Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (3) (2022), 632 – 645
DOI : 10.15672/hujms.801550

Research Article

A mathematical interpretation on special tube surfaces in Galilean 3-space

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Abstract

In this paper, we study the special tube surfaces generated by rectifying curves with respect to the Darboux frame in terms of the geodesic curvature, the normal curvature and the geodesic torsion in Galilean 3-space. During this study we establish some definite results of geodesics on specific tube surfaces with the help of Clairauts theorem in detail and we compute the Gaussian curvature and the mean curvature of the special tube surfaces with respect to the Darboux frame. After that, considering the geodesic conditions and the curvatures of the special tube surface, we give some theorems for the rectifying curves with $v-$parameter (and $w-$parameter) being a geodesic curve and an asymptotic curve, respectively.

Mathematics Subject Classification (2020). 53A35, 53A05, 53C22

Keywords. Galilean space, tube surface, rectifying curves, Darboux frame, geodesic curve

1. Introduction

Recently, curves and surfaces in Galilean space have been a current research topic for many researchers. For example, the rectifying curves play some important roles in mechanics, kinematics as well as in defining the curve of constant precession, and the position vector of a rectifying curve is always in the direction of the Darboux vector.

Geodesics are widely studied in Riemannian geometry through metric geometry and general relativity. In other words, the curves with stationary arc length between two given points X and Y are called the geodesic lines, and they are determined by the solutions of geodesic differential equations, to construct these differential equations we shall use variational calculus and Lagrange equations. Also, geodesic equations are given with constancy of motion in the form of energy with many approaches externalizing the significant use of energy, as introduced in many books [12, 14, 16].

In considering the mathematical problem of geodesics on the tube surface generated by rectifying curve with the Darboux frame, there is an important advantage conceptually that derives from taking a physicist's point of view by interpreting parametrized geodesics as the paths traced out in time by the motion of a point on the tube surface by identifying

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Received: 29.09.2020; Accepted: 19.11.2021

the time parameter. This combination of the constants of the motion is of course also constant along a geodesic. The existence of this constant is a conclusion of the one-parameter rotational group of symmetries of the tube surface, like this, a constant of the movement introduces a new thing when the surface is invariant under any one-parameter group of symmetries, which is seen in the variational approximate to the geodesic equations easily. Mathematically, this quantity is a constant obtained by Clairaut for geodesic movement on the surface defined in a coordinate system adapted to this one-parameter group of symmetries [14].

Many studies of tube surfaces, including rectifying curves, the Darboux frame, geodesic curve, Mean curvature, Gaussian curvature, have received much attention from our researchers. Among them, we can cite our work [4], we described the rotational surfaces using curves and matrices which are the subgroups of rotating a selected axis in Galilean 4-space. We also refer to [19]. We examined the tube surfaces generated by the curve in Galilean 3-space and gave certain results of describing the geodesics on the surfaces [3,5]. Besides, we obtained conditions being geodesic on the tubular surface with the help of Clairauts theorem. We also studied the linear Weingarten surfaces and HK-quadric surfaces, harmonic surfaces using the Gaussian and mean curvatures of tubular surfaces generated by rectifying curves in Galilean 3-space [2]. In our study [6] we expressed the specific kinetic energy, the specific angular momentum, and conditions being geodesic on rotational surface generated by a magnetic curve with the help of Clairaut's theorem with the Killing magnetic field. We also refer to [7]. Dede defined the tubular surfaces and the differential properties of tube surfaces in Galilean space [8]. Ali determined the position vector ofan arbitrary curve with respect to the Frenet frame in Galilean 3-space. Also, the author deduced in terms of the curvature and torsion, the natural representation of the position vector of an arbitrary curve being a plane curve, helix, general helix, Salkowski curves and anti-Salkowski curves in Galilean space, respectively [1]. Milin-Šipuš and Divjak [13] developed the local theory of surfaces immersed in the pseudo-Galilean space, a special type of Cayley-Klein spaces, and they studied surfaces of constant curvatures. Kasap and et al. [10], analyzed a family of surfaces from a given space-like(or time-like) geodesic curve using the Frenet frame of the curve in Minkowski space, and they expressed the surface family of a linear combination of the components of this framework. Also, they gave necessary and sufficient conditions for the coefficients to meet both geodesic and isoparametric requirements. Kim and Yoon [11] classified the ruled surfaces in L^3-spaces which satisfy some algebraic equations in terms of the Gaussian curvature. Karacan and et al. [9], Saad and et al. [17] studied the geodesics and surfaces in Minkowski space.

The aim of this work is to introduce the special tube surfaces produced by rectifying curves with respect to the Darboux frame in Galilean 3-space using Clairaut's theorem. We organise this study as follows: Section 2 provides some preliminary definitions and theorem useful for the reader. In Section 3, we characterize an isotropic rectifying curve generated by the Galilean Darboux frame in G_3. In Section 4, we give some characterizations for the specific tube surfaces generated by a rectifying curve with the Darboux frame in Galilean 3-space with help of the Clairaut's theorem. Furthermore, by using the Gaussian and the mean curvatures of specific tube surfaces and the Euler-Lagrange equations, we investigate the relation between the rectifying curves with $v-$parameter (and $w-$parameter) and special curves such as geodesics and asymptotic curves on the special tube surfaces.

2. Preliminaries

The classical context of Euclidean space is the source of results that can be transferred to some other geometries. One way of describing new geometries is Cayley-Klein spaces. Projective spaces are expressed as P_nR, and this is expressed by an absolute shape that is a subset of P_nR which occurs as a sequence of quadric and planes 1. The projective space, P_nR, has invariants as the absolute shape definition for the subgroup of projections. It is called the Cayley-Klein space motion group. Thanks to the absolute shape, metric connections are defined and invariant under the motion group.

The scalar product of the vectors $U = (u_1, u_2, u_3)$, $V = (v_1, v_2, v_3)$ in G_3 is defined as follows

$$\langle U, V\rangle = \begin{cases} u_1 v_1 \,, & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0 \\ u_2 v_2 + u_3 v_3, & \text{if } u_1 = 0, v_1 = 0 \end{cases}. \tag{2.1}$$

The cross product is given in Galilean space as

$$U \times V = \begin{cases} (0, v_1 u_3 - v_3 u_1, v_2 u_1 - v_1 u_2)\,, & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0 \\ (v_3 u_2 - v_2 u_3, 0, 0)\,, & \text{if } u_1 = 0, \; v_1 = 0 \end{cases}. \tag{2.2}$$

Let $\beta : I \subset \mathbb{R} \to G_3$ be a curve parametrized by arc length with curvatures $\kappa > 0, \tau$. Then, for the curve $\beta(w) = (w, y(w), z(w))$ the vectors of the Frenet-Serret frame are defined by

$$t(w) = \beta'(w) = (1, y'(w), z'(w)); n(w) = \frac{t'(w)}{\kappa(w)}; b(w) = \frac{n'(w)}{\tau(w)},$$

where the real-valued function $\kappa(w) = \| t'(w) \|$ is given as the first curvature of the curve β, the second curvature function is defined as $\tau(w) = \| n'(w) \|$. For the curve in G_3, Frenet-Serret equations can be written as follows

$$t' = \kappa n, n' = \tau b, b' = -\tau n. \tag{2.3}$$

Furthermore, a surface $\Theta = \Theta(w, v)$ in G_3 is given by

$$\Theta(w, v) = (x(w, v), y(w, v), z(w, v)). \tag{2.4}$$

Then, the unit isotropic normal vector field η on $\Theta(w, v)$ can be expressed as

$$\eta = \frac{\Theta_{,w} \times \Theta_{,v}}{\|\Theta_{,w} \times \Theta_{,v}\|}, \tag{2.5}$$

where the partial differentiation with respect to parameters w and v have the following forms

$$\Theta_{,w} = \frac{\partial \Theta(w, v)}{\partial w}; \Theta_{,v} = \frac{\partial \Theta(w, v)}{\partial v}. \tag{2.6}$$

The isotropic unit vector δ is expressed on the tangent plane of the surface given by

$$\delta = \frac{x_{,v}\Theta_{,w} - x_{,w}\Theta_{,v}}{\widetilde{w}}, \tag{2.7}$$

where $x_{,w} = \frac{\partial x(w,v)}{\partial w}$, $x_{,v} = \frac{\partial x(w,v)}{\partial v}$ and $\widetilde{w} = \|\Theta_{,w} \times \Theta_{,v}\|$.

Let us define

$$g_1 = x_{,w}, g_2 = x_{,v}, g_{ij} = g_i g_j; g^1 = \frac{x_{,v}}{w}; g^2 = \frac{x_{,w}}{w}; g^{ij} = g^i g^j; i, j = 1, 2 \tag{2.8}$$

$$h_{11} = \left\langle \Theta^*_{,w}, \Theta^*_{,w} \right\rangle, h_{12} = \left\langle \Theta^*_{,w}, \Theta^*_{,v} \right\rangle; h_{22} = \left\langle \Theta^*_{,v}, \Theta^*_{,v} \right\rangle, \tag{2.9}$$

where $\Theta^*_{,w}$ and $\Theta^*_{,v}$ are vector projections $\Theta_{,w}$ and $\Theta_{,v}$ onto the yz-plane. The distance square form ds^2 is given on the surface $\Theta(w, v)$ as

$$ds^2 = ds_1^2 + ds_2^2 = (g_1 dw + g_2 dv)^2 + \varepsilon \left(h_{11} dw^2 + 2h_{12} dw dv + h_{22} dv^2 \right), \tag{2.10}$$

herein

$$\varepsilon = \begin{cases} 0, & dw : dv \text{ non-isotropic} \\ 1, & dw : dv \text{ isotropic} \end{cases}, \tag{2.11}$$

[12, 14, 15].

The coefficients of ds^2 are denoted by g^*_{ij}, which can be represented in terms of g_i and h_{ij} as follows

$$\widetilde{w}^2 = g_1^2 h_{22} - 2g_1 g_2 h_{12} + g_2^2 h_{11}.$$

The Gaussian curvature and the mean curvature of a surface are defined by the coefficient L_{ij} of the second fundamental form, which are the normal components of $\Theta_{,i,j}(i, j = 1, 2)$

$$\Theta_{,i,j} = \sum_{k=1}^{2} \Gamma^k_{ij} \Theta_{,k} + L_{ij}\eta, \tag{2.12}$$

where Γ^k_{ij} is the Christoffel symbols and L_{ij} are given as follows

$$L_{ij} = \frac{1}{g_1} \left\langle g_1 \Theta^*_{,i,j} - g_{i,j}\Theta^*_{,1}, \eta \right\rangle = \frac{1}{g_2} \left\langle g_2 \Theta^*_{,i,j} - g_{i,j}\Theta^*_{,2}, \eta \right\rangle. \tag{2.13}$$

From this, the Gaussian curvature K and the mean curvature H are given as

$$K = \frac{L_{11}L_{22} - L_{12}^2}{\widetilde{w}^2}, H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{\widetilde{w}^2}, \tag{2.14}$$

[16].

Definition 2.1 ([18])**.** Let $\beta : I \subset \mathbb{R} \to S \subset G_3$ be a unit-speed curve and let $\overrightarrow{t}, \overrightarrow{Q}, \overrightarrow{n}$ be the Darboux frame fields. Then, the system $\{\overrightarrow{t}, \overrightarrow{Q}, \overrightarrow{n}\}$ is an orthonormal frame and the vectors of the Galilean Darboux frame are given as

$$t' = k_g Q + k_n n; Q' = \tau_g n; n' = -\tau_g Q, \tag{2.15}$$

here k_g and k_n are called the tangential and normal component of the curvature vector respectively. Also, these functions are called as the geodesic curvature and the normal curvature respectively. These vectors yield a unit tangent vector field t of the curve β on S and units of normal vector field n at the point $\beta(w)$ of β and $Q = n \times_{G_3} t$, the frame $\{t, Q, n\}$ is called the Darboux frame or the tangential-normal frame field.

Definition 2.2 ([12])**.** A vector $x = (x_1, x_2, x_3)$ is called non-isotropic if $x_1 \neq 0$. All units isotropic vectors are of the form $x = (1, x_2, x_3)$. For isotropic vectors, $x_1 = 0$ holds.

Theorem 2.3 ([14])**.** *(Clairauts Theorem) Let β be a geodesic on a surface of rotation S, let ρ be the distance function of a point of S from the axis of rotation, and let θ be the angle between β and the meridians of S. The $\rho sin\theta$ is constant along β. Conversely, if $\rho sin\theta$ is constant along the curve β on the surface, and if no part of β is part of some parallel of S, then β is a geodesic curve.*

3. Characterization of isotropic rectifying curves with the Darboux frame in G_3

In this section, we characterize an isotropic rectifying curve with vector fields tangential component and binormal component in terms of their curvatures using the Galilean Darboux frame in G_3. By definition, the position vector of the curve satisfies the equation

$$\beta(w) = \Sigma_0 \overrightarrow{t} + \Sigma_1 \overrightarrow{Q}; \Sigma_0(w), \Sigma_1(w) \in C^\infty, \tag{3.1}$$

and differentiating the equation (3.1) with respect to w and using the Frenet equations (2.15), we obtain

$$\overrightarrow{t} = \dot{\Sigma}_0 \overrightarrow{t} + \left(\Sigma_0 k_g + \dot{\Sigma}_1\right) \overrightarrow{Q} + (\Sigma_0 k_n + \Sigma_1 \tau_g)\, \overrightarrow{n}. \tag{3.2}$$

It follows that

$$\dot{\Sigma}_0 = 1; \Sigma_0 k_g + \dot{\Sigma}_1 = 0; \Sigma_0 k_n + \Sigma_1 \tau_g = 0, \tag{3.3}$$

and therefore

$$\Sigma_0 = c + w; \Sigma_1 = -\int k_g\left(c+w\right)dw + c_2 \text{ and } \Sigma_1 = \frac{-k_n(w+c)}{\tau_g}, \tag{3.4}$$

where $c, c_2 \in \mathbb{R}_0$. In this way $\Sigma_0(w)$ and $\Sigma_1(w)$ are expressed in terms of curvature functions k_g, k_n and τ_g of the rectifying curve β. Furthermore, by using the last equation in (3.3) and relation (3.4), we easily find that the curvatures k_g, k_n and τ_g satisfy the equation

$$\tau_g\left\{\int k_g\left(c+w\right)dw + c_2\right\} - k_n(w+c) = 0. \tag{3.5}$$

Conversely, assume that equation (3.5) is satisfied. Then, we can write the curve

$$\alpha(w) = \beta(w) - \Sigma_0\overrightarrow{t} - \Sigma_1\overrightarrow{Q}$$

with the functions $\Sigma_0(w), \Sigma_1(w) \in C^\infty$ as in the equation (3.4). Since $\alpha^{'} = 0$, we can say that β is congruent to an isotropic rectifying curve.

Hence, we can write the position vector as

$$\beta(w) = (c+w)\overrightarrow{t} + \left(-\int k_g\left(c+w\right)dw + c_2\right)\overrightarrow{Q}$$

or

$$\beta(w) = (c+w)\overrightarrow{t} + \left(\frac{-k_n(w+c)}{\tau_g}\right)\overrightarrow{Q}\ . \tag{3.6}$$

If the curve β is a geodesic, $k_g = 0$ and from (3.5) we get

$$c_2\tau_g\left(w\right) - k_n\left(w\right)\left(w+c\right) = 0 \Rightarrow k_n = c_2\frac{\tau_g}{w+c}, \tag{3.7}$$

and since the equation $\kappa^2 = k_g^2 + k_n^2$ is satisfied for the Galilean Darboux frame, by considering the equations $k_g = 0$ and $k_n = c_2\frac{\tau_g}{w+c}$ the curvature κ of the curve β is found as

$$\kappa = \pm c_2\frac{\tau_g}{w+c}. \tag{3.8}$$

Therefore, from (3.8) and (3.7) we get $\kappa = \pm k_n$.

In this way, we obtain the following theorem.

Theorem 3.1. *Let $\beta : I \subset \mathbb{R} \to G_3$ be a smooth isotropic curve with curvatures $\kappa(w) \geq 0$, τ in G_3. If β is a rectifying curve generated by the Galilean Darboux frame, then the following statements hold:*

(1) *The position vector of the curve β is given by*

$$\beta(w) = (c+w)\overrightarrow{t} + \left(-\int k_g(w)\left(c+w\right)dw + c_2\right)\overrightarrow{Q}$$

or

$$\beta(w) = (c+w)\overrightarrow{t} - \left(\frac{k_n\left(w\right)\left(w+c\right)}{\tau_g}\right)\overrightarrow{Q}.$$

(2) *β is congruent to an isotropic rectifying curve, if and only if*

$$\tau_g\left(w\right)\left\{\int k_g(w)\left(c+w\right)dw + c_2\right\} - k_n\left(w\right)\left(w+c\right) = 0, c, c_2 \in \mathbb{R}_0.$$

(3) If the curve *β is a geodesic curve, the normal curvature and the curvature κ are given as*

$$k_n = \pm\kappa = \pm c_2\frac{\tau_g}{w+c}.$$

4. The special tube surfaces generated by the rectifying curve with the Darboux frame in G_3

In this section, we present tube surfaces generated by a rectifying curve with the Darboux frame in three-dimensional Galilean space. Furthermore, we compute the Gaussian and mean curvature of tubular surface with the Darboux frame, by using Clairaut's theorem we investigate the relation between the curves with $v-$parameter (and the curves with $w-$parameter) and special curves such as geodesics, asymptotic curves on the specific tube surfaces with the Darboux frame.

A canal surface is expressed as one-parameter set of spheres, whose center is described by a radius function ρ and the orbit $\beta(w)$, in addition to parametrizing the spine curve via the Frenet frame. If the radius function ρ is constant, the canal surface is called the tube or tubular surface [8].

We know that a tube surface $\Theta(w,v)$ of radius ρ around $\beta(w)$ is the set of points at a distance A from $\beta(w)$, such that $\beta(w)$ is a center curve on the surface $\Theta(w,v)$, R is a point on the surface $\Theta(w,v)$ and $\{\overrightarrow{t},\overrightarrow{Q},\overrightarrow{n}\}$ is Frenet frame at $R\in\Theta(w,v)$. Then, since the characteristic circles of canal surface lie in the plane which is perpendicular to the tangent of center curve $\beta(w)$, we can write tube surface with Darboux frame in G_3 as

$$R=\beta(w)+\rho\Rightarrow\rho=A(\cos v\overrightarrow{n}+\sin v\overrightarrow{b}), \tag{4.1}$$

where v is the Euclidean angle between isotropic vectors; $\overrightarrow{n}$ and $\overrightarrow{\rho}$ lie in the Euclidean normal plane of the curve $\beta(w)$.

Theorem 4.1. *Let β be an isotropic rectifying curve with the curvatures $\kappa\geq 0,\tau$ in G_3, and let $\Theta(w,v)$ be the tube surface generated by the curve β with the Darboux frame. Then, the following statements hold:*

(1) *The tube surface $\Theta(w,v)$ with the Darboux frame is parametrized by*

$$\Theta(w,v)=(w+c)\overrightarrow{t}+A\cos v\overrightarrow{n}+(-\frac{k_n(w+c)}{\tau_g}+A\sin v)\overrightarrow{b}.$$

(2) *The Gaussian curvature K and the mean curvature H are respectively given as*

$$K(w,v)=\frac{-\sin v\left(2k_g+(w+c)\frac{dk_g}{dw}\right)-\cos v\,(k_n+(w+c)k_g\tau_g)}{A};H=\frac{-1}{2A},$$

where this family of the tube surface has constant mean curvature.

(3) *The first fundamental form of the surface Θ is given by*

$$I=2\dot{w}^2+\left(\frac{1}{2H}\right)^2\dot{v}^2.$$

(4) *If $\beta(w(s))$ is a geodesic curve on $\Theta(w(s),v)$, the following statements satisfy:*

(i) *If the curve β is a geodesic with $v-$parameter on the surface $\Theta(w,v)$, then*

$$v=2c_2H^2s+d_2 \text{ or } v=2H\int\sin\theta ds.$$

(ii) *If the curve β is a geodesic with $w-$parameter on the surface $\Theta(w,v)$, then*

$$w=\int\cos\theta ds \text{ or } w=\frac{c_1}{4}s+d_1.$$

(iii) *If the rectifying curve β is a geodesic on the surface $\Theta(w(s),v)$, then*

$$K(w,v)=2H\kappa(w)\cos v,$$

where $c_i,d_i\in\mathbb{R}$ and θ is the angle between the meridian $\dot{\beta}$ and $N_{w(s)}$.

Proof. Let us first assume that $\Theta(w,v)$ is the tube surface generated by the rectifying curve β with the Darboux frame. From (4.1), the surface is given by the parametrization

$$\Theta(w,v) = \beta(w) + A(\cos v\overrightarrow{n} + \sin v\overrightarrow{Q}), \tag{4.2}$$

where v is the angle between the isotropic vectors $\overrightarrow{n}$ and $\overrightarrow{A}$. Also, from the equation (3.1) satisfying the condition (3.5), the tube surface is written as

$$\Theta(w,v) = (w+c)\overrightarrow{t} + A\cos v\overrightarrow{n} + (-\int k_g\,(c+w)\,dw + c_2 + A\sin v)\overrightarrow{Q} \tag{4.3}$$

or

$$\Theta(w,v) = (w+c)\overrightarrow{t} + A\cos v\overrightarrow{n} + (-\frac{k_n(w+c)}{\tau_g} + A\sin v)\overrightarrow{Q}. \tag{4.4}$$

The tangent space of the tube surface $\Theta(w,v)$ given by the equation (4.4) at an arbitrary point of $\Theta(w,v)$ is spanned by

$$\Theta_w = \overrightarrow{t} + ((w+c)k_g - A\cos v\tau_g)\,\overrightarrow{Q} + (\tau_g A\sin v)\,\overrightarrow{n} = N_s; \tag{4.5}$$

$$\Theta_v = A(\cos v\overrightarrow{Q} - \sin v\overrightarrow{n}) = AN_v. \tag{4.6}$$

Subsequently, the vector cross product is obtained as

$$\Theta_w \times_{G_3} \Theta_v = \begin{pmatrix} 0 & e_2 & e_3 \\ 1 & \begin{pmatrix} (w+c)k_g \\ -A\cos v\tau_g \end{pmatrix} & \tau_g A\sin v \\ 0 & A\cos v & -A\sin v \end{pmatrix} = A(0,\sin v,\cos v); \tag{4.7}$$

$$\left\|\Theta_w \times_{G_3} \Theta_v\right\| = A. \tag{4.8}$$

By using equations (4.7) and (4.8), the unit isotropic normal vector η of $\Theta(w,v)$ can be expressed as follows

$$\eta = \sin v\overrightarrow{Q} + \cos v\overrightarrow{n}. \tag{4.9}$$

Furthermore, from (2.7) we obtain the equation $\delta = \frac{-\Theta_v}{A} = \sin v\overrightarrow{n} - \cos v\overrightarrow{Q}$, and since $\overrightarrow{n}$ and $\overrightarrow{Q}$ are the isotropic vectors, the Galilean Frenet frame consideration leads to,

$$x(w,v) = w + c; x_w = 1 = g_1; x_v = 0 = g_2;$$

$$g_{11} = 1, g_{12} = 0, g_{22} = 0; g^1 = 0, g^2 = \frac{-1}{A}; \tag{4.10}$$

$$h_{11} = 1, h_{12} = 0, h_{22} = A^2. \tag{4.11}$$

It is possible to calculate the second fundamental form of $\Theta(w,v)$, which leads to the following equations

$$\begin{aligned}\Theta_{ww} &= \left(2k_g + (w+c)\frac{dk_g}{dw} - A\cos v\frac{d\tau_g}{dw} - \tau_g^2 A\sin v\right)\overrightarrow{Q} \\ &+ \left(k_n + (w+c)k_g\tau_g - A\tau_g^2\cos v + \frac{d\tau_g}{dw}A\sin v\right)\overrightarrow{n};\end{aligned} \tag{4.12}$$

$$\Theta_{vv} = A(-\sin v\overrightarrow{Q} - \cos v\overrightarrow{n}); \Theta_{wv} = \tau_g A\sin v\overrightarrow{Q} + \tau_g A\cos v\overrightarrow{n}. \tag{4.13}$$

The second fundamental form coefficients can be calculated from (2.13) by using the equations (4.9), (4.12), (4.13). Then, we get

$$L_{11} = \Theta_{ww}\cdot_{G_3}\eta = \sin v\left(2k_g + (w+c)\frac{dk_g}{dw}\right) + \cos v\,(k_n + (w+c)k_g\tau_g) - A\tau_g^2;$$

$$L_{22} = -A; L_{12} = \tau_g A. \tag{4.14}$$

Hence, from (2.14) the Gaussian curvature K and the mean curvature H are obtained as

$$K(w,v) = \frac{-\sin v\left(2k_g + (w+c)\frac{dk_g}{dw}\right) - \cos v\left(k_n + (w+c)k_g\tau_g\right)}{A}; \tag{4.15}$$

$$H = \frac{g_2^2 L_{11} - 2g_1g_2L_{12} + g_1^2 L_{22}}{2w^2} = \frac{-1}{2A}. \tag{4.16}$$

For the geodesic rectifying curve since $\frac{k_n(w+c)}{\tau_g}$ =constant and $k_g = 0$, we can write

$$K(w,v) = 2Hk_n \cos v = 2H\kappa(w)\cos v.$$

After substituting (4.10) and (4.11) into (2.10), for the first fundamental form of the tube surface the coefficients can be obtained as follows:

$$I = dw^2 + \varepsilon\left(dw^2 + \left(\frac{1}{2H}\right)^2 dv^2\right),$$

by means of Galilean geometry, we get

$$I = 2dw^2 + \left(\frac{1}{2H}\right)^2 dv^2; \varepsilon = 1. \tag{4.17}$$

Since $\tau_g \neq 0$, for $\frac{(w+c)k_n}{d} = \tau_g$ the first fundamental form has two variable parameters. It is also important to note that the parametrization coordinates are orthogonal since the first fundamental form is diagonal and Lagrangian can be obtained from the first fundamental form. Then, we have

$$2\dot{w}^2 + \left(\frac{1}{2H}\right)^2 \dot{v}^2 = L. \tag{4.18}$$

The trajectories of moving particles on $\Theta(w(s), v)$ are determined by the following equations:

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w(s)}{\partial s}}\right) = \frac{\partial L}{\partial w(s)}; \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v}. \tag{4.19}$$

These equations are called as the Euler-Lagrange equations. The particular solution providing the initial value of differential equations in (4.19) is a geodesic β passing through initial point $(w(s_0), v(s_0))$ and the end point $(w(s_1), v(s_1))$.

From Theorem 2.3, the geodesics on the tube surface can be obtained by using the Euler-Lagrange differential equations as follows:

(1) For $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w(s)}{\partial s}}\right) = \frac{\partial L}{\partial w(s)} = 0$, we obtain $\frac{\partial L}{\frac{\partial w(s)}{\partial s}} = 4\dot{w}$ =constant, which means

$$w(s) = \frac{c_1}{4}s + d_1 \text{ or } \dot{w}(s) = \frac{c_1}{4}. \tag{4.20}$$

(2) For $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v} = 0$, we can obtain $\frac{\partial}{\partial s}\left(2\left(\frac{1}{2H}\right)^2\dot{v}\right) = 0$, which means $2\left(\frac{1}{2H}\right)^2\dot{v}$ is constant along the geodesic and leading to

$$v = 2H^2c_2s + d_2 \text{ or } \dot{v} = 2H^2c_2. \tag{4.21}$$

Let $\beta(w)$ be a geodesic on the surface $\Theta(w(s), v)$. Meanwhile, let θ be the angle between $\dot{\beta}$ which is a meridian and $N_{w(s)}$, and N_v is the vector pointing along parallels of Θ. Thus, it can be said that $\{N_w, N_v\}$ has an orthonormal basis and the unit tangent vector $\dot{\beta}$ can be written as

$$\dot{\beta} = N_w\cos\theta + N_v\sin\theta = \dot{w}(s)\Theta_w + \dot{v}\Theta_v = \dot{w}(s)N_w - \dot{v}\frac{1}{2H}N_v.$$

It can be seen that $\frac{1}{2H}\dot{v} = -\sin\theta$, and we can also write as $\frac{1}{(2H)^2}\dot{v} = -\frac{1}{2H}\sin\theta$=constant along $\beta(w(s))$. On the contrary, $\beta(w(s))$ is a rectifying curve satisfying the condition

$\frac{1}{(2H)^2}\dot{v}$ =constant, then the second Euler-Lagrange equation is satisfied and differentiating the equation L, substituting in the first Euler Lagrange equation. After that we obtain

$$v = -2H \int \sin\theta ds. \tag{4.22}$$

Furthermore, the equation $w = \frac{c_1}{4}s + d_1$ can be written as $\dot{w} = \frac{c_1}{4}$. It is also possible to see that $\dot{w}(s) = \cos\theta$. Hence, we can write $4\dot{w} = 4\cos\theta$=constant along the rectifying curve $\beta(w(s))$. If $\beta(w(s))$ is a rectifying curve satisfying the condition $4\cos\theta$ =constant, then the first Euler-Lagrange equation is satisfied and derivative of the equation L is taken, substituting in the second Euler Lagrange equation. Hence, we get

$$w = \int \cos\theta ds, \tag{4.23}$$

where $c_i, d_i \in \mathbb{R}_0$. □

Theorem 4.2. *The general equations of geodesics on the tube surface $\Theta(w, v)$ in G_3 are given by the following equations:*

(1) *For the parameters $v = 2H^2c_2s + d_2$ or $v = -2H\int \sin\theta ds$, the following equations are satisfied*

$$\frac{dw}{dv} = \frac{1}{2\sqrt{2}c_3H^2}\sqrt{L_1 - H^2c_3} \text{ or } \frac{dw}{dv} = \frac{-1}{2H\sqrt{2}\sin\theta}\sqrt{L_1 - \sin^2\theta}; c_3 \in \mathbb{R}_0 \tag{4.24}$$

and the curve β is also geodesic with $v-$parameter.

(2) *For the parameters $w = \int \cos\theta ds$ or $w = \frac{c_1}{4}s + d_1$, the following equations are satisfied*

$$\frac{dv}{dw} = -c_5H\sqrt{L_2 - \frac{c_4}{2}} \text{ or } \frac{dv}{dw} = \frac{-2H}{\cos\theta}\sqrt{L_2 - 2\cos^2\theta}; c_5, c_4 \in \mathbb{R}_0 \tag{4.25}$$

and the curve β is also geodesic with $w-$parameter, where H is the mean curvature on the tube surface $\Theta(w, v)$, θ is the angle between the meridian $\dot{\beta}$ and $N_{w(s)}$.

Proof. In order to obtain the general equation of geodesics, we can taken into consideration the Euler-Lagrange equations in (4.19) together with metric on the tube surfaces in Galilean 3-space. From the solving of the differential equations in (4.19), we have the following cases:

(1) For the parameters $v = 2H^2c_2s + d_2$ or $v = -2H\int \sin\theta ds$, the geodesic equation can be obtained by solving the differential equation $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v}$, leading to $\dot{v} = 2H^2c_2$ or $\dot{v} = -2H\sin\theta$. Also, if we substitute $\dot{v}$ into the equation $2\dot{w}^2 + \left(\frac{1}{2H}\right)^2\dot{v}^2 = L_{1,2}$, then we get

$$2\left(\frac{dw}{dv}\frac{dv}{ds}\right)^2 + \left(\frac{1}{2H}\right)^2\left(\frac{dv}{ds}\right)^2 = L_1.$$

Furthermore, we obtain the general equations of geodesic as $\frac{dw}{dv} = \frac{\sqrt{L_1 - H^2c_3}}{2\sqrt{2}c_3H^2}$ or $\frac{dw}{dv} = \frac{-\sqrt{L_1 - \sin^2\theta}}{2\sqrt{2}H\sin\theta}$.

(2) For the parameters $w = \int\cos\theta ds$ or $w = \frac{c_1}{4}s + d_1$ obtained by solving the differential equation $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w}$, we can write $\dot{w} = \frac{c_1}{4}; \dot{w} = \cos\theta$. If $\dot{w}$ is added to the Lagrangian equation, the following equation can be written

$$2\left(\frac{dw}{ds}\right)^2 + A^2\left(\frac{dv}{dw}\frac{dw}{ds}\right)^2 = L_2.$$

Hence, the general equations of geodesic on Θ is written as $\frac{dv}{dw} = -c_5 H \sqrt{L_2 - \frac{c_4}{2}}$ or $\frac{dv}{dw} = \frac{-2H\sqrt{L_2 - 2\cos^2\theta}}{\cos\theta}$, where $c_i, d_i \in \mathbb{R}$.

$\square$

Corollary 4.3. *Let β be a regular rectifying curve in G_3 with curvatures $\kappa \geq 0, \tau$ and let $\Theta(w,v)$ be the tube surface generated by the rectifying curve β with the Darboux frame. Then, the following statements hold:*

(a) *The rectifying curves with $v-$parameter on $\Theta(w,v)$ are also geodesic curves.*
(b) *The rectifying curves with $w-$parameter on $\Theta(w,v)$ are geodesic curves if and only if k_g, k_n and τ_g satisfy the following condition*

$$\tau_g(w) = -2H \int \left(\left(2k_g + (w+c)\frac{dk_g}{dw}\right)\cos v - (k_n + (w+c)k_g\tau_g)\sin v \right) dw,$$

where H is the mean curvature on the surface Θ.

Proof. Let's find the expressions $\Theta_{vv} \times_{G_3} \eta$, $\Theta_{ww} \times_{G_3} \eta$ for the curve β with $v-$parameter(and $w-$parameter) on the tube surface $\Theta(w,v)$.

If the curve with $v-$ parameter is a geodesic curve, then the equation $\Theta_{vv} \times_{G_3} \eta = 0$ is satisfied. Hence, we get $\Theta_{vv} \times_{G_3} \eta = 0$ by using the expression $\Theta_{vv} \times_{G_3} \eta$.

If the curve β on the surface is the geodesic curve with w-parameter, from the equations (4.12) and (4.9), we get

$$\Theta_{ww} \times_{G_3} \eta = \begin{pmatrix} \left(2k_g + (w+c)\frac{dk_g}{dw} - \frac{d^2}{dw^2}\left(\frac{k_n(w+c)}{\tau_g}\right)\right)\cos v \\ -\left(k_n + (w+c)k_g\tau_g - \tau_g\frac{d}{dw}\left(\frac{k_n(w+c)}{\tau_g}\right)\right)\sin v - \frac{d\tau_g}{dw}A \end{pmatrix} \overrightarrow{t},$$

and for the geodesic curve β with $w-$parameter, the last equation is equal to zero. So, we obtain

$$\begin{aligned} 0 = &\left(2k_g + (w+c)\frac{dk_g}{dw} - \frac{d^2}{dw^2}\left(\frac{k_n(w+c)}{\tau_g}\right)\right)\cos v \\ &- \left(k_n + (w+c)k_g\tau_g - \tau_g\frac{d}{dw}\left(\frac{k_n(w+c)}{\tau_g}\right)\right)\sin v - \frac{d\tau_g}{dw}A. \end{aligned} \tag{4.26}$$

Moreover, for a rectifying curve we obtain the following equation

$$-2H \int \left(\left(2k_g + (w+c)\frac{dk_g}{dw}\right)\cos v - (k_n + (w+c)k_g\tau_g)\sin v \right) dw = \tau_g(w). \tag{4.27}$$

$\square$

Theorem 4.4. *Let β be an isotropic rectifying curve in G_3 with curvatures $\kappa \geq 0, \tau$ and let $\Theta(w,v)$ be the tube surface generated by the rectifying curve with the Darboux frame. Then, the following statements hold:*

(a) *If the curve β with $w-$parameter on $\Theta(w,v)$ is a geodesic curve, then the following equations are satisfied*

$$\begin{aligned} A &= \frac{-\kappa\sin v}{\tau_w}; \\ \tau(w) &= 2H\int \kappa(w)\sin v dw; \\ \kappa(w) &= \frac{\tau_w}{2H\sin v}; \\ K(w,v) &= \tau_w \cot v. \end{aligned}$$

(b) *If the curve β with $w-$parameter on $\Theta(w,v)$ is an asymptotic curve, then the following equation are satisfied*

$$K\left(w,v\right)=2H\kappa(w)\sin v\frac{-f(w)\sin v-g(w)\cos v}{f(w)\cos v-g(w)\sin v},$$

where $f(w)=\left(2\kappa+(w+c)\kappa_{w}\right),g(w)=(w+c)\kappa\tau$.

Proof. (a) If the rectifying curve β with $w-$parameter on the tube surface $\Theta(w,v)$ is a geodesic curve, then $k_g=0,\ k_n=\kappa,\tau_g=\tau$. So, from (4.26) we get

$$-\kappa\sin v=\frac{d\tau}{dw}A;A=\frac{-1}{2H}$$

and from the previous equation we write

$$2H\int\kappa\left(w\right)\sin vdw=\tau\left(w\right)$$

or

$$\frac{-\kappa\sin v}{\tau_{w}}=A. \tag{4.28}$$

Also, from (4.15) we have

$$K\left(w,v\right)=\frac{-\kappa\cos v}{A}=\tau_{w}\cot v; \tag{4.29}$$

$$\frac{-\kappa\sin v}{\tau_{w}}=\frac{-1}{2H}\Rightarrow\kappa\left(w\right)=\frac{\tau_{w}}{2H\sin v}. \tag{4.30}$$

(b) If the curve β with $w-$parameter on the tube surface $\Theta(w,v)$ is an asymptotic curve, then $k_n=0\ k_g=\kappa,\tau_g=\tau$ and from (4.26), we get

$$\frac{(2\kappa+(w+c)\kappa_{w})\cos v-(w+c)\kappa\tau\sin v}{\tau_{w}}=A \tag{4.31}$$

and for the equations $f(w)=\left(2\kappa+(w+c)\kappa_{w}\right),g(w)=(w+c)\kappa\tau$, the Gaussian curvature $K\left(w,v\right)$ is obtained as

$$\begin{aligned}K\left(w,v\right)&=\frac{-\left(2\kappa+(w+c)\kappa_{w}\right)\sin v-(w+c)\kappa\tau\cos v}{A};\\K\left(w,v\right)&=\tau_{w}\frac{-f(w)\sin v-g(w)\cos v}{f(w)\cos v-g(w)\sin v};\end{aligned}$$

$$K\left(w,v\right)=2H\kappa(w)\sin v\frac{-f(w)\sin v-g(w)\cos v}{f(w)\cos v-g(w)\sin v}. \tag{4.32}$$

□

Theorem 4.5. *Let Θ be the tube surface generated by the rectifying curve β in G_3. Then, the following statements hold:*

(1) *If the rectifying curve β with $w-$parameter is an asymptotic curve if and only if the following condition is satisfied.*

$$A=\frac{\left(2k_{g}+(w+c)\frac{dk_{g}}{dw}\right)\sin v+(k_{n}+(w+c)k_{g}\tau_{g})\cos v}{\tau_{g}^{2}}$$

In this case,

(a_1) *For the condition $A=\frac{\kappa(w)\cos v}{\tau(w)^{2}}$, if the rectifying curve β with $w-$parameter is also a geodesic curve, then the Gaussian curvature is given by*

$$K\left(w,v\right)=-\tau\left(w\right)^{2}\left(1+(w+c)\tau\left(w\right)\right).$$

(a_2) *If the rectifying curve β with $w-$parameter is an asymptotic curve satisfying the following equation*

$$A=\frac{(2\kappa+(w+c)\kappa_w)\sin v+(w+c)\kappa\tau\cos v}{\tau^2},$$

then the Gaussian curvature is given by

$$K\left(w,v\right)=-\tau\left(w\right)^2.$$

(2) *There is no asymptotic rectifying curve β with $v-$parameter on the tube surface Θ.*

Proof. (1) Let Θ be the tube surface generated by the rectifying curve β with $w-$parameter in G_3. If the curve β is an asymptotic, then the equation $\Theta_{ww}\cdot_{G_3}\eta=0$ is satisfied. In this case, from the equations (4.9) and (4.12), we get

$$\begin{aligned}\Theta_{ww}\cdot_{G_3}\eta &= \left(2k_g+(w+c)\frac{dk_g}{dw}-\frac{d^2}{dw^2}\left(\frac{k_n(w+c)}{\tau_g}\right)\right)\sin v\\ &+\left(k_n+(w+c)k_g\tau_g-\tau_g\frac{d}{dw}\left(\frac{k_n(w+c)}{\tau_g}\right)\right)\cos v-A\tau_g^2.\end{aligned}$$

Hence, from the previous equation we obtain

$$A=\frac{\left(2k_g+(w+c)\frac{dk_g}{dw}\right)\sin v+(k_n+(w+c)k_g\tau_g)\cos v}{\tau_g^2}. \tag{4.33}$$

Thus,

(a_1) If the rectifying curve β is a geodesic curve on the surface, then $k_g=0, k_n=\kappa$, $\tau_g=\tau$, $\frac{k_n(w+c)}{\tau_g}$ =constant. If we replace these in (4.33) it follows that

$$A=\frac{\kappa\cos v}{\tau^2}, \tag{4.34}$$

and by a calculation, the Gaussian curvature is

$$K\left(w,v\right)=\frac{-\left(\kappa+(w+c)\kappa\tau\right)\cos v}{A}=-\tau\left(w\right)^2\left(1+(w+c)\tau\left(w\right)\right). \tag{4.35}$$

(a_2) If the rectifying curve β is an asymptotic curve, then $k_g=\kappa, k_n=0$, $\tau_g=\tau$. If we replace in (4.33) we get

$$A=\frac{(2\kappa+(w+c)\kappa_w)\sin v+(w+c)\kappa\tau\cos v}{\tau^2} \tag{4.36}$$

and by a calculation, the Gaussian curvature is

$$K\left(w,v\right)=\frac{-\left(2\kappa+(w+c)\kappa_w\right)\sin v-(w+c)\kappa\tau\cos v}{A}=-\tau\left(w\right)^2. \tag{4.37}$$

(2) If the curve β is an asymptotic curve with $v-$parameter, the equation $\Theta_{vv}\cdot_{G_3}\eta=0$ is satisfied. In this case, we obtain $\Theta_{vv}\cdot_{G_3}\eta\neq 0$. Therefore, there is no asymptotic rectifying curve β with $v-$parameter.

□

5. Conclusion

In this paper, we investigate the rectifying curves with Darboux frame in Galilean 3-space. Then, by using the rectifying curves with Darboux frame we introduce the special tube surface in Galilean 3-space. Besides, we compute the Gaussian and mean curvature of tube surface with Darboux frame and we give some characterizations for the geodesic curves and the asymptotic curves with $v-$parameter(and $w-$parameter) by using the Gaussian and mean curvatures. Furthermore, we express the general equations of geodesics on special tube surfaces with the Darboux frame by using the Euler-Lagrange equations with the help of Clairauts theorem.

Acknowledgment. The authors would like to thank the literature authors for the scientific aspects and ideas provided for this study. We also request a huge thanks to the reviewers for their constructive comments and input to improve the quality of our work.

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