Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (1) (2023), 151 – 162
DOI : 10.15672/hujms.1074783

Research Article

Polynomially partial isometric operators

Dijana Mosić

Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia

Abstract

In order to extend the notion of semi-generalized partial isometries and partial isometries, we introduce a new class of operators called polynomially partial isometries. Since this new class of operators contains semi-generalized partial isometries, partial isometries, isometries and co-isometries, we proposed a wider class of operators. Several basic properties of polynomially partial isometries and some invariant subspaces of corresponding operators are presented. We study decomposition theorems and spectral theorems for polynomially partial isometries, generalizing some well-known results for partial isometries and semi-generalized partial isometries to polynomially partial isometries. Applying polynomially partial isometries, we solve some equations.

Mathematics Subject Classification (2020). 47A05, 47A15, 47B40

Keywords. partial isometry, isometry, decomposition theorem, Hilbert space

1. Introduction

Let H represent a complex Hilbert space, and let $\mathcal{B}(H)$ stand for the set of all bounded linear operators on H. For $T \in \mathcal{B}(H)$, we denote by T^* the adjoint operator of T, by $\mathcal{R}(T)$ its range and by $\mathcal{N}(T)$ its kernel. The spectrum, the point spectrum and the approximate point spectrum of T will be denoted by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$, respectively.

We use Poly to denote the set of all complex polynomials in one variable. Taking conjugate coefficients of $p \in$ Poly, we obtain $\overline{p} \in$ Poly, precisely: if $z \in \mathbb{C}$, then $\overline{p}(z) := \overline{p(\overline{z})}$.

Extending the notion of normal operators ($TT^* = T^*T$), Alzraiqi and Patel [2] introduced n-normal operators as: $T \in B(H)$ is n-normal ($n \in \mathbb{N}$) if $T^nT^* = T^*T^n$. The class of n-normal operators includes n-nilpotent operators $T^n = 0$, Hermitian ($T = T^*$) and unitary operators ($TT^* = T^*T = I$). Observe that T is n-normal if and only if T^n is normal. For recent results concerning n-normal operators see [7, 8, 10].

Recently, polynomially normal operators were defined in [9], generalizing the notion of n-normal operators. Let $T \in \mathcal{B}(H)$ and $p \in$ Poly be nontrivial. If $p(T)T^* = T^*p(T)$, then T is p-normal. According to [9], T is p-normal if and only if $p(T)$ is normal.

Let us recall that $T \in \mathcal{B}(H)$ is an isometry if $\|Tx\| = \|x\|$, for all $x \in H$. Note that T is an isometry if and only if $T^*T = I$. For $T \in \mathcal{B}(H)$, if T^* is an isometry, then T is co-isometry. In the case that $\|T\| \leq 1$, T is a contraction.

Email address: dijana@pmf.ni.ac.rs
Received: 16.02.2022; Accepted: 21.07.2022

An operator $T \in \mathcal{B}(H)$ is a partial isometry if $\|Tx\| = \|x\|$, for all $x \in \mathcal{N}(T)^{\perp}$ (or equivalently $T|_{\mathcal{N}(T)^{\perp}} : \mathcal{N}(T)^{\perp} \to H$ is an isometry). Recall that T is a partial isometry if and only if $TT^*T = T$ if and only if T^* is a partial isometry if and only if TT^* (or T^*T) is the orthogonal projection onto $\mathcal{R}(T)$ ($\mathcal{N}(T)$). Partial isometries present an important class of operators which contains isometries, co-isometries, unitary operators and orthogonal projections. Since every square matrix can be represented as a linear combination of mutually orthogonal partial isometries with positive coefficients [12], many authors were motivated to investigate partial isometries [14]. Operators similar to partial isometries were studied in [5]. Various results for partial isometries for Banach algebra, C^*-algebra and ring elements can be found in [1, 4, 18, 19, 21]. Some extensions of partial isometries were presented in [15, 16, 20].

As a generalization of partial isometries and nilpotent operators, Garbouj and Skhiri [13] introduced a new class of operators called semi-generalized partial isometries in the following way: let $T \in \mathcal{B}(H)$ and $n \in \mathbb{N}$. Then:

(i) T is a n-left generalized partial isometry if $T^nT^*T = T^n$;
(ii) T is a n-right generalized partial isometry if $TT^*T^n = T^n$;
(iii) T is a semi-generalized partial isometry if T is k-left or k-right generalized partial isometry for some $k \in \mathbb{N}$.

Motivated by the previous research about semi-generalized partial isometries [13] and partial isometries as well as the results from [9], our goal is to continue studying generalizations of partial isometries. We present a new class of operators called polynomially partial isometries including semi-generalized partial isometries, partial isometries, isometries, co-isometries, which means that we proposed a wider class of operators. We give some elementary properties of polynomially partial isometries and some invariant subspaces of corresponding operators. Since there are polynomially partial isometries which are not partial isometries, we develop conditions for polynomially partial isometries to be partial isometries. Several decomposition theorems for polynomially partial isometries are established. We also prove spectral theorems for this class of operators. Thus, we generalize some well-known results for partial isometries and semi-generalized partial isometries to polynomially partial isometries.

2. Polynomially partial isometries

As an extension of semi-generalized partial isometries and partial isometries, we define a new class of operators called polynomially partial isometries.

Definition 2.1. Let $T \in \mathcal{B}(H)$ and $p \in \text{Poly}$ be nontrivial. If

(i) $p(T)T^*T = p(T)$, then T is called left p-partial isometry;
(ii) $TT^*p(T) = p(T)$, then T is called right p-partial isometry;
(iii) T is both left and right p-partial isometry, then T is called p-partial isometry;
(iv) $S \in \mathcal{B}(H)$ is q-partial isometry for some $q \in \text{Poly}$, then S is called polynomially partial isometry.

Obviously, for arbitrary $p \in \text{Poly}$, every isometry is left p-partial isometry, and every co-isometry is right p-partial isometry. The converse is not true in general as we can see below in Example 2.2.

Also, if $T \in \mathcal{B}(H)$ is a left p-partial isometry and $p(T)$ is one-to-one, then T is an isometry. When T is a right p-partial isometry and $p(T)$ is onto, then T is a co-isometry. In particular, if T is a p-partial isometry and $p(T)$ is invertible, then T is unitary.

Remark that, if $T \in \mathcal{B}(H)$ is a left (or right) p-partial isometry for $p(t) = t^n$, $n \in \mathbb{N}$, then T is a n-left (n-right) generalized partial isometry. In the case that T is a left or right p-partial isometry for $p(t) = t$, then T is a partial isometry. Hence, the set of all polynomially partial isometric operators contains n-left and n-right generalized partial isometries from [13] and partial isometries.

We now give two examples of left p-partial isometries which are not partial isometries.

Example 2.2. Consider an operator T on $H = \mathbb{C}^3$ given by

$$T = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and a polynomial $p(t) = t^3 + t^2 - 3t - 3$. Using

$$T^2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^3 = \begin{bmatrix} 0 & 9 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^*T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$T^2T^*T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^3T^*T = \begin{bmatrix} 0 & 81 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad TT^*T = \begin{bmatrix} 0 & 27 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we get

$$p(T) = T^3 + T^2 - 3T - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} = p(T)T^*T.$$

Hence, T is a left p-partial isometry, but $TT^*T \neq T$.

Example 2.3. For an operator T on $H = \mathbb{C}^3$ defined by

$$T = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and a polynomial $p(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_3t^3 + a_2t^2$, $a_2, \ldots, a_n \in \mathbb{C}$, we observe that $TT^*T \neq T$,

$$T^*T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T^kT^*T, \qquad 2 \leq k \leq n.$$

Thus, $p(T) = a_nT^n + a_{n-1}T^{n-1} + \cdots + a_3T^3 + a_2T^2 = p(T)T^*T$.

Under additional assumptions, partial isometries can become polynomially partial isometries. If we suppose, for $p(t) = a_nt^n + \cdots + a_1t + a_0$, where $a_0, a_1, \ldots, a_n \in \mathbb{C}$, that $p(0) = 0$, i.e. $a_0 = 0$, we easily obtain the next conclusion.

Corollary 2.4. *If $T \in \mathcal{B}(H)$ is a partial isometry and $p \in \mathrm{Poly}$ such that $p(0) = 0$, then T is a p-partial isometry.*

The fact that T is a partial isometry if and only if T^* is a partial isometry, can be generalized for polynomially partial isometries in the following way.

Lemma 2.5. *Let $T \in \mathcal{B}(H)$ and $p \in \mathrm{Poly}$. Then:*

(i) *T is a left p-partial isometry if and only if T^* is a right $\overline{p}$-partial isometry;*
(ii) *T is a right p-partial isometry if and only if T^* is a left $\overline{p}$-partial isometry;*
(iii) *T is a p-partial isometry if and only if T^* is a $\overline{p}$-partial isometry.*
(iv) *for an unitary operator $S \in \mathcal{B}(H)$, T is a left (or right) p-partial isometry if and only if STS^* is a left (right) p-partial isometry.*

Proof. (i) Notice that

$$\begin{aligned} T \text{ is left } p\text{-partial isometry} &\Leftrightarrow p(T)T^*T = p(T) \\ &\Leftrightarrow T^*T\overline{p}(T^*) = \overline{p}(T^*) \\ &\Leftrightarrow T^* \text{ is right } \overline{p}\text{-partial isometry.} \end{aligned}$$

(ii) Applying part (i) for T^* and $\overline{p}$, we prove this part.
(iii) It follows by parts (i) and (ii).
(iv) Since S is unitary, then $p(STS^*) = Sp(T)S^*$ and so

$$\begin{aligned} p(STS^*)(STS^*)^*STS^* = p(STS^*) &\Leftrightarrow Sp(T)S^*ST^*S^*STS^* = Sp(T)S^* \\ &\Leftrightarrow p(T)T^*T = p(T). \end{aligned}$$

□

By properties of polynomials, we immediately check the next elementary result.

Lemma 2.6. *Let $T \in \mathcal{B}(H)$, $\lambda \in \mathbb{C}\backslash\{0\}$ and $p, q \in$ Poly.*

(i) *If T is a left (or right) p-partial isometry, then T is a left (right) λp-partial isometry.*
(ii) *If T is both left p-partial isometry and left q-partial isometry, then T is a left $p+q$-partial isometry.*
(iii) *If T is both right p-partial isometry and right q-partial isometry, then T is a right $p+q$-partial isometry.*

For a closed subspace M of H and $T \in \mathcal{B}(H)$, recall that $T(M) \subseteq M$ if and only if $T^*(M^\perp) \subseteq M^\perp$. Using this equivalence, we get the following invariant subspaces of corresponding operators.

Lemma 2.7. *Let $T \in \mathcal{B}(H)$ and $p \in$ Poly. Then*

(i) *$T(\mathcal{N}(p(T))) \subseteq \mathcal{N}(p(T))$ and $T(\mathcal{R}(p(T))) \subseteq \mathcal{R}(p(T))$;*
(ii) *$T^*(\mathcal{N}(p(T))^\perp) \subseteq \mathcal{N}(p(T))^\perp$ and $T^*(\mathcal{R}(p(T))^\perp) \subseteq \mathcal{R}(p(T))^\perp$;*
(iii) *for a left p-partial isometry T,*

$$T^*T(\mathcal{N}(p(T))) \subseteq \mathcal{N}(p(T)) \quad \text{and} \quad T^*T(\mathcal{N}(p(T))^\perp) \subseteq \mathcal{N}(p(T))^\perp;$$

(iv) *for a right p-partial isometry T,*

$$TT^*(\mathcal{R}(p(T))) \subseteq \mathcal{R}(p(T)) \quad \text{and} \quad TT^*(\mathcal{R}(p(T))^\perp) \subseteq \mathcal{R}(p(T))^\perp;$$

It is well-known, for a partial isometry $T \in \mathcal{B}(H)$, that $T^*T|_{\mathcal{N}(T)^\perp} = I|_{\mathcal{N}(T)^\perp}$ and $T^*T|_{\mathcal{R}(T)} = I|_{\mathcal{R}(T)}$. For $n \in \mathbb{N}$, notice that T is a n-left (or n-right) generalized partial isometry if and only if $T^*T|_{\mathcal{N}(T^n)^\perp} = I|_{\mathcal{N}(T^n)^\perp}$ $(T^*T|_{\overline{\mathcal{R}(T^n)}} = I|_{\overline{\mathcal{R}(T^n)}})$ [13]. This result is now generalized for left and right p-partial isometries.

Theorem 2.8. *Let $T \in \mathcal{B}(H)$ and $p \in$ Poly. Then:*

(i) *T is a left p-partial isometry if and only if $T^*T|_{\mathcal{N}(p(T))^\perp} = I|_{\mathcal{N}(p(T))^\perp}$;*
(ii) *T is a right p-partial isometry if and only if $TT^*|_{\overline{\mathcal{R}(p(T))}} = I|_{\overline{\mathcal{R}(p(T))}}$.*

Proof. (i) Suppose that T is a left p-partial isometry and $x \in \mathcal{N}(p(T))^\perp$. Then, by Lemma 2.7, $T^*Tx - x \in \mathcal{N}(p(T))^\perp$. Because $p(T)(T^*Tx - x) = 0$, we have $T^*Tx - x \in \mathcal{N}(p(T)) \cap \mathcal{N}(p(T))^\perp$ which gives $T^*Tx - x = 0$.

If $T^*T|_{\mathcal{N}(p(T))^\perp} = I|_{\mathcal{N}(p(T))^\perp}$, then $T^*T(\mathcal{N}(p(T))^\perp) \subseteq \mathcal{N}(p(T))^\perp$. Since $(T^*T)^* = T^*T$, we have $T^*T(\mathcal{N}(p(T))) \subseteq \mathcal{N}(p(T))$. For $x \in \mathcal{N}(p(T))^\perp$ and $y \in \mathcal{N}(p(T))$, we observe that $p(T)T^*Tx = p(T)x$ and $p(T)T^*Ty = 0 = p(T)y$. Thus, for all $z = x + y \in H = \mathcal{N}(p(T))^\perp \oplus \mathcal{N}(p(T))$, $p(T)T^*Tz = p(T)z$.

(ii) Assume that T is a right p-partial isometry and $y \in \mathcal{R}(p(T))$. There exists $x \in H$ such that $p(T)x = y$. So, $TT^*y = TT^*p(T)x = p(T)x = y$ and, by continuity of TT^*, we deduce that $TT^*|_{\overline{\mathcal{R}(p(T))}} = I|_{\overline{\mathcal{R}(p(T))}}$.

The equality $TT^*|_{\overline{\mathcal{R}(p(T))}} = I|_{\overline{\mathcal{R}(p(T))}}$ implies $TT^*p(T)x = p(T)x$, for all $x \in H$. □

A subspace M of H is a reducing subspace for $T \in \mathcal{B}(H)$ if both M and $M^\perp$ are invariant subspaces for T (or equivalently if M is invariant for both T and T^*). Recall that, if M is a reducing closed space for T, $(T|_M)^* = T^*|_M$. By Theorem 2.8, we have the next consequence.

Corollary 2.9. *Let $T \in \mathcal{B}(H)$ and $p \in \mathrm{Poly}$.*

(i) *If $\mathcal{N}(T)$ is a reducing subspace for T, then*

$$T \textit{ is a left } p-\textit{partial isometry } \Leftrightarrow\ T|_{\mathcal{N}(p(T))^\perp} \textit{ is an isometry.}$$

(ii) *If $\mathcal{R}(T)$ is a reducing subspace for T, then*

$$T \textit{ is a right } p-\textit{partial isometry } \Leftrightarrow\ T|_{\overline{\mathcal{R}(p(T))}} \textit{ is an co}-\textit{isometry.}$$

For a non-zero partial isometry $T \in \mathcal{B}(H)$, by [17, Corollary 32], note that $\|T\| = 1$ and $\mathcal{R}(T)$ is closed. It is interesting to consider these properties for left and right p-partial isometries.

Lemma 2.10. *Let $T \in \mathcal{B}(H)$ and $p \in \mathrm{Poly}$. If $\|T\| < 1$, then*

$$T \textit{ is a left (or right) } p-\textit{partial isometry } \Leftrightarrow\ p(T) = 0.$$

Proof. If T is a left p-partial isometry, then

$$\|p(T)\| = \|p(T)T^*T\| \leq \|p(T)\|\|T^*T\| = \|p(T)\|\|T\|^2.$$

In the case that $p(T) \neq 0$, notice that $1 \leq \|T\|^2$ which is a contradiction.

The equality $p(T) = 0$ yields that T is a left p-partial isometry.

The proof is similar for a right p-partial isometry. □

Theorem 2.11. *Let $T \in \mathcal{B}(H)$ and $p \in \mathrm{Poly}$.*

(i) *If T is a left p-partial isometry, then*

$$T(\mathcal{N}(p(T))^\perp) \perp T(\mathcal{N}(p(T))).$$

In addition, if $p(T) \neq 0$, then

$$\|T\| = \max\{1, \|T|_{\mathcal{N}(p(T))}\|\}.$$

(ii) *If T is a right p-partial isometry, then*

$$T^*(\overline{\mathcal{R}(p(T))}) \perp T^*(\overline{\mathcal{R}(p(T))}^\perp).$$

In addition, if $p(T) \neq 0$, then

$$\|T\| = \max\{1, \|T^*|_{\mathcal{N}(\overline{p}(T^*))}\|\}.$$

Proof. (i) By Theorem 2.8, T is a left p-partial isometry implies that $T^*T|_{\mathcal{N}(p(T))^\perp} = I|_{\mathcal{N}(p(T))^\perp}$. For $x \in \mathcal{N}(p(T))^\perp$ and $y \in \mathcal{N}(p(T))$, $< Tx, Ty >=< T^*Tx, y >=< x, y >= 0$ and thus $T(\mathcal{N}(p(T))^\perp) \perp T(\mathcal{N}(p(T)))$.

Let $z = x + y$, $x \in \mathcal{N}(p(T))^\perp$ and $y \in \mathcal{N}(p(T))$. Applying Lemma 2.7, $\mathcal{N}(p(T))$ is a reducing subspace of T^*T. Therefore,

$$\begin{aligned} \|Tz\|^2 &= < T^*T(x+y), x+y >=< T^*Tx, x > + < T^*Ty, y > \\ &= \|x\|^2 + \|Ty\|^2 \leq \max\{1, \|T|_{\mathcal{N}(p(T))}\|^2\}\|z\|^2 \end{aligned}$$

which gives $\|T\| \leq \max\{1, \|T|_{\mathcal{N}(p(T))}\|\}$. Using the assumption $p(T) \neq 0$ and Lemma 2.10, we deduce that $\max\{1, \|T|_{\mathcal{N}(p(T))}\|\} \leq \|T\|$ implying $\|T\| = \max\{1, \|T|_{\mathcal{N}(p(T))}\|\}$.

(ii) It follows by Lemma 2.8, duality and part (i). □

In the case that $\mathcal{N}(T) \subseteq \mathcal{N}(p(T))$, we present one equivalent condition for a left p-partial isometry T to be a partial isometry. Notice that the hypothesis $\mathcal{N}(T) \subseteq \mathcal{N}(p(T))$ can be replaced with $p(0) = 0$.

Corollary 2.12. *Let $p \in \mathrm{Poly}$ and $T \in \mathcal{B}(H)$ be a left p-partial isometry. If $\mathcal{N}(T) \subseteq \mathcal{N}(p(T))$, then the following statements are equivalent:*

(i) *T is a partial isometry;*

(ii) *$\|Tx\| = \|x\|$, for all $x \in \mathcal{N}(p(T)) \cap \mathcal{N}(T)^{\perp}$.*

Proof. Using Theorem 2.8 and the hypothesis T is a left p-partial isometry, we obtain, for $x \in \mathcal{N}(p(T))^{\perp}$,

$$\|Tx\|^2 =< T^*Tx, x >=< x, x >= \|x\|^2.$$

Because $\mathcal{N}(T)^{\perp} = \mathcal{N}(p(T))^{\perp} \oplus (\mathcal{N}(p(T)) \cap \mathcal{N}(T)^{\perp})$, we can write $x \in \mathcal{N}(T)^{\perp}$ as

$$x = x_1 + x_2 \in \mathcal{N}(p(T))^{\perp} \oplus (\mathcal{N}(p(T)) \cap \mathcal{N}(T)^{\perp}).$$

By Theorem 2.11,

$$\|Tx\|^2 = \|Tx_1\|^2 + \|Tx_2\|^2 = \|x_1\|^2 + \|Tx_2\|^2.$$

Hence, T is a partial isometry if and only if $\|Tx_2\|^2 = \|x_2\|^2$. $\square$

By Corollary 2.12 and duality, we verify the next result related to a right p-partial isometry.

Corollary 2.13. *Let $p \in \mathrm{Poly}$ and $T \in \mathcal{B}(H)$ be a right p-partial isometry. If $\overline{\mathcal{R}(p(T))} \subseteq \overline{\mathcal{R}(T)}$, then the following statements are equivalent:*

(i) *T is a partial isometry;*

(ii) *$\|T^*x\| = \|x\|$, for all $x \in \overline{\mathcal{R}(p(T))}^{\perp} \cap \mathcal{R}(T)$.*

The next example contains some polynomials for which the conditions $\mathcal{N}(T) \subseteq \mathcal{N}(p(T))$ and $\overline{\mathcal{R}(p(T))} \subseteq \overline{\mathcal{R}(T)}$ of Corollary 2.13 and Corollary 2.13, respectively, are satisfied.

Example 2.14. For $p \in \mathrm{Poly}$ such that $p(0) = 0$, note that $p(t) = tq(t)$ for some $q \in \mathrm{Poly}$. Therefore, if $T \in \mathcal{B}(H)$, $p(T) = Tq(T) = q(T)T$ implies $\mathcal{N}(T) \subseteq \mathcal{N}(p(T))$ and $\overline{\mathcal{R}(p(T))} \subseteq \overline{\mathcal{R}(T)}$.

In the following theorem, we obtain the decomposition for a left p-partial isometry.

Theorem 2.15. *Let $T \in \mathcal{B}(H)$ and $p \in \mathrm{Poly}$. Then the following statements are equivalent:*

(i) *T is a left p-partial isometry;*

(ii) *there exist $B \in \mathcal{B}(H)$ and a partial isometry $A \in \mathcal{B}(H)$ such that*

$$T = A + B, \qquad AB = A^*B = BA^* = 0, \qquad p(B) = p(0)A^*A.$$

Proof. (i) $\Rightarrow$ (ii): Suppose that T is a left p-partial isometry and $Q \in \mathcal{B}(H)$ is the orthogonal projection onto $\mathcal{N}(p(T))^{\perp}$. Set $A = TQ$ and $B = T(I - Q)$. By Lemma 2.7, $\mathcal{N}(p(T))$ is an invariant subspace of T and T^*T. Then we can easily check that $AB = A^*B = BA^* = 0$, $(I-Q)T(I-Q) = T(I-Q)$ and $B^n = T^n(I-Q)$, for $n \in \mathbb{N}$. Using Theorem 2.8, we get $T^*TQ = Q$ which yields $A^*A = QT^*TQ = Q$, $AA^*A = TQ = A$ and $p(B) = p(T)(I - Q) + p(0)Q = p(0)A^*A$.

(ii) $\Rightarrow$ (i): Let $p(z) = \sum\limits_{i=0}^{n} a_i z^i$. From $AB = A^*B = BA^* = 0$ and $p(B) = a_0A^*A$, we firstly obtain

$$p(T) = a_n \sum_{i=0}^{n} B^i A^{n-i} + a_{n-1} \sum_{i=0}^{n-1} B^i A^{n-1-i} + \cdots + a_1(A + B) + a_0 I$$

and

$$\begin{aligned} p(T)B^*B &= a_n B^n B^*B + a_{n-1}B^{n-1}B^*B + \cdots + a_1 BB^*B + a_0 B^*B \\ &= p(B)B^*B = a_0 A^*AB^*B = 0. \end{aligned}$$

Further, the equality $AA^*A = A$ gives

$$\begin{aligned} p(T)A^*A &= a_n \sum_{i=0}^{n-1} B^i A^{n-i} + a_{n-1} \sum_{i=0}^{n-2} B^i A^{n-1-i} + \cdots + a_1 A + a_0 A^*A \\ &= p(T) - p(B) + a_0 A^*A = p(T). \end{aligned}$$

Thus,

$$p(T)T^*T = p(T)(A^*A + B^*B) = p(T)A^*A + p(T)B^*B = p(T).$$

□

By duality and Theorem 2.15, we obtain the decomposition for a right p-partial isometry.

Theorem 2.16. *Let $T \in \mathcal{B}(H)$ and $p \in \mathrm{Poly}$. Then the following statements are equivalent:*

(i) *T is a right p-partial isometry;*
(ii) *there exist $B \in \mathcal{B}(H)$ and a partial isometry $A \in \mathcal{B}(H)$ such that*

$$T = A + B, \qquad BA = A^*B = BA^* = 0, \qquad p(B) = AA^*p(0).$$

Corollary 2.17. *Let $T \in \mathcal{B}(H)$ and $p \in \mathrm{Poly}$.*

(i) *If T is a left p-partial isometry, then $T(\mathcal{N}(p(T))^\perp)$ is closed and*

$$\mathcal{R}(T) \text{ is closed } \Leftrightarrow \mathcal{R}(T|_{\mathcal{N}(p(T))}) \text{ is closed.}$$

In addition, if $\dim \mathcal{N}(p(T)) < +\infty$, then $\mathcal{R}(T)$ is closed.

(ii) *If T is a right p-partial isometry, then $T^*(\overline{\mathcal{R}(p(T))})$ is closed and*

$$\mathcal{R}(T) \text{ is closed } \Leftrightarrow \mathcal{R}(T^*|_{\overline{\mathcal{R}(p(T))}^\perp}) \text{ is closed.}$$

In addition, if $\dim \overline{\mathcal{R}(p(T))}^\perp < +\infty$, then $\mathcal{R}(T)$ is closed.

Proof. (i) Using the same notations as in the proof of Theorem 2.15, we have that A is a partial isometry and $\mathcal{R}(A) = T(\mathcal{N}(p(T))^\perp)$ is closed. By the equality $\mathcal{R}(T) = \mathcal{R}(A) \oplus \mathcal{R}(B)$, we obtain

$$\mathcal{R}(T) \text{ is closed } \Leftrightarrow \mathcal{R}(B) \text{ is closed.}$$

In the case that $\dim \mathcal{N}(p(T)) < +\infty$, we have $\dim \mathcal{R}(B) < +\infty$ which gives that $\mathcal{R}(T)$ is closed.

(ii) By duality and part (i), we prove this part. □

For a partial isometry $T \in \mathcal{B}(H)$, it is well-known that $\|Tx\| = \|x\|$, for all $x \in \mathcal{N}(T)^\perp$ (or $\|T^*x\| = \|x\|$, for all $x \in \mathcal{R}(T)$). Now, we generalize this result for left and right p-partial isometries.

Corollary 2.18. *Let $T \in \mathcal{B}(H)$ and $p \in \mathrm{Poly}$. Then*

(i) *T is a left p-partial isometry if and only if $\|Tx\| = \|x\|$ for all $x \in \mathcal{N}(p(T))^\perp$ and $T^*T(\mathcal{N}(p(T))) \subseteq \mathcal{N}(p(T))$.*
(ii) *T is a right p-partial isometry if and only if $\|T^*x\| = \|x\|$, for all $x \in \overline{\mathcal{R}(p(T))}$ and $TT^*(\overline{\mathcal{R}(p(T))}) \subseteq \overline{\mathcal{R}(p(T))}$.*

Proof. (i) If T is a left p-partial isometry, by Lemma 2.7, $\mathcal{N}(p(T))$ is an invariant subspace for T^*T and, by Theorem 2.8,

$$\|Tx\|^2 =< T^*Tx, x >=< x, x >= \|x\|^2,$$

for all $x \in \mathcal{N}(p(T))^\perp$.

On the other hand, let $Q \in \mathcal{B}(H)$ be the orthogonal projection onto $\mathcal{N}(p(T))^\perp$ and $x \in H$. We can write $x = Qx + (I - Q)x$. Since $\|TQx\| = \|Qx\|$ and $\mathcal{R}(I - Q)$ is invariant for T^*T, then

$$\begin{aligned} < T^*TQx, x > &= < T^*TQx, Qx + (I - Q)x > \\ &= < T^*TQx, Qx > + < Qx, T^*T(I - Q)x > \\ &= < TQx, TQx > + < Qx, (I - Q)x > \\ &= < Qx, Qx > +0 =< Qx, x >, \end{aligned}$$

which yields $T^*TQ = Q$. So, by Theorem 2.8, T is a left p-isometry.

(ii) This part follows from (i) by duality. □

We can easily check the following auxiliary result.

Lemma 2.19. *Let $T \in \mathcal{B}(H)$ and $p \in$ Poly. If T is a left (or right) p-partial isometry and $M \subset H$ is an invariant closed subspace of T and T^*T, then $T|_M$ is left (right) p-partial isometry.*

Recall that a n-left generalized partial isometry can be written as a direct sum of an isometry and a nilpotent operator of degree n [13, Theorem 3.6]. When $\mathcal{N}(T) \subseteq \mathcal{N}(p(T))$ (or $p(0) = 0$), we prove that a left p-partial isometry is the direct sum of an isometry and an operator for which the value of polynomial p is zero.

Theorem 2.20. *Let $T \in \mathcal{B}(H)$ and $p \in$ Poly. If $\mathcal{N}(T) \subseteq \mathcal{N}(p(T))$, then T is a left p-partial isometry if and only if there exist two closed subspaces M and N of H such that*

(i) *$H = M \oplus N$ and $T^*T(M) \subseteq M$,*

(ii) *$T|_M$ is an isometry, $T(N) \subseteq N$ and $N \subseteq \mathcal{N}(p(T))$.*

Proof. Suppose that T is a left p-partial isometry, $M = \mathcal{N}(p(T))^\perp$ and $N = \mathcal{N}(p(T))$. Applying Lemma 2.7 and Corollary 2.18, we obtain that (i) and (ii) hold.

Conversely, by $T^*T(M) \subseteq M$, $M \perp N$ and $H = M \oplus N$, we have that N is an invariant subspace of T^*T. Then, for $x = x_1 + x_2$, $x_1 \in M$ and $x_2 \in N$, we get $T^*Tx_1 = x_1$ and $T^*Tx_2 \in \mathcal{N}(p(T))$, which implies

$$p(T)T^*Tx = p(T)T^*Tx_1 + p(T)T^*Tx_2 = p(T)x_1 = p(T)x.$$

□

Using Theorem 2.20, we obtain the next theorem related to right p-partial isometries.

Theorem 2.21. *Let $T \in \mathcal{B}(H)$ and $p \in$ Poly. If $\overline{\mathcal{R}(p(T))} \subseteq \overline{\mathcal{R}(T)}$, then T is a right p-partial isometry if and only if there exist two closed subspaces M and N of H such that*

(i) *$H = M \oplus N$ and $TT^*(M) \subseteq M$,*

(ii) *$T^*|_M$ is an isometry, $T^*(N) \subseteq N$ and $N \subseteq \mathcal{N}((p(T))^*)$.*

We now study the decomposition of an operator which is both p-normal and left (or right) p-partial isometry. For $T \in \mathcal{B}(H)$, we can consider the following condition proposed by Apostol [3]:

$$\lim_n \|T^*T^n - T^nT^*\|^{\frac{1}{n}} = 0. \tag{2.1}$$

Also, denote by

$$H_0 = \{x \in H : \lim_n \|T^nx\|^{\frac{1}{n}} = 0\}.$$

Observe that H_0 is subspace of H and it is invariant under T.

Theorem 2.22. *Let $p \in \mathrm{Poly}$ and $T \in \mathcal{B}(H)$ be p-normal.*

(i) *If T is a left p-partial isometry, then T is decomposed by $\mathcal{N}(p(T))^{\perp}$ and $\mathcal{N}(p(T))$ in the direct sum $T = S \oplus A$, where S is an isometry and $p(A) = 0$. In addition, if T satisfies* (2.1) *and $\mathcal{N}(p(T)) = \overline{H_0}$, then S is unitary.*

(ii) *If T is a right p-partial isometry, then T is decomposed by $\overline{\mathcal{R}(p(T))}$ and $\overline{\mathcal{R}(p(T))}^{\perp}$ in the direct sum $T = S \oplus A$, where S is a co-isometry and $p(A) = 0$. In addition, if T^* satisfies* (2.1) *and $\overline{\mathcal{R}(p(T))}^{\perp} = \overline{H_0}$, then S is unitary.*

Proof. (i) From the equality $p(T)T^* = T^*p(T)$, we have that $\mathcal{N}(p(T))$ is a reducing subspace for T. By Corollary 2.9, we deduce that $S = T|_{\mathcal{N}(p(T))^{\perp}}$ is an isometry. It is clear that $p(A) = 0$ for $A = T|_{\mathcal{N}(p(T))}$.

Using [3] (or [11, Proposition 2]), if T satisfies (2.1) and $\mathcal{N}(p(T)) = \overline{H_0}$, we deduce that $S = T|_{\overline{H_0}^{\perp}}$ is normal an so S is unitary.

(ii) This part is evident by (i) and duality. □

Theorem 2.23. *Let $p \in \mathrm{Poly}$, $T \in \mathcal{B}(H)$ satisfy* (2.1) *and $\mathcal{N}(p(T)) \subseteq \overline{H_0}$. Then T is a left p-partial isometry if and only if there exist three subspaces $M_1, M_2, M_3 \subseteq H$ such that*

(i) *M_1, M_2, M_3 are reducing subspaces of T^*T;*

(ii) *$H = M_1 \oplus M_2 \oplus M_3$;*

(iii) *M_1 is invariant under T, $M_1 \subseteq \mathcal{N}(P(T))$, $T|_{M_2}$ is isometry, M_3 reduce T and $T|_{M_3}$ is unitary.*

Proof. Suppose that T is a left p-partial isometry. By the hypothesis $\mathcal{N}(p(T)) \subseteq \overline{H_0}$, for $M_1 = \mathcal{N}(p(T))$, $M_2 = \mathcal{N}(p(T))^{\perp} \cup \overline{H_0}$ and $M_3 = \overline{H_0}^{\perp}$, we can write $\mathcal{N}(p(T))^{\perp} = M_2 \oplus M_3$ and $H = M_1 \oplus M_2 \oplus M_3$. Using Lemma 2.7 and [3] (or [11, Proposition 2]), we have that M_1, M_2, M_3 are reducing subspaces of T^*T and M_3 is a reducing subspace of T. Applying Theorem 2.8, Lemma 2.19, [11, Proposition 2] and $\overline{H_0}^{\perp} \subseteq \mathcal{N}(p(T))^{\perp}$, we conclude that $T|_{M_2}$ is isometry and $T|_{M_3}$ is unitary.

Conversely, let $x = x_1 + x_2 + x_3 \in M_1 \oplus M_2 \oplus M_3$. Then, by $(T^*T)|_{M_k} = T^*|_{M_k} T|_{M_k}$, $k = \overline{1, 3}$,

$$\begin{aligned} p(T)T^*Tx &= p(T)T^*|_{M_1}T|_{M_1}x_1 + p(T)T^*|_{M_2}T|_{M_2}x_2 + p(T)T^*|_{M_3}T|_{M_3}x_3 \\ &= p(T)(x_2 + x_3) = p(T)x. \end{aligned}$$

So, T is a left p-partial isometry. □

Recall that, $T \in \mathcal{B}(H)$ is quasicommuting [11] if

$$\lim_n \|T^*T^n - T^nT^*\| = 0. \tag{2.2}$$

Set

$$H_1 = \{x \in H : \lim_n \|T^n x\| = 0\}.$$

Theorem 2.24. *Let $p \in \mathrm{Poly}$, $T \in \mathcal{B}(H)$ be a left (or right) p-partial isometry satisfying* (2.2) *and $\mathcal{N}(p(T)) \subseteq H_1$. If there is a number $M > 0$ such that $\|T^n\| \le M$ for all n, then*

$$T = U \oplus V,$$

where U is unitary and, for x in the domain of V, $\lim_n \|V^n x\| = 0$ ($\lim_n \|(V^*)^n x\| = 0$).

Proof. By [11, Lemma 2 and Theorem 8], H_1 is a reducing closed subspace of T. Set $U = T|_{H_1^{\perp}}$ and $V = T|_{H_1}$. Using $H_1^{\perp} \subseteq \mathcal{N}(p(T))^{\perp}$, [11, Theorem 8] and Theorem 2.8, we have that U is unitary. □

We now extend some well-known results related to the spectral theory for partial isometries [11] and semi-generalized partial isometries [13] to left p-partial isometries. For $T \in \mathcal{B}(H)\backslash\{0\}$, define

$$\Gamma_T = \{\lambda \in \mathbb{C} : \frac{1}{\|T\|} \le |\lambda| \le \|T\|\}$$

and if $T = 0$, let $\Gamma_T = \emptyset$. Notice that $\Gamma_T = \Gamma_{T^*}$ and $\Gamma_T = \emptyset$ when $\|T\| < 1$.

Theorem 2.25. *Let $p \in \mathrm{Poly}$ and $T \in \mathcal{B}(H)$ be p-normal. If T is a left p-partial isometry, then*

$$\sigma_p(T) \subset \Gamma_T \cup \{\lambda \in \mathbb{C} : p(\lambda) = 0\} \cup \{0\}.$$

Proof. If $\lambda \in \sigma_p(T)\backslash\{0\}$, then exists a nonzero vector $x \in H$ such that $Tx = \lambda x$. Hence, $T^n x = \lambda^n x$, for $n \in \mathbb{N}$, $p(T)x = p(\lambda)x$ and $x = \lambda^{-1}Tx$. Since T is p-normal and left p-partial isometry, we get

$$\begin{aligned} 0 &= T^*p(T)x - p(T)T^*x = p(\lambda)T^*x - \lambda^{-1}p(T)T^*Tx \\ &= p(\lambda)T^*x - \lambda^{-1}p(T)x = p(\lambda)T^*x - \lambda^{-1}p(\lambda)x \\ &= p(\lambda)(T^*x - \lambda^{-1}x) \end{aligned}$$

which gives $p(\lambda) = 0$ or $T^*x = \lambda^{-1}x$, i.e. $\lambda^{-1} \in \sigma_p(T^*)$. Because $|\lambda| \le \|T\|$ and $|\lambda^{-1}| \le \|T\|$, we have $\lambda \in \Gamma_T$. $\square$

Theorem 2.26. *Let $p \in \mathrm{Poly}$ and $T \in \mathcal{B}(H)$ be p-normal. If T is a left p-partial isometry, then*

$$\sigma_{ap}(T) \subset \Gamma_T \cup \{\lambda \in \mathbb{C} : p(\lambda) = 0\} \cup \{0\}.$$

Proof. Berberian [6] introduced an extension of a Hilbert space H to a Hilbert space K and reduced the problem of the approximate point spectrum of an operator T on H to the point spectrum problem of the corresponding operator T' on K. Recall that, for $T, S \in \mathcal{B}(H)$, $(T^*)' = (T')^*$, $I' = I$, $(\lambda T)' = \lambda T'$, $(S+T)' = S' + T'$, $(TS)' = T'S'$ and $\|T'\| = \|T\|$. Because T' is p-normal and left p-partial isometry, then, by [6, Theorem 1] and Theorem 2.25,

$$\sigma_{ap}(T) = \sigma_p(T') \subset \Gamma_T \cup \{\lambda \in \mathbb{C} : p(\lambda) = 0\} \cup \{0\}.$$

$\square$

In the end, we apply polynomially partial isometries to solve some equations.

Remark 2.27. Let $T \in \mathcal{B}(H)$, $p \in \mathrm{Poly}$ and

$$S_n = (T^*T)^nT^* = T^*(TT^*)^n, \qquad n \in \mathbb{N}.$$

(i) If T is a left p-partial isometry, then, for all $n \in \mathbb{N}$,

$$p(T)(T^*T)^n = p(T) \quad \text{and} \quad p(T)S_nT = p(T).$$

For arbitrary $U, V, R, W \in \mathcal{B}(H)$, set

$$E = U - T^*TU + Q_1W \quad \text{and} \quad F = RQ_2,$$

where Q_1 is a projection onto $\mathcal{N}(p(T))$ and Q_2 is a projection onto $\overline{\mathcal{R}(T)}^{\perp}$. The equalities $p(T)E = 0$ and $FT = 0$ imply that $(T^*T)^n + E$ is the solution of the equation $p(T)S = p(T)$ and $S_n + E + F$ is the solution of the equation $p(T)ST = p(T)$, for all $n \in \mathbb{N}$.

(ii) If T is a right p-partial isometry, then, for all $n \in \mathbb{N}$,

$$(TT^*)^np(T) = p(T) \quad \text{and} \quad TS_np(T) = p(T).$$

For arbitrary $U, V, R, W \in \mathcal{B}(H)$, let

$$E = U - UTT^* + WQ_1 \quad \text{and} \quad F = Q_2R,$$

where Q_1 is a projection onto $\overline{\mathcal{R}(p(T))}^{\perp}$ and Q_2 is a projection onto $\mathcal{N}(T)$. From $Ep(T) = 0$ and $TF = 0$, we have that $(TT^*)^n + E$ is the solution of the equation $Sp(T) = p(T)$ and $S_n + E + F$ is the solution of the equation $TSp(T) = p(T)$, for all $n \in \mathbb{N}$.

Acknowledgment. The author is supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, grant no. 451-03-68/2022-14/200124, and the bilateral project between Serbia and Slovenia (Generalized inverses, operator equations and applications), grant number 337-00-21/2020-09/32.

References

[1] A. Alahmari, M. Mabrouk and M.A. Taoudi, *Discussions on partial isometries in Banach spaces and Banach algebras*, Bull. Korean Math. Soc. **54** (2), 485-495, 2017.

[2] S.A. Aluzuraiqi and A.B. Patel, *On n-normal operators*, General Math. Notes **1**, 61-73, 2010.

[3] C. Apostol, *Propriétés de certains operateurs bornés des espaces de Hilbert II*, Rev. Roum. Math. Purs Appl. **12**, 759-762, 1967.

[4] M.L. Arias and M. Mbekhta, *On partial isometries in C*–algebras*, Studia Math. **205** (1), 71-82, 2011.

[5] C. Badea and M. Mbekhta, *Operators similar to partial isometries*, Acta Sci. Math. (Szeged) **71**, 663-680, 2005.

[6] S.K. Berberian, *Approximate proper vectors*, Proc. Amer. Math. Soc. **13**, 111-114, 1962.

[7] M. Chō, J.E. Lee, K. Tanahashi and A. Uchiyama, *Remarks on n-normal operators*, Filomat **32** (15), 5441-5451, 2018.

[8] M. Chō and B. Načevska Nastovska, *Spectral properties of n-normal operators*, Filomat **32** (14), 5063-5069, 2018.

[9] D.S. Djordjević, M. Chō and D. Mosić, *Polynomially normal operators*, Ann. Funct. Anal. **11**, 493-504, 2020.

[10] B.P. Duggal and I.H. Kim, *On nth roots of normal operators*, Filomat **34** (8), 2797-2803, 2020.

[11] I. Erdelyi and F.R. Miller, *Decomposition theorems for partial isometries*, J. Math. Anal. Appl. **30**, 665-679, 1970.

[12] F.J. Fernández-Polo and A. Peralta, *Partial isometries: a survey*, Adv. Oper. Theory **3** (1), 75-116, 2018.

[13] Z. Garbouj and H. Skhiri, *Semi-generalized partial isometries operators*, Results Math. **75**, 15, 2020.

[14] P.R. Halmos and J.E. McLaughlin, *Partial isometries*, Pac. J. Appl. Math. **13**, 585-596, 1963.

[15] P.R. Halmos and L.J. Wallen, *Powers of partial isometries*, J. Math. Mech. **19**, 657-663, 1969/1970.

[16] O.A. Mahmoud Sid Ahmed, *Generalization of m-partial isometries on a Hilbert space*, Int. J. Pure Appl. Math. **104** (4), 599-619, 2015.

[17] M. Mbekhta, *Partial isometries and generalized inverses*, Acta Sci. Math. (Szeged) **70**, 767-781, 2004.

[18] D. Mosić and D.S. Djordjević, *Partial isometries and EP elements in Banach algebras*, Abstr. Appl. Anal. **2011**, Article ID 540212, 9 pages, 2011.

[19] D. Mosić and D.S. Djordjević, *Weighted partial isometries and weighted–EP elements in C^*-algebra*, Appl. Math. Comput. **265**, 17-30, 2015.

[20] A. Saddi and O.A. Mahmoud Sid Ahmed, *m-partial isometries on Hilbert spaces*, Internat. J. Functional Analysis, Operator Theory and Applications **2** (1), 67-83, 2010.

[21] C. Schmoeger, *Partial isometries on Banach spaces*, Seminar LV, No. **20**, 13 pp. 28.02.2005.