Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (2) (2023), 356 – 366
DOI : 10.15672/hujms.1095437

Research Article

Smallest maximal matchings of graphs

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Abstract

Let $G = (V_G, E_G)$ be a simple and connected graph. A set $M \subseteq E_G$ is called a matching of G if no two edges of M are adjacent. The number of edges in M is called its size. A matching M is maximal if it cannot be extended to a larger matching in G. The smallest size of a maximal matching is called the saturation number of G. In this paper we are concerned with the saturation numbers of lexicographic product of graphs. We also address and solve an open problem about the size of maximum matchings in graphs with a given maximum degree Δ.

Mathematics Subject Classification (2020). 05C76, 05C70, 05C69

Keywords. maximal matching, product graph, saturation number, independent domination number

1. Introduction

Matchings in graphs serve as useful models of many problems in engineering and in natural and social sciences. Whenever we have a collection $\mathcal{G}$ of interacting elements or entities capable of forming exclusive "monogamous" pairings by choosing one of several possible mates, we can study it by studying matchings in the corresponding graph G: The entities are represented by its vertices, their potential pairings by its edges, and actually formed pairings by matchings in G. Examples range from human marriages to adsorption of dimers on structured substrates. The exclusivity of pairings is reflected in the fact that no two edges from a matching can have a vertex in common. Another immediate consequence of the exclusivity condition is that the maximum number of formed pairs cannot exceed one half of the number of entities. The number of pairs is called the size of the matching and it is one of the most important matching-related graph invariants.

In most cases, small matchings are not interesting. As a rule, one is almost always interested in large matchings. A matching can be large in at least two different senses. One way to have a large matching is to have many edges in it. Matchings containing the largest possible number of edges are called maximum matchings. Another way to have a large matching is to have a matching that cannot be extended to a valid matching by

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Received: 13.04.2022; Accepted: 24.08.2022

adding an edge. Such matchings are called maximal matchings. Obviously, any maximum matching is also maximal, but the opposite is generally not true.

There is a huge difference between maximum and maximal matchings. The maximum ones are well researched and well understood. There is a well developed structural theory for them and they are (reasonably) easy to construct. No such theory is available for maximal matchings so far.

All maximum matchings are of the same size. This size reflects the highest possible yield of any process modeled by matchings on a given graph. Maximal matchings, however, usually come in a range of different sizes. At the upper end of this range are maximum matchings. The lower end is far more interesting, since it provides the information on the lowest possible efficiency of the considered process. Unfortunately, it is algorithmically difficult to compute the cardinality of a smallest maximal matching even for quite restricted classes of graphs. Hence, it is of considerable interest to know this quantity, known as the saturation number, for as many classes of graphs as possible.

In this paper we study the saturation number of graphs that arise via binary operation known as the composition or the lexicographic product of two graphs. Besides, we improve a previously known lower bound on the saturation number in terms of maximum degree.

2. Preliminaries

In this section we give definitions and notations that will be used in the paper. For more information on matching-related stuff we recommend the classical monograph by Lovász and Plummer [18], and for general references we recommend [7] or [20].

All graphs considered in this paper are connected and simple, i.e., they do not have loops and multiple edges.

For a graph G we denote by V_G the set of its vertices and by E_G the set of its edges. The number of vertices of G is the **order** of G. The **degree** of a vertex v of G is the number of edges of G incident to v. The maximum degree and the minimum degree of a graph G is usually denoted by Δ (or Δ_G) and δ (or δ_G), respectively. A graph whose vertices can be divided into two disjoint sets U and W such that each edge has one end in U and the other in W is called **bipartite**. We denote the path and cycle of order n by P_n and C_n, respectively. The cardinality of a set S is denoted by $|S|$.

Let $G = (V_G, E_G)$ be a graph. A collection of edges $M_G \subseteq E_G$ is called a **matching** of G if no two edges of M_G are adjacent. The vertices incident to the edges of a matching M_G are said to be **saturated** by M_G (or M_G-saturated); the others are said to be **unsaturated** (or M_G-unsaturated). A matching whose edges meet all vertices of G is called a **perfect matching** of G. If there does not exist a matching M'_G in G such that $|M_G| < |M'_G|$, then M_G is called a **maximum matching** of G. A matching M_G is **maximal** if it cannot be extended to a larger matching in G. The cardinality of any maximum matching, $\nu(G)$, and the cardinality of any smallest maximal matching in G, $s(G)$, are called the **matching number** and the **saturation number** of G, respectively.

If all maximal matchings in G are also maximum matchings (i.e., if $s(G) = \nu(G)$), then the graph G is called **equimatchable**. If any maximal matching in G is also perfect, then G is called **randomly matchable**. For any given even $n \geq 4$, there are only two randomly matchable graphs, K_n and $K_{n/2,n/2}$; equimatchable graphs are more numerous and far more interesting.

As mentioned above, smallest maximal matchings have a wide range of applications. For example, one can see [21] for application of smallest maximal matchings related to a telephone switching network and see [8] for applications of smallest maximal matchings related to the stable marriage problem which involves a set of institutions and applicants. See also [12] for an enumeratively oriented approach to smallest maximal matchings in a chemical context and [10] for some generalizations.

Yannakakis and Gavril [21] proved that finding a smallest maximal matching is NP-hard even for bipartite (or planar) graphs with maximum degree 3. Thirteen years later, Horton and Kilakos [17] extended their results to planar bipartite graphs and planar cubic graphs. After that, a similar work on the class of k-regular bipartite graphs was done by Demange and T. Ekim [8]. The saturation number is, hence, difficult to compute. It can be, however, easily approximated within a factor of two. This follows from bounds $\frac{\nu(G)}{2} \leq s(G) \leq \nu(G)$ and the fact that $\nu(G)$ can be efficiently computed (see [18]). See [5,19,22] for more details on the lower bound $s(G) \geq \frac{\nu(G)}{2}$. Another lower bound, $s(G) \geq \frac{n-\alpha(G)}{2}$, is easy to prove, but not very useful, since it is not easy to compute the independence number $\alpha(G)$ for general graphs.

The NP-hardness of finding smallest maximal matchings justifies efforts to study them in particular classes of graphs. We concentrate here on the graphs that arise from simpler graphs via lexicographic product.

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. The **lexicographic product** of G and H, denoted by $G[H]$, is the graph whose vertex set is $V_G \times V_H$ and two vertices (g, h) and (g', h') are adjacent when (gg' is an edge of G) or (hh' is an edge of H and g is equal to g'), see Figs. 1 and 2 and also [16] for more details. The graph $G[H]$ is sometimes referred to as the **composition** of G and H.

Graph products play an important role in study of computer architecture, parallel algorithms and also in other branches of science.

A set of pairwise non-adjacent vertices of a graph G is called the **independent set**. The size of a largest independent set is called the **independence number** of G and denoted by $\alpha(G)$.

For a graph $G = (V_G, E_G)$, we say $D_G \subseteq V_G$ is an **independent dominating set** in G if D_G is an independent set and each vertex of $V_G \setminus D_G$ is adjacent to at least one vertex in D_G. The **independent domination number** of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set of G. If we drop the requirement of independence, we obtain **dominating sets**, and the smallest cardinality of a dominating set in G is the **domination number** of G, denoted by $\gamma(G)$.

A vertex with degree zero is called an isolated vertex and a vertex with degree one is called a pendent vertex.

For a nonempty set X of edges of a graph G, the **subgraph** $G(X)$ **induced by** X has X as its edge set and a vertex v belongs to $G(X)$ if v is incident with at least one edge in X. Also, for a nonempty set Y of vertices of a graph G, the **subgraph** $G(Y)$ **induced by** Y is the subgraph of G whose vertex set is Y and whose edge set consists of all edges of G which have both ends in Y. (This definition is taken from [7], and the original notation $G[X]$ is slightly modified in order to avoid confusion with the lexicographic product.)

3. Lexicographic product

In this section we present upper bounds on the saturation number of the lexicographic product of two graphs in terms of independence, matching and saturation numbers of the factors.

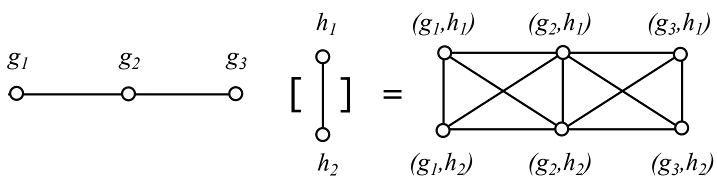

Figure 1. $P_3[P_2]$.

Let G and H be two graphs with $V(G) = \{g_1, \dots, g_n\}$ and $V(H) = \{h_1, \dots, h_m\}$. Clearly there are n copies of H in $G[H]$. Now we investigate copies of G in $G[H]$. For considering i-th vertex of a copy of G, there are m possible cases $(g_i, h_1), (g_i, h_2), \dots, (g_i, h_m)$ and so $G[H]$ contains m^n copies of G. Now suppose that $G_i = G[H](V_i)$, $i \in \{1, \dots, m\}$, where $V_i = (g_1, h_i) \dots (g_n, h_i)$. Clearly G_is are m copies of G without common vertices in $G[H]$. Let $G_{1_j}, \dots, G_{l_j}$ be a sequence of copies of G in $G[H]$ where $l_j > m$. All of these copies have a vertex in $\{(g_1, h_1), (g_1, h_2), \dots, (g_1, h_m)\}$. Then the number of these copies, l_j, is more than the number of the vertices of $\{(g_1, h_1), (g_1, h_2), \dots, (g_1, h_m)\}$; hence by the pigeonhole principle we can conclude that there are not more than m copies of G with distinct vertices in $G[H]$.

For more illustration, consider graphs $P_3 := g_1g_2g_3$ and $P_2 := h_1h_2$ and set

$$\begin{aligned}
&G_1 = P_3^1 := (g_1, h_1)(g_2, h_1)(g_3, h_1),\ P_3^2 := (g_1, h_1)(g_2, h_1)(g_3, h_2),\\
&P_3^3 := (g_1, h_1)(g_2, h_2)(g_3, h_1),\ P_3^4 := (g_1, h_1)(g_2, h_2)(g_3, h_2),\\
&G_2 = P_3^5 := (g_1, h_2)(g_2, h_2)(g_3, h_2),\ P_3^6 := (g_1, h_2)(g_2, h_2)(g_3, h_1),\\
&P_3^7 := (g_1, h_2)(g_2, h_1)(g_3, h_1),\ P_3^8 := (g_1, h_2)(g_2, h_1)(g_3, h_2).
\end{aligned}$$

One can check that $P_3^1, \dots, P_3^8$ are copies of P_3, and G_1 and G_2 are copies of P_3 with distinct vertices in $P_3[P_2]$.

Note that, in the following, we will work with m copies of G with distinct vertices.

As a warm-up, we establish a best possible lower bound on the saturation number of $G[H]$.

Theorem 3.1. *Let G and H be two graphs with n and m vertices, respectively. Then*

$$s(G[H]) \geq \frac{1}{2}\left(mn - \alpha(G)\alpha(H)\right).$$

Proof follows immediately by noting that $\alpha(G[H]) = \alpha(G)\alpha(H)$ ([13]) and $s(G) \geq (V(G) - \alpha(G))/2$ for any graph G. The above bound is sharp if the maximum independent set I constructed by choosing $\alpha(H)$ independent vertices in each of $\alpha(G)$ copies of H corresponding to the vertices of any maximum independent set of G is **nice**, i.e., if the graph induced by $V(G[H]) \backslash I$ has a perfect matching. That happens, for example, for $P_5[C_4]$, but not for $K_{1,3}[C_4]$.

Let M be a matching in a graph G. An M-augmenting path in G is a path whose edges are alternately in M and $E_G \setminus M$ such that neither its origin nor its terminus is covered by M.

Theorem 3.2 (Berge's theorem [6]). *A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.*

In the rest of this section we employ the same idea to construct upper bounds on the saturation number of $G[H]$.

Theorem 3.3. *Suppose G and H are two graphs with n and m vertices, respectively. If at least one of them has a perfect matching, then*

$$\begin{aligned}
s(G[H]) \leq \min\{&\alpha(G)s(H) + (m - 2\nu(H))\nu(G) + (n - \alpha(G))\nu(H),\\
&\alpha(H)s(G) + (n - 2\nu(G))s(H) + (m - \alpha(H))\nu(G)\}.
\end{aligned}$$

Proof. Suppose G and H are two graphs. Set

$$\begin{aligned}
&A = \{(g,h)(g,h') \mid (g \in X \text{ and } hh' \in M_H)\},\\
&B = \{(g,h)(g,h') \mid (g \in X^c \text{ and } hh' \in M'_H)\},\\
&C = \Big\{(g,h)(g',h') \mid gg' \in M'_G \text{ and } \Big((g,g' \in X^c \ \&\ h = h' \in V_H \setminus V_{H(M'_H)})\\
&\quad \text{or } (g \in X, g' \in X^c \ \&\ (h,h') \in W\Big)\Big\},\\
&M = A \cup B \cup C,
\end{aligned}$$

where X is a largest independent set of G, $X^c = V_G \setminus X$, M_H is a smallest maximal matching of H, M'_G and M'_H are maximum matchings of G and H, respectively, and W is a largest subset of $(V_H \setminus V_{H(M_H)}) \times (V_H \setminus V_{H(M'_H)})$ with this property that $h \neq h''$ and $h' \neq h'''$ for each $(h,h'),(h'',h''') \in W$. (Our strategy to define A, B and C is that edges of $A \cup B$ are in copies of H and edges of C are in copies of G. Each vertex (g,h) such that $g \in X$ and $h \in V_H \setminus V_{H(M_H)}$ is unsaturated by A. Also each vertex (g,h) such that $g \in X^c$ and $h \in V_H \setminus V_{H(M'_H)}$ is unsaturated by B. Thus two ends of each edge of C are unsaturated by $A \cup B$. Note that if there exist $(h,h'),(h'',h''') \in W$ such that $h = h''$ or $h' = h'''$, then we have two edges $(g,h)(g',h')$ and $(g,h'')(g',h''')$ with a common vertex which is not suitable for this aim that M be a matching in $G[H]$.)

Set $A \cup B$ is formed by different copies of M_H and M'_H and so no two edges of $A \cup B$ are adjacent. Also, set C is constructed by different copies of M'_G and consequently no two edges of C are adjacent. Moreover, edges of C saturate vertices of $G[H]$ which are not saturated by edges of $A \cup B$. Therefore, $M = A \cup B \cup C$ is a matching in $G[H]$. It remains to show that it is maximal. We consider first the case when H has a perfect matching. In that case, since M'_H is perfect, all vertices (g,h) such that $g \in X^c$ are saturated by B. Furthermore, since all vertices (g,h) such that $g \in X^c$ are saturated, $C = \emptyset$ and all vertices unsaturated by $A \cup B$ are in the copies of H corresponding to the vertices from X. On ther other hand, M_H is a maximal matching in H and consequently no edge of $G[H]$ exists between vertices unsaturated by A. Hence no edge of $G[H]$ exists between vertices unsaturated by $M = A \cup B$ and M cannot be extended to a larger matching in $G[H]$.

Now, if H does not have a perfect matching, then G must have one. Again, it suffices to show that M saturates all vertices in the copies of H corresponding to the vertices in X^c.

Let $V_H \setminus V_{H(M_H)} = \{h_{i_1}, \ldots, h_{i_t}\}$ and $V_H \setminus V_{H(M'_H)} = \{h_{k_1}, \ldots, h_{k_l}\}$. Since M'_H is maximum matching of H, then $t \geq l$. Set $M_j = \{(g,h)(g',h') \mid gg' \in M'_G \ \&\ h = h_{i_j}$ if $g \in X$ otherwise $h = h_{k_j} \ \&\ h' = h_{i_j}$ if $g' \in X$ otherwise $h' = h_{k_j}\}$ for $j \in \{1, \ldots, l\}$. It is not difficult to see that M_j is a copy of M'_G in $G[H]$. Also, according to this fact that M'_G is perfect and definition of M_js, it is clear that $\cup_{j=1}^{l} M_j$ saturates all vertices in the copies of H corresponding to the vertices in X^c. On the other hand, since $\cup_{j=1}^{l} M_j \subseteq M$, then M saturates all vertices in the copies of H corresponding to the vertices in X^c. In other words, all vertices of $G[H]$ not saturated by M remain isolated in the copies of H indexed by X and no edge between them can be added to M. Hence, M is a maximal matching in $G[H]$.

The first expression in the theorem statement now follows by noting that $|A \cup B| = \alpha(G)s(H) + (n - \alpha(G))\nu(H)$ and $|C| = (m - 2\nu(H))\nu(G)$. Therefore,

$$s(G[H]) \leq \alpha(G)s(H) + (m - 2\nu(H))\nu(G) + (n - \alpha(G))\nu(H).$$

To establish the remaining claim, take

$$
\begin{aligned}
A' &= \{(g,h)(g',h) \mid (h \in X' \text{ and } gg' \in M_G) \text{ or } (h \in V_H \setminus X' \text{ and } gg' \in M'_G)\}\\
B' &= \{(g,h)(g,h') \mid hh' \in M_H \text{ and } g \in V_G \setminus V_{G(M'_G)}\},\\
M' &= A' \cup B',
\end{aligned}
$$

where X' is a largest independent set of H, M'_G is a maximum matching of G, and M_G and M_H are smallest maximal matchings of G and H, respectively. We investigate M' in two cases:

Case 1: G has a perfect matching. Then M'_G is perfect and consequently each vertex (g,h) such that $h \in V_H \setminus X'$ is saturated by A'. This concludes that no edge between the vertices of $G[H]$ not saturated by A' in the copies of H can be added to M' and this means $B' = \emptyset$. Further, no edge between the vertices (g_1,h) and (g_2,h) such that $h \in X'$ can be added to M' because of $\{(g,h)(g',h) \mid h \in X' \text{ and } gg' \in M_G\} \subseteq A'$. Moreover, if (g_1,h_1) and (g_2,h_2) are unsaturated vertices by M', then (g_1,h_1) and (g_2,h_2) are not adjacent in $G[H]$ because otherwise g_1g_2 are in $E(G)$ and so g_1 or g_2 is saturated by M_G and consequently (g_1,h_1) or (g_2,h_2) is saturated by $\{h \in V_H \setminus X' \text{ and } gg' \in M'_G\} \subseteq A'$. Hence all vertices of $G[H]$ not saturated by M' remain isolated in the copies of H indexed by X. Therefore, in this case, M' is a maximal matching of $G[H]$.

Case 2: G does not have a perfect matching. Thus H has a perfect matching. In this case, we need to prove $(V_G \setminus V_{G(M'_G)}) \subseteq (V_G \setminus V_{G(M_G)})$.
Consider a smallest maximal matching M_G in G. By using argument applied in the proof of Berge's theorem [6], we reach to a maximum matching M'_G of G with this property that $(V_G \setminus V_{G(M'_G)}) \subseteq (V_G \setminus V_{G(M_G)})$. If G contains no M_G-augmenting path, then set $M'_G := M_G$. Otherwise, consider a longest M_G-augmenting path P in G and set $M_1 := M_G \Delta E(P)$. If G contains no M_1-augmenting path, then set $M'_G := M_1$. By repeating this construction as long as necessary, we arrive at a maximum matching M'_G such that $(V_G \setminus V_{G(M'_G)}) \subseteq (V_G \setminus V_{G(M_G)})$.
Now we prove M' is a maximal matching in $G[H]$. According to the structure of M_G ang M'_G, it is not difficult to check that M' is matching. It remains to prove M' is maximal. Since M_G and M'_G are smallest maximal matching and a maximum matching in G, respectively, then no edge between the vertices (g,h) and (g',h) can be added to M'. Also, according to structure of M_G and M'_G and the fact that M_H is perfect, no edge between the vertices (g,h) and (g',h') such that $h \neq h'$ can be added to M'. Moreover, no edge between the vertices (g,h) and (g',h) such that $h \in X'$ can be added to M' because of B'. Therefore, M' is maximal matching in $G[H]$.

Hence, in each case, M' is a maximal matching in $G[H]$ and

$$s(G[H]) \leq \alpha(H)s(G) + (n - 2\nu(G))s(H) + (m - \alpha(H))\nu(G).$$

This completes our proof. □

We illustrate our result with some examples. In our first example, we look at a composition of two trees, $P_4[K_{1,3}]$, shown in Fig 2. Here one component, $K_{1,3}$, is equimatchable, while the other, P_4, is not. This asymmetry will be reflected in different sizes of M and M'. The sets X, X^c, M_H, M'_H, M_G, A, B, W, X', M'_G, A', and B', are as follows:
$X = \{g_1, g_3\}$, $X^c = \{g_2, g_4\}$, $M_H = \{h_1h_2\}$, $M'_H = \{h_1h_3\}$, $M_G = \{g_2g_3\}$,

$W = \{(h_3, h_2)(h_4, h_4)\}$, $X' = \{h_2, h_3, h_4\}$, $M'_G = \{g_1g_2, g_3g_4\}$. Hence,

$$A = \{(g_1, h_1)(g_1, h_2), (g_3, h_1)(g_3, h_2)\},$$
$$B = \{(g_2, h_1)(g_2, h_3), (g_4, h_1)(g_4, h_3)\},$$
$$C = \{(g_1, h_3)(g_2, h_2), (g_3, h_3)(g_4, h_2), (g_1, h_4)(g_2, h_4), (g_3, h_4)(g_4, h_4)\},$$
$$A' = \{(g_2, h_2)(g_3, h_2), (g_2, h_3)(g_3, h_3), (g_2, h_4)(g_3, h_4), (g_1, h_1)(g_2, h_1), (g_3, h_1)(g_4, h_1)\},$$
$$B' = \emptyset.$$

Maximal matchings $M = A \cup B \cup C$ and $M' = A' \cup B'$ constructed from the above sets are shown in Fig. 2 (a) and (b), respectively. The size of M' attains the bound of Theorem 3.3. It can be easily seen that no smaller maximal matching exists in $P_4[K_{1,3}]$: Any matching N of size 4 would leave 8 vertices unsaturated. As at most three of them can belong to the same copy of $K_{1,3}$, at least two adjacent copies of $K_{1,3}$ must contain unsaturated vertices. But then they could be saturated by the edge connecting them, and hence, N cannot be maximal. So, the upper bound of Theorem 3.3 is again sharp.

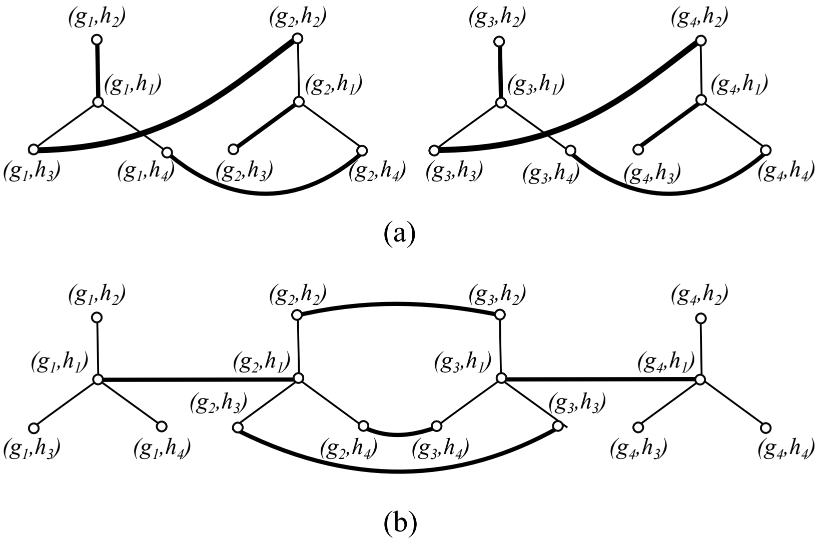

Figure 2. a) Maximal matching M in $P_4[K_{1,3}]$. b) A smaller (and also smallest) maximal matching M' in $P_4[K_{1,3}]$. The edges between neighboring copies of K_4 not participating in the matching are omitted.

This example could serve as a starting point toward more general results and maybe even explicit formulas for the saturation number of $P_n[G]$ for more general classes of G, but we leave the details to the interested reader.

As another example we take the lexicographic product of two (even) complete graphs $K_n[K_m]$. The resulting graph is again complete, K_{mn}, and its saturation number is equal to $\frac{mn}{2}$, exactly the value obtained by plugging the corresponding values in Theorem 3.3.

Our final example shows that the difference between the upper bounds of Theorem 3.3 and actual $s(G[H])$ can be arbitrarily large. Take $G = P_n$ and $H = K_4$. Then $\alpha(P_n) = \lceil \frac{n}{2} \rceil$, $\nu(P_n) = \lfloor \frac{n}{2} \rfloor$, $s(P_n) = \left\lceil \frac{n-1}{3} \right\rceil$, $\alpha(K_4) = 1$, and $\nu(K_4) = s(K_4) = 2$. By plugging these values into the formula of Theorem 3.3 one obtains

$$s(P_n[K_4]) \leq \min\{2n, \left\lceil \frac{n-1}{3} \right\rceil + 2\beta + 3 \left\lfloor \frac{n}{2} \right\rfloor\}$$

Here $\beta = n - 2\nu(P_n)$ is equal to either 0 or 1, depending on the parity of n. One can see that the second term is approximately of the order of $\frac{11n}{6}$, again depending on the combination of residues of n modulo 2 and 3. On the other hand, it is easy to construct a maximal matching in $P_n[K_4]$ of size (again approximately) at most $\frac{7n}{4}$ (an example is shown in Fig. 3). By denoting the number of vertices of $P_n[K_4]$ by $N = 4n$, one can see

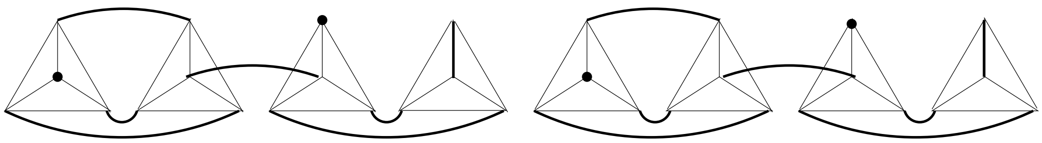

Figure 3. A maximal matching in $P_n[K_4]$ of size $\frac{7n}{4}$. The edges between neighboring copies of K_4 not participating in the matching are omitted.

that the difference between the size of the constructed maximal matching and the upper bound of Theorem 3.3 is (roughly) $\frac{n}{12}$ and can be made arbitrarily large by choosing large enough n.

The expressions from Theorem 3.3 simplify when both factors have a perfect matching.

Corollary 3.4. *Let G and H be two graphs of order n and m, respectively, with perfect matchings. Then*

$$s(G[H]) \leq \min\left\{\frac{mn}{2} - \alpha(G)\left(\frac{m}{2} - s(H)\right), \frac{mn}{2} - \alpha(H)\left(\frac{n}{2} - s(G)\right)\right\}.$$

It would be interesting to characterize graphs for which the upper bound of Theorem 3.3 is sharp.

4. Saturation number as the independent edge domination number

For a graph G, the **line graph** of G, denoted $L(G)$, is a graph whose vertices are edges of G and two vertices are adjacent in $L(G)$ if and only if their corresponding edges are adjacent in G. It is clear that matchings in G correspond to independent sets in $L(G)$.

It can be easily shown that smallest maximal matchings in a graph G are in a one-to-one correspondence with smallest independent dominating sets in $L(G)$. Indeed, if V' is a smallest independent dominating set in $L(G)$, then the edges corresponding to the vertices of V' form a matching M' in G. This matching is maximal, since all remaining edges of G are adjacent to (i.e., dominated by) edges from M'. If there was a maximal matching M'' of smaller size in G, the corresponding vertices in $L(G)$ would form an independent dominating set V'' of cardinality smaller than $|V'|$, a contradiction.

Hence, we have established that $s(G) = i(L(G))$ for any graph G. We can, in fact, dispense with the requirement of independence, since line graphs are claw-free (i.e., they cannot have $K_{1,3}$ as an induced subgraph), and for claw-free graphs we have $i(G) = \gamma(G)$ [1]. This is actually an improvement of the above result, since $\gamma(G) \leq i(G) \leq \alpha(G)$ for all graphs. We refer the reader to a survey by Goddard and Henning for the proof of this fact and for more detail on independent domination number [14]. Hence we have the following result.

Theorem 4.1.

$$s(G) = \gamma(L(G)).$$

Now we can use another classical result, cited below, to answer an open problem posed by Biedl et al. in [5].

Proposition 4.2 ([4]). *For a graph G with n vertices and maximum degree Δ,*

$$\left\lceil \frac{n}{\Delta + 1} \right\rceil \leq i(G) \leq n - \Delta.$$

In [5], a lower bound for the saturation number of graph G with n edges and maximum degree Δ was given as $s(G) \geq \frac{n}{2\Delta - 1}$. Now, by combining Theorem 4.1 and Proposition 4 and using the fact that $\Delta_{L(G)} \leq 2\Delta_G - 2$, we can improve this bound.

Theorem 4.3. *If G is a graph with n edges and Δ is the maximum degree of $L(G)$, then*

$$\left\lceil \frac{n}{\Delta + 1} \right\rceil \leq s(G) \leq n - \Delta.$$

This answers an open question by Biedl et al.: "What can be said about the size of maximum matchings in graphs with maximum degree Δ, for some fixed $\Delta \geq 4$? Can we obtain a bound better than $\frac{n}{2\Delta - 1}$?"

By combining Theorem 4.3 and the fact that $s(G) \leq \nu(G)$, we can answer that $\left\lceil \frac{n}{\Delta_{L(G)} + 1} \right\rceil$ is a better bound than $\frac{n}{2\Delta - 1}$.

We illustrate the power of Theorem 4.3 by establishing lower bounds on the saturation number of two classes of chemically relevant graphs used for modeling fullerenes and benzenoid compounds.

A **fullerene graph** is a planar 3-connected cubic graph twelve of whose faces are pentagons and any remaining faces are hexagons.

Proposition 4.4. *Let F_n be a fullerene graph with n vertices. Then*

$$s(F_n) \geq \frac{3n}{10}.$$

Proof. By definition of fullerene graphs, it is clear that they have $3n/2$ edges if original fullerene graph has n vertices. Further, their line graphs are 4-regular. The claim now follows by plugging these numbers into Theorem 4.3. □

The established lower bound is sharp; there are exactly 11 fullerene graphs for which it is achieved. However, it is never sharp for fullerene graphs on more than 60 vertices. It was shown in [3] that $s(F_n) \sim \frac{n}{3}$, improving thus upper bounds of [9] and [2].

A **benzenoid graph** B is obtained by taking a cycle with simply connected interior in the regular hexagonal tiling of plane. All vertices of the tiling lying on the cycle and (if any) in its interior are taken as vertices of the graph, and edges of tiling between those vertices are the edges of B. If B has no internal vertices, we say that B is **catacondensed**. (The terminology is almost completely borrowed from the theory of polycyclic aromatic hydrocarbons, a vast and important class of organic chemical compounds. We refer the reader to a classical monograph by Gutman and Cyvin [15] for a thorough introduction.)

It can be easily shown that a catacondensed benzenoid graph B_h with h hexagons has $4h + 2$ vertices and $5h + 1$ edges. It is also clear that, as soon as $h > 1$, the largest degree of a vertex in $L(B_h)$ is equal to 4. By plugging those values into Theorem 4.3 we obtain the following lower bound.

Proposition 4.5. *Let B_h be a catacondensed benzenoid graph with h hexagons. Then $s(B_h) \geq h + 1$.*

This bound is also sharp; it is achieved for any $h \geq 1$ by the linear polyacene of length h, i.e., by the straight linear chain of h hexagons. See also [11] and [19] for more on saturation numbers of benzenoid graphs.

5. Concluding remarks

Saturation number of a graph is an invariant that is difficult to compute and that frequently arises in applications. In this paper we have investigated its behavior under lexicographic product. We have obtained (sharp) bounds. It would be of interest to further investigate the quality of the obtained bounds and to characterize the graphs that satisfy them with equality. Another interesting thing would be to investigate how the bounds are affected when one of the components of $G[H]$ is equimatchable.

Acknowledgment. We thank the anonymous referee for careful reading and useful suggestions. Partial support of the Croatian Science Foundation via research project LightMol (Grant no. IP-2016-06-1142) and of Slovenian ARRS (Grant no. J1-3002) is gratefully acknowledged by T. Došlić.

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