homeomorphic to $[0, 1] \times D$, where D is a discrete space with $|D| = \omega_1$. Observe that $[0, 1] \times D$ (and hence X) is not star-Menger.

Theorem 4.11. *If $X = \cup_{\alpha\in\Lambda} X_\alpha$, where each X_α is a closed locally star-Menger subspace of X and the collection $\{X_\alpha : \alpha \in \Lambda\}$ is locally finite in X, then X is locally star-Menger.*

Proof. Let $Y = \oplus_{\alpha\in\Lambda} X_\alpha$. By Proposition 4.9(2), Y is locally star-Menger. Define $f : Y \to X$ by $f(x, \alpha) = x$ and also for each α define $\varphi_\alpha : X_\alpha \to Y$ by $\varphi_\alpha(x) = (x, \alpha)$. We now show that f is a perfect mapping. Clearly, f is continuous. Let F be closed in Y. Since for each α $\varphi_\alpha^{-1}(F)$ is closed in X, it follows that $f(F) = \cup_{\alpha\in\Lambda} \varphi_\alpha^{-1}(F)$ is also closed in X as $\{X_\alpha : \alpha \in \Lambda\}$ is locally finite in X. Thus f is closed. Now let $x \in X$. Choose an open set V containing x in X that intersects only finitely many members of $\{X_\alpha : \alpha \in \Lambda\}$, say $\{X_{\alpha_i} : 1 \le i \le k\}$. Now $f^{-1}(x) = \oplus_{\{\alpha_i : 1\le i\le k\}} \{x\}$ and hence $f^{-1}(x)$ is a compact subspace of Y. Thus f is a perfect mapping and the conclusion now follows from Corollary 4.5. □

We end this section with a similar assertion in the context of a P-space. First we need the following observation.

Lemma 4.12 (Folklore)**.** *Let X be a P-space. If $\{X_\alpha : \alpha \in \Lambda\}$ is a locally countable family of closed sets in X, then $\cup_{\alpha\in\Lambda} X_\alpha$ is also closed in X.*

Theorem 4.13. *If $X = \cup_{\alpha\in\Lambda} X_\alpha$ is a P-space, where each X_α is a closed locally star-Menger subspace of X and the collection $\{X_\alpha : \alpha \in \Lambda\}$ is locally countable in X, then X is locally star-Menger.*

4.2. Decomposition of locally star-Menger spaces

We say that a space X decomposes into the collection $\mathcal{D}$ of subsets of X if $\mathcal{D}$ forms a partition of X. The function $\varphi : X \to \mathcal{D}$ defined by $\varphi(x) = D_x$, where $x \in D_x$, is called the decomposition mapping. The topology on $\mathcal{D}$ is defined as follows. A subset $\mathcal{U}$ of $\mathcal{D}$ is open in $\mathcal{D}$ if and only if $\varphi^{-1}(\mathcal{U})$ is open in X. In this case we say that $\mathcal{D}$ is a decomposition space (or in short, a decomposition) of X [33]. The decomposition mapping $\varphi : X \to \mathcal{D}$ is clearly surjective and continuous. The natural decomposition in X of a mapping $f : X \to Y$ is the collection of disjoint sets $f^{-1}(y)$, $y \in Y$.

Definition 4.14. We say that a space X belongs to the class $\mathfrak{M}^\star$ if there exists a cover $\mathcal{M}$ of X consisting of star-Menger subspaces and a subset U of X is open if $U \cap M$ is open in M for each $M \in \mathcal{M}$ (or, equivalently a subset F of X is closed if $F \cap M$ is closed in M for each $M \in \mathcal{M}$).

Clearly, every locally star-Menger space belongs to the class $\mathfrak{M}^\star$.

Lemma 4.15. *Let $\mathcal{D}$ be a decomposition of X. If X belongs to the class $\mathfrak{M}^\star$, then $\mathcal{D}$ also belongs to the class $\mathfrak{M}^\star$.*

Proof. Choose a cover $\mathcal{M}$ of X consisting of star-Menger subspaces of X. Let $\varphi : X \to \mathcal{D}$ be the decomposition mapping. Clearly, $\varphi(\mathcal{M}) = \{\varphi(M) : M \in \mathcal{M}\}$ is a cover of $\mathcal{D}$ consisting of star-Menger subspaces. Let $\mathcal{U} \subseteq \mathcal{D}$ be such that $\mathcal{U} \cap \varphi(M)$ is open in $\varphi(M)$ for each $M \in \mathcal{M}$. The proof will be complete if we show that $\mathcal{U}$ is open in $\mathcal{D}$. Let $M \in \mathcal{M}$. Since $\varphi^{-1}(\mathcal{U} \cap \varphi(M)) = \varphi^{-1}(\mathcal{U}) \cap \varphi^{-1}(\varphi(M))$, it follows that $\varphi^{-1}(\mathcal{U}) \cap M = \varphi^{-1}(\mathcal{U} \cap \varphi(M)) \cap M$. It is easy to see that $\varphi^{-1}(\mathcal{U} \cap \varphi(M))$ is open in $\varphi^{-1}(\varphi(M))$. Also, since $M \subseteq \varphi^{-1}(\varphi(M))$, we have $\varphi^{-1}(\mathcal{U}) \cap M$ is open in M. Clearly, $\varphi^{-1}(\mathcal{U})$ is open in X as the choice of $M \in \mathcal{M}$ is arbitrary. Thus $\mathcal{U}$ is open in $\mathcal{D}$. □

Theorem 4.16. *Every member of the class $\mathfrak{M}^\star$ is homeomorphic to a decomposition of some locally star-Menger space.*

Proof. Let $\mathcal{M} = \{X_\alpha : \alpha \in \Lambda\}$ be a cover of X which witnesses $X \in \mathfrak{M}^\star$. Define $Y = \cup_{\alpha \in \Lambda} Y_\alpha$, where Y_α's are pairwise disjoint open star-Menger subspaces of Y such that for each α there is a homeomorphism $h_\alpha : Y_\alpha \to X_\alpha$. Clearly, Y is locally star-Menger. The mapping $f : Y \to X$ given by $f(y) = h_\alpha(y)$ for $y \in Y_\alpha$ is continuous and surjective. Also, since $X \in \mathfrak{M}^\star$, if $f^{-1}(V)$ is open in Y, then V is also open in X. Now $\mathcal{D} = \{f^{-1}(x) : x \in X\}$ is a decomposition of Y. The mapping that sends each element x of X to $f^{-1}(x)$ is clearly a homeomorphism. □

Corollary 4.17. *The class of all spaces which are obtained as a decomposition of locally star-Menger spaces is identical with the class $\mathfrak{M}^\star$.*

We now present a characterization of locally star-Menger spaces under bi-quotient mappings. First recall that bi-quotient image of a second countable space is again second countable.

Theorem 4.18. *A zero-dimensional Hausdorff space X is obtained as the image of a locally star-Menger metrizable space under a bi-quotient mapping if and only if X is locally star-Menger and locally metrizable.*

Proof. Let Y be locally star-Menger metrizable and $g : Y \to X$ be a bi-quotient mapping. By Theorem 4.4(2), X is locally star-Menger. We now show that X is locally metrizable. Since Y is locally star-Menger, choose an open cover $\mathcal{U} = \{V_y : y \in Y\}$ of Y such that $y \in V_y \subseteq M_y$, where M_y is star-Menger for each y. Let $x \in X$ and choose $y \in g^{-1}(x)$. Since g is a bi-quotient mapping, there exist a finite subset $\{V_{y_i} : 1 \leq i \leq k\}$ of $\mathcal{U}$ and an open set U in X containing x such that $U \subseteq \cup_{i=1}^k g(V_{y_i})$. Clearly, $M = \cup_{i=1}^k M_{y_i}$ is a metrizable Menger subspace of Y and hence M is second countable. Thus $g(M)$ is second countable and hence a metrizable subspace of X, as zero-dimensionality implies complete regularity. Thus U is the required metrizable subspace of X.

For the other direction, assume that X is locally star-Menger and locally metrizable. By Theorem 3.2, X has a base consisting of clopen star-Menger neighbourhoods. It is easy to observe that there exists a cover $\{X_\alpha : \alpha \in \Lambda\}$ of X such that for each α, X_α is a clopen metrizable star-Menger subspace of X. Similarly as in the proof of Theorem 4.16, we define a space Y as follows.

(1) $Y = \cup_{\alpha\in\Lambda} Y_\alpha$ with each Y_α is star-Menger and metrizable.
(2) For each α Y_α is open in Y and $Y_\alpha \cap Y_\beta = \emptyset$ for $\alpha \neq \beta$.
(3) For each α there exists a homeomorphism $h_\alpha : Y_\alpha \to X_\alpha$.

Clearly, such Y is metrizable and locally star-Menger as well. The mapping $f : Y \to X$ defined by $f(y) = h_\alpha(y)$ for $y \in Y_\alpha$ is open, continuous and surjective. Thus f is a bi-quotient mapping. □

Corollary 4.19. *A zero-dimensional Hausdorff space X is obtained as the image of a locally star-Menger metrizable space under an open continuous mapping if and only if X is locally star-Menger and locally metrizable.*

5. Remainder of locally star-Menger spaces

All spaces in this section are assumed to be Tychonoff. By a remainder of a space X we mean the subspace $bX \setminus X$ of a compactification bX of X. Recall that a space X is a p-space [2, 14] if in any (in some) compactification bX of X there exists a countable family $\{\mathcal{U}_n : n \in \mathbb{N}\}$ with for each n $\mathcal{U}_n$ is a collection of open sets in bX such that for each $x \in X$, $x \in \cap\{St(x, \mathcal{U}_n) : n \in \mathbb{N}\} \subseteq X$. It is well known that every metrizable space is a p-space [1, 3]. In [23], continuous images of Lindelöf p-spaces are called as Lindelöf Σ-spaces. A space X is said to be an s-space if there exists a countable open source for X in any (in some) compactification bX of X [5, 6]. It is well known that every Lindelöf p-space is an s-space [6]. Also, if X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space [3, Theorem 2.1].
In this section we present some observations about the remainder of locally star-Menger spaces. We start with a basic observation about p-spaces.

Proposition 5.1. *The property of being a p-space is hereditary for closed subspaces.*

Proof. Let F be a closed set in a p-space X. Choose a compactification bX of X and a countable family $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that satisfies the condition of p-spaces. Since $Y = \overline{F}^{bX}$ is compact and F is dense in Y, it follows that Y is a compactification of F. For each n let $\mathcal{V}_n = \{U \cap Y : U \in \mathcal{U}_n\}$. The collection $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is countable, where each $\mathcal{V}_n$ consists of open subsets of Y. Let $y \in F$. We now show that $y \in \cap\{St(y, \mathcal{V}_n) : n \in \mathbb{N}\} \subseteq F$. Clearly, $y \in St(y, \mathcal{U}_n)$ for each n. Thus for each n there exists a $U_n \in \mathcal{U}_n$ such that $y \in U_n$, i.e. $y \in U_n \cap Y \in \mathcal{V}_n$. Thus $y \in \cap\{St(y, \mathcal{V}_n) : n \in \mathbb{N}\}$. Observe that $St(y, \mathcal{V}_n) = St(y, \mathcal{U}_n) \cap Y$ holds for each n and also by the given condition $\cap\{St(y, \mathcal{U}_n) : n \in \mathbb{N}\} \cap Y \subseteq X \cap Y$, i.e. $\cap\{St(y, \mathcal{V}_n) : n \in \mathbb{N}\} \subseteq X \cap Y$. Since $F = \overline{F}^X = Y \cap X$, we can conclude that $y \in \cap\{St(y, \mathcal{V}_n) : n \in \mathbb{N}\} \subseteq F$. Hence F is a p-space. □

Let Y be a subspace of a space X. We say that X has the property $\mathcal{P}$ outside of Y if each closed set F in X with $Y \cap F = \emptyset$ has the property $\mathcal{P}$.

Theorem 5.2. *Let $\mathcal{U}$ be an open cover of a paracompact p-space X such that $\overline{U}$ is star-Menger for each $U \in \mathcal{U}$. If Y is a remainder of X, then there exists a compact subspace K of Y such that Y is a Lindelöf p-space outside of K (and hence Y is an s-space outside of K).*

Proof. Let $Y = bX \setminus X$, where bX is a compactification of X. For each $U \in \mathcal{U}$ choose an open set V_U in bX such that $V_U \cap X = U$. Let $W = \cup\{V_U : U \in \mathcal{U}\}$ and $K = bX \setminus W$. Then K is compact and W is open in bX with $X \subseteq W$ and $K \subseteq Y$. We now show that Y is a Lindelöf p-space outside of K. Let F be a closed set in Y such that $K \cap F = \emptyset$. Clearly, $\overline{F}^{bX} \subseteq W$ as $K \cap \overline{F}^{bX} = \emptyset$. Since $\{V_U : U \in \mathcal{U}\}$ is a cover of $\overline{F}^{bX}$ by open sets in bX and $\overline{F}^{bX}$ is compact, we can find a finite set $\{V_{U_i} : 1 \leq i \leq k\}$ such that

$\overline{F}^{bX} \subseteq \cup_{i=1}^{k} V_{U_i}$. Now $M = \cup_{i=1}^{k} \overline{U}_i^X$ is a closed star-Menger subspace of X and hence is a Menger subspace of X (since X is paracompact). By Proposition 5.1, M is a p-space. If $Z = \overline{M}^{bX}$, then Z is compact and also M is dense in Z. Clearly, Z is a compactification of M and $Z \cap Y$ is the remainder of M in Z. Thus $Z \cap Y$ being a remainder of a Lindelöf p-space, is again a Lindelöf p-space. Observe that $\cup_{i=1}^{k} V_{U_i} \cap \overline{X}^{bX} \subseteq \overline{\cup_{i=1}^{k} V_{U_i} \cap X}^{bX}$, i.e. $\cup_{i=1}^{k} V_{U_i} \subseteq \overline{\cup_{i=1}^{k} U_i}^{bX} \subseteq \overline{M}^{bX} = Z$, i.e. $\overline{F}^{bX} \subseteq Z$. Since $F = \overline{F}^{Y} = \overline{F}^{bX} \cap Y$, we have $F \subseteq Z \cap Y$. Evidently F is closed in $Z \cap Y$ and hence F is a Lindelöf p-space. This completes the proof. □

Corollary 5.3. *For a locally star-Menger space X the following statements hold.*

(1) *If X is a zero-dimensional paracompact p-space, then for every remainder Y of X there exists a compact subspace K of Y such that Y is a Lindelöf p-space outside of K (and hence Y is an s-space outside of K).*

(2) *If X is a metrizable space, then for every remainder Y of X there exists a compact subspace K of Y such that Y is a Lindelöf p-space outside of K (and hence Y is an s-space outside of K).*

Recall that a space X is said to be homogeneous if for any $x, y \in X$ there exists a homeomorphism $f : X \to X$ such that $f(x) = y$.
The result that if a space X is the union of a finite collection of its closed s-subspaces, then X is an s-space (see [31]) will be used in our next finding.

Theorem 5.4. *Let $\mathcal{U}$ be an open cover of a paracompact p-space X such that $\overline{U}$ is star-Menger for each $U \in \mathcal{U}$. If X has a homogeneous remainder Y, then Y is an s-space.*

Proof. By Theorem 5.2, we can find a compact subspace K of Y such that Y is a Lindelöf p-space outside of K. If $Y = K$, then the proof follows. Otherwise for each $y \in Y \setminus K$ there exists an open set U_y in Y such that $y \in U_y \subseteq \overline{U_y}^Y \subseteq Y \setminus K$ as $Y \setminus K$ is open in Y. Clearly, each such $\overline{U_y}^Y$ is a Lidelöf p-space. We now show that for each $x \in Y$ there exists an open set V in Y such that $\overline{V}^Y$ is a Lindelöf p-space. Let $x \in Y$ and fix a $y \in Y \setminus K$. Let $f : Y \to Y$ be a homeomorphism such that $f(y) = x$. Now choose an open set $y \in U_y$ in Y with $\overline{U_y}^Y$ is a Lindelöf p-space. Clearly, $V = f(U_y)$ is the required open set in Y. Consequently, we obtain a cover $\mathcal{W}$ of K by open sets in Y such that for each $W \in \mathcal{W}$, $\overline{W}^Y$ is an s-space. Choose a finite set $\{W_i : 1 \leq i \leq k\} \subseteq \mathcal{W}$ such that $K \subseteq \cup_{i=1}^{k} W_i$. It is now clear that $Y \setminus \cup_{i=1}^{k} W_i$ is an s-space. Thus $Y = \cup_{i=1}^{k} \overline{W_i}^Y \cup (Y \setminus \cup_{i=1}^{k} W_i)$ is also an s-space. □

Corollary 5.5. *For a locally star-Menger space X the following statements hold.*

(1) *If X is a zero-dimensional paracompact p-space with a homogeneous remainder Y, then Y is an s-space.*

(2) *If X is a metrizable space with a homogeneous remainder Y, then Y is an s-space.*

For the next result we need the following observation about s-spaces from [4].

Theorem 5.6 ([4, Theorem 2.7])**.** *If X is an s-space, then any (some) remainder of X in a compactification of X is a Lindelöf Σ-space.*

Theorem 5.7. *Let $\mathcal{U}$ be an open cover of a paracompact p-space X such that $\overline{U}$ is star-Menger for each $U \in \mathcal{U}$. If X has a homogeneous remainder, then $X = L \cup Z$, where L is a closed Lindelöf Σ-subspace and Z is an open locally compact subspace of X.*

Proof. Let $Y = bX \setminus X$ be a homogeneous remainder of X, where bX is a compactification of X. By Theorem 5.4, Y is an s-space. Also, $bY = \overline{Y}^{bX}$ is a compactification of Y. Now choose $L = bY \cap X$. Clearly, L is closed in X and $L = bY \setminus Y$, i.e. L is the remainder of Y

in bY. Again by Theorem 5.6, L is a Lindelöf Σ-subspace of X. Since $Z = bX \setminus bY$ is open in bX, for each $x \in Z$ there exists an open set U_x in bX such that $x \in U_x \subseteq \overline{U_x}^{bX} \subseteq Z$. It now follows that Z is locally compact and $X = L \cup Z$. □

Corollary 5.8. *Let $\mathcal{U}$ be an open cover of a paracompact p-space X such that $\overline{U}$ is star-Menger for each $U \in \mathcal{U}$. If X has a homogeneous remainder and in addition X is nowhere locally compact, then X is a Lindelöf Σ-space.*

Proof. We have $X = L \cup Z$, where L is a closed Lindelöf Σ-subspace and Z is an open locally compact subspace of X. Clearly, $Z = \emptyset$ as X is nowhere locally compact. □

Corollary 5.9. *For a locally star-Menger space X the following statements hold.*

(1) *If X is a zero-dimensional paracompact p-space with a homogeneous remainder, then $X = L \cup Z$, where L is a closed Lindelöf Σ-subspace and Z is an open locally compact subspace of X.*

(2) *If X is a metrizable space with a homogeneous remainder, then $X = L \cup Z$, where L is a closed Lindelöf Σ-subspace and Z is an open locally compact subspace of X.*

Corollary 5.10. *For a locally star-Menger space X the following statements hold.*

(1) *If X is a zero-dimensional nowhere locally compact paracompact p-space with a homogeneous remainder, then X is a Lindelöf Σ-space.*

(2) *If X is a nowhere locally compact metrizable space with a homogeneous remainder, then X is a Lindelöf Σ-space.*

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