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OSCILLATION RESULTS FOR SECOND-ORDER QUASI-LINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract

In this paper, some new oscillation criteria are obtained for the second-order quasi-linear neutral delay differential equation

$$\Big(r(t)\big|\big(x(t)+p(t)x(\tau(t))\big)'\big|^{\alpha-1}\big(x(t)+p(t)x(\tau(t))\big)'\Big)' + f\big(t, x(\sigma(t))\big) = 0,\ t \geq t_0$$

under the case when $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\mathrm{d}t < \infty$. Our results improve and supplement some known results in the literature. An example is also provided to illustrate the main results.

Keywords: Oscillation, Neutral delay differential equation, Second-order

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1. Introduction

This paper is concerned with the oscillation problem of the second-order quasi-linear neutral delay differential equation

$$\big(r(t)|z'(t)|^{\alpha-1}z'(t)\big)' + f\big(t, x(\sigma(t))\big) = 0,\ t \ge t_0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$ and $\alpha > 0$ is a constant.

Throughout this paper, we will assume that the following conditions hold.

(A_1) $r \in C^1([t_0,\infty),\mathbb{R})$, $r(t) > 0$, $p \in C([t_0,\infty),\mathbb{R})$ and $0 \le p(t) \le p_1 < 1$;

(A_2) $\tau \in C([t_0,\infty),\mathbb{R})$, $\tau(t) \le t$, $\lim_{t\to\infty}\tau(t) = \infty$, $\sigma \in C^1([t_0,\infty),\mathbb{R})$, $\sigma(t) \le t$, $\sigma'(t) > 0$ and $\lim_{t\to\infty}\sigma(t) = \infty$;

(A_3) $f(t,u) \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and there exists a function $q \in C\big([t_0,\infty),[0,\infty)\big)$ such that $q(t)$ is not identically zero on any ray of the form $[t_x,\infty)$ for any $t_* \ge t_0$ and

$$f(t,u)\mathrm{sgn}\ u \ge q(t)|u|^{\alpha},\ \text{ for } u \ne 0 \text{ and } t \ge t_0.$$

By a solution of (1.1), we mean a function $x \in C([T_x,\infty),\mathbb{R})$ for some $T_x \ge t_0$ which has the property that $r(t)|z'(t)|^{\alpha-1}z'(t) \in C^1([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$, otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all of its nonconstant solutions are oscillatory.

We note that neutral delay differential equations find numerous applications in electric networks. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high speed computers where the lossless transmission lines are used to interconnect switching circuits; see Hale [1].

In the last few years, many studies have been made on the oscillatory behavior of solutions of differential equations, we refer to the recent papers [2–21] and the references cited therein.

Agarwal et al. [2], Chern et al. [3], Džurina and Stavroulakis [4], Kusano et al. [5, 6] and Mirzov [7] observed some similar properties between

$$\big(r(t)|x'(t)|^{\alpha-1}x'(t)\big)' + q(t)|x[\sigma(t)]|^{\alpha-1}x[\sigma(t)] = 0,\ t \ge t_0 \tag{1.2}$$

and the corresponding linear equation

$$\big(r(t)x'(t)\big)' + q(t)x(t) = 0,\ t \ge t_0.$$

Very recently, Džurina and Hudáková [8], Baculíková and Lacková [9], Liu and Bai [11], Xu and Meng [12], Dong [13] and Ye and Xu [14] established some oscillation criteria for (1.2) with neutral term under the condition when

$$\int_{t_0}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(t)}\mathrm{d}t = \infty.$$

Especially, [12] obtained some sufficient conditions which guarantee that every solution of

$$\begin{aligned}\Big(r(t)\big|\big(x(t)+p(t)x(\tau(t))\big)'\big|^{\alpha-1}\big(x(t)+p(t)x(\tau(t))\big)'\Big)'&\\ + q(t)|x(\sigma(t))|^{\alpha-1}x(\sigma(t)) = 0,\ t \ge t_0&\end{aligned} \tag{1.3}$$

is either oscillatory or tends to zero for the case when

$$\int_{t_0}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(t)}\mathrm{d}t < \infty. \tag{1.4}$$

Han et al. [15] found a mistake in [14]. In order to correct the mistake, they examined the oscillation of (1.1) for the case when (1.4), $\tau(t) = t - \tau$, $p'(t) \ge 0$ and $\sigma(t) \le t - \tau$. Obviously, $\tau(t) = t - \tau$, $p'(t) \ge 0$ and $\sigma(t) \le t - \tau$ are restrictions. To the best of our knowledge, there are no results which ensure that every solution of (1.1) oscillates under the case when $p'(t) \le 0$.

Motivated by the papers [9, 12, 14, 15], the aim of this paper is to further study the oscillation of (1.1). We establish some new criteria and our results improve and complement those results obtained in [12, 15].

In what follows, all functional inequalities considered in this paper are assumed to hold eventually, that is they are satisfied for all sufficiently large t.

2. Oscillation criteria

In this section, we will derive some oscillation criteria for (1.1). For the sake of convenience, we let

$$\pi(t) := \int_t^\infty \frac{1}{r^{\frac{1}{\alpha}}(s)} \mathrm{d}s \text{ and } d_+(t) := \max\{0, d(t)\}.$$

2.1. Theorem. *Suppose that* (1.4) *holds and there exists a constant* $k > 0$ *such that* $p(t) \le k\pi(t)$. *Moreover, assume that there exists a real-valued function* $\rho \in C^1\big([t_0,\infty),(0,\infty)\big)$ *such that*

$$\text{(2.1)} \qquad \limsup_{t\to\infty} \int_{t_0}^t \Big[\rho(s)q(s)\big(1-p(\sigma(s))\big)^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(\sigma(s))((\rho'(s))_+)^{\alpha+1}}{\rho^\alpha(s)(\sigma'(s))^\alpha}\Big] \mathrm{d}s = \infty.$$

If

$$\text{(2.2)} \qquad \limsup_{t\to\infty} \int_{t_0}^t \left[Kq(s)\pi^\alpha(\sigma(s))\pi^\alpha(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{\pi(s)r^{\frac{1}{\alpha}}(s)}\right] \mathrm{d}s = \infty$$

holds for all constants $K > 0$, *then every solution of* (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$, such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \ge t_1$. Then $z(t) \ge x(t) > 0$ for $t \ge t_1$ and it follows from (1.1) that

$$\text{(2.3)} \qquad \big(r(t)|z'(t)|^{\alpha-1}z'(t)\big)' \le -q(t)x^\alpha(\sigma(t)) \le 0.$$

Thus, $r(t)|z'(t)|^{\alpha-1}z'(t)$ is a nonincreasing function. Now we have two possible cases for $z'(t)$: (i) $z'(t) > 0$ eventually, (ii) $z'(t) < 0$ eventually.

(i) Suppose that $z'(t) > 0$ for $t \ge t_2 \ge t_1$. Then, proceeding as in the proof of Theorem 2.1 in [14], we can get a contradiction to (2.1).

(ii) Assume that $z'(t) < 0$ for $t \ge t_2 \ge t_1$. We define the function ω by

$$\text{(2.4)} \qquad \omega(t) = \frac{r(t)(-z'(t))^{\alpha-1}z'(t)}{z^\alpha(t)},\ t \ge t_1.$$

Then $\omega(t) < 0$ for $t \ge t_1$. Noting $(r(t)|z'(t)|^{\alpha-1}z'(t))' \le 0$, then $r(t)|z'(t)|^{\alpha-1}z'(t)$ is nonincreasing and

$$\text{(2.5)} \qquad z'(s) \le \frac{r^{\frac{1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(s)} z'(t),\ s \ge t.$$

Integrating (2.5) from t to l, we get

$$z(l) \le z(t) + r^{\frac{1}{\alpha}}(t)z'(t) \int_t^l \frac{\mathrm{d}s}{r^{\frac{1}{\alpha}}(s)},\ l \ge t.$$

Letting $l \to \infty$ in the above inequality and using $\lim_{t\to\infty} z(t) = c \geq 0$ (c is finite), we obtain

$$z(t) + r^{\frac{1}{\alpha}}(t)z'(t)\pi(t) \geq c. \tag{2.6}$$

From (2.4) and (2.6), we have

$$\omega(t)\pi^{\alpha}(t) \geq -1. \tag{2.7}$$

On the other hand, from (2.3), we see that there exists a constant $c_1 > 0$, such that

$$-r^{\frac{1}{\alpha}}(t)z'(t) \geq c_1. \tag{2.8}$$

Substituting (2.8) into (2.6), we obtain

$$z(t) \geq c_1\pi(t) + c. \tag{2.9}$$

If $\lim_{t\to\infty} z(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$ due to $0 < x(t) \leq z(t)$. Thus, for $\varepsilon = c_1/(2k)$, we have $x(\tau(t)) < c_1/(2k)$. So

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - \frac{c_1}{2k}p(t) \geq c_1\pi(t) - \frac{c_1}{2k}p(t) \geq \frac{c_1}{2}\pi(t). \tag{2.10}$$

If $\lim_{t\to\infty} z(t) = c > 0$, then for any $\varepsilon > 0$, we have $c + \varepsilon > z(t) > c$. Pick $0 < \varepsilon < (c(1-p_1))/p_1$. Then from the definition of z and (2.9), we get

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - p_1(c+\varepsilon) \geq mz(t) \geq mc_1\pi(t), \tag{2.11}$$

where

$$m = \frac{c - p_1(c+\varepsilon)}{c+\varepsilon} > 0.$$

It follows from (2.10) and (2.11) that there exists a constant $M > 0$ such that

$$x(t) \geq M\pi(t). \tag{2.12}$$

Now, differentiating (2.4), we see that

$$\omega'(t) = \frac{(r(t)(-z'(t))^{\alpha-1}z'(t))'z^{\alpha}(t) - \alpha r(t)(-z'(t))^{\alpha-1}z'(t)z^{\alpha-1}(t)z'(t)}{z^{2\alpha}(t)}.$$

From the above equality and (2.3), we have

$$\omega'(t) \leq -q(t)\frac{x^{\alpha}(\sigma(t))}{z^{\alpha}(t)} - \frac{\alpha r(t)(-z'(t))^{\alpha-1}z'(t)z^{\alpha-1}(t)z'(t)}{z^{2\alpha}(t)}. \tag{2.13}$$

Note that $z'(t) < 0$. Then there exists a constant $M_1 > 0$ such that $z(t) \leq M_1$. Thus from (2.4), (2.12) and (2.13) we obtain

$$\omega'(t) + \left(\frac{M}{M_1}\right)^{\alpha} q(t)\pi^{\alpha}(\sigma(t)) + \frac{\alpha}{r^{\frac{1}{\alpha}}(t)}(-\omega(t))^{\frac{\alpha+1}{\alpha}} \leq 0,\ t \geq t_3 \geq t_2. \tag{2.14}$$

Multiplying (2.14) by $\pi^{\alpha}(t)$ and integrating it from t_3 to t yields

$$\begin{aligned}\pi^{\alpha}(t)\omega(t) - \pi^{\alpha}(t_3)\omega(t_3) + \alpha\int_{t_3}^{t} r^{-\frac{1}{\alpha}}(s)\pi^{\alpha-1}(s)\omega(s)\mathrm{d}s + \\ \left(\frac{M}{M_1}\right)^{\alpha}\int_{t_3}^{t} q(s)\pi^{\alpha}(\sigma(s))\pi^{\alpha}(s)\mathrm{d}s + \alpha\int_{t_3}^{t}\frac{\pi^{\alpha}(s)}{r^{\frac{1}{\alpha}}(s)}(-\omega(s))^{\frac{\alpha+1}{\alpha}}\mathrm{d}s \leq 0.\end{aligned} \tag{2.15}$$

Set $p = (\alpha+1)/\alpha$, $q = \alpha+1$, and

$$a = -(\alpha+1)^{\frac{\alpha}{\alpha+1}}\pi^{\frac{\alpha^2}{\alpha+1}}(t)\omega(t),\ b = \frac{\alpha}{(\alpha+1)^{\frac{\alpha}{\alpha+1}}}\pi^{-\frac{1}{\alpha+1}}(t).$$

Using Young's inequality

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q,\ a,\ b \in \mathbb{R},\ p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1,$$

we get

$$-\alpha\pi^{\alpha-1}(t)\omega(t) \leq \alpha\pi^{\alpha}(t)(-\omega(t))^{\frac{\alpha+1}{\alpha}} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(t)}.$$

Therefore, we have

$$-\alpha\frac{\pi^{\alpha-1}(t)\omega(t)}{r^{\frac{1}{\alpha}}(t)} \leq \alpha\frac{\pi^{\alpha}(t)(-\omega(t))^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(t)} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(t)r^{\frac{1}{\alpha}}(t)}.$$

It follows from the last inequality and (2.15) that

$$\begin{aligned}(2.16)\qquad \int_{t_3}^{t}\left[\left(\frac{M}{M_1}\right)^{\alpha} q(s)\pi^{\alpha}(\sigma(s))\pi^{\alpha}(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(s)r^{\frac{1}{\alpha}}(s)}\right]\mathrm{d}s \leq \\ \pi^{\alpha}(t_3)\omega(t_3) - \pi^{\alpha}(t)\omega(t) \leq \pi^{\alpha}(t_3)\omega(t_3) + 1\end{aligned}$$

due to (2.7), which contradicts (2.2). This completes the proof.

Now, we will give a criterion which insure that every solution x of (1.1) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

2.2. Theorem. *Suppose that* (1.4) *holds. Further, assume that there exists a real-valued function* $\rho \in C^1\big([t_0,\infty),(0,\infty)\big)$ *such that* (2.1) *holds. If*

$$(2.17)\qquad \limsup_{t\to\infty}\int_{t_0}^{t}\left[Kq(s)\pi^{\alpha}(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(s)r^{\frac{1}{\alpha}}(s)}\right]\mathrm{d}s = \infty$$

holds for all constants $K > 0$, *then every solution of* (1.1) *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$, such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) \geq x(t) > 0$ for $t \geq t_1$. In view of (1.1), we get (2.3) and there exist two possible cases of the sign of $z'(t)$.

(*i*) Assume that $z'(t) > 0$ for $t \geq t_2 \geq t_1$. Then, proceeding as in the proof of Theorem 2.1 in [14], we can get a contradiction to (2.1).

(*ii*) Suppose that $z'(t) < 0$ for $t \geq t_2 \geq t_1$. We define the function ω as in (2.4), then we obtain (2.5)–(2.7) and (2.13). Obviously, $\lim_{t\to\infty} z(t) = c \geq 0$, where c is finite. If $\lim_{t\to\infty} z(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$ due to $0 < x(t) \leq z(t)$. If $\lim_{t\to\infty} z(t) = c > 0$, proceeding as in the proof of Theorem 2.1, we can get (2.11). That is, there exists a constant $m > 0$ such that $x(t) \geq mz(t)$. Thus

$$\frac{x^{\alpha}(\sigma(t))}{z^{\alpha}(t)} = \frac{x^{\alpha}(\sigma(t))}{z^{\alpha}(\sigma(t))}\frac{z^{\alpha}(\sigma(t))}{z^{\alpha}(t)} \geq m^{\alpha}.$$

It follows from the above inequality, (2.4) and (2.13) that

$$\omega'(t) + m^{\alpha}q(t) + \frac{\alpha}{r^{\frac{1}{\alpha}}(t)}(-\omega(t))^{\frac{\alpha+1}{\alpha}} \leq 0,\ t \geq t_3 \geq t_2.$$

The rest of the proof is similar to that of Theorem 2.1, and so is omitted. The proof is complete.

Next, we will establish another oscillation criterion for (1.1).

2.3. Theorem. *Assume that* (1.4) *holds and there exists a constant* $k > 0$ *such that* $p(t) \leq k\pi(t)$. *Furthermore, assume that there exists a function* $\rho \in C^1\big([t_0, \infty),\ (0, \infty)\big)$ *such that* (2.1) *holds. If*

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(v)} \left[\int_{t_0}^{v} q(u)\pi^{\alpha}(\sigma(u))\mathrm{d}u\right]^{\frac{1}{\alpha}} \mathrm{d}v = \infty, \tag{2.18}$$

then every solution of (1.1) *is oscillatory.*

Proof. Assume the converse. Let x be a nonoscillatory solution of (1.1). Without loss of generality we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Then $z(t) \geq x(t) > 0$ for $t \geq t_1$. Similar to the proof of Theorem 2.1 we have (2.3) and there exist two possible cases of the sign of $z'(t)$.

If $z'(t) > 0$ for $t \geq t_2 \geq t_1$, then we back to the proof of Theorem 2.1 in [14], and we can get a contradiction to (2.1).

If $z'(t) < 0$ for $t \geq t_2 \geq t_1$, proceeding as in the proof of Theorem 2.1, we obtain (2.12) for some constant $M > 0$. Hence, from (2.3) and (2.12), we have

$$(r(t)(-z'(t))^{\alpha})' \geq q(t)x^{\alpha}(\sigma(t)) \geq M^{\alpha}q(t)\pi^{\alpha}(\sigma(t)).$$

Integrating the above inequality from t_3 ($t_3 \geq t_2$) to t, we get

$$r(t)(-z'(t))^{\alpha} \geq r(t_3)(-z'(t_3))^{\alpha} + M^{\alpha}\int_{t_3}^{t} q(u)\pi^{\alpha}(\sigma(u))\mathrm{d}u \geq M^{\alpha}\int_{t_3}^{t} q(u)\pi^{\alpha}(\sigma(u))\mathrm{d}u. \tag{2.19}$$

Integrating the last inequality from t_3 to t, we obtain

$$z(t_3) - z(t) \geq M\int_{t_3}^{t} \frac{1}{r^{\frac{1}{\alpha}}(v)} \left[\int_{t_3}^{v} q(u)\pi^{\alpha}(\sigma(u))\mathrm{d}u\right]^{\frac{1}{\alpha}} \mathrm{d}v,$$

which contradicts (2.18). This completes the proof.

In the following, we obtain a sufficient condition which guarantee that every solution x of (1.1) oscillates or satisfies $\lim_{t\to\infty} x(t) = 0$.

2.4. Theorem. *Assume that* (1.4) *holds. Moreover, assume that there exists a function* $\rho \in C^1\big([t_0, \infty), (0, \infty)\big)$ *such that* (2.1) *holds. If*

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(v)} \left[\int_{t_0}^{v} q(u)\mathrm{d}u\right]^{\frac{1}{\alpha}} \mathrm{d}v = \infty, \tag{2.20}$$

then every solution of (1.1) *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$, such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) \geq x(t) > 0$ for $t \geq t_1$. In view of (1.1), we get (2.3) and there exist two possible cases of the sign of $z'(t)$.

(*i*) Suppose that $z'(t) > 0$ for $t \geq t_2 \geq t_1$. Then, proceeding as in the proof of Theorem 2.1 in [14], we can get a contradiction to (2.1).

(*ii*) Assume that $z'(t) < 0$ for $t \geq t_2 \geq t_1$. Clearly, $\lim_{t\to\infty} z(t) = c \geq 0$, where c is finite. If $\lim_{t\to\infty} z(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$ due to $0 < x(t) \leq z(t)$. If $\lim_{t\to\infty} z(t) = c > 0$, proceed as in the proof of Theorem 2.1, we can get (2.11). That is, there exists a constant $m > 0$ such that $x(t) \geq mz(t)$. Note that $z(t) \geq c > 0$. From (2.3) and (2.11), we obtain

$$(r(t)(-z'(t))^{\alpha})' \geq q(t)x^{\alpha}(\sigma(t)) \geq (mc)^{\alpha}q(t).$$

The rest of the proof is similar to that of Theorem 2.3, and so is omitted. The proof is complete.

3. Applications

In this section, we shall give some applications to illustrate our results.

In 2006, Xu and Meng [12] studied (1.3) and obtain some oscillatory criteria. For example

3.1. Theorem. [12, Theorem 2.3] *Assume that* (1.4) *holds,* $0 \le p(t) < 1$, $p'(t) \ge 0$, $\lim_{t\to\infty} p(t) = A$. *Further, assume that there exists a function* $\xi \in C^1\big([t_0,\infty),(0,\infty)\big)$ *such that* $\xi'(t) \ge 0$ *and*

$$\int^{\infty}\left(\frac{1}{r(t)\xi(t)}\int^{t}\xi(s)q(s)\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}t = \infty.$$

If (2.1) *holds for* $\rho(t) = \int_{t_0}^{t}\big(1/(r^{1/\alpha}(s))\big)\mathrm{d}s$, *then every solution of* (1.3) *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Clearly, when $0 \le p(t) \le p_1 < 1$, Theorem 2.4 improves results of [12, Theorem 2.3], since we remove the conditions $\tau(t) = t - \tau$, $p'(t) \ge 0$ and $\lim_{t\to\infty} p(t) = A$.

In the following, we will give an example to illustrate our results.

Example 3.1 Consider the equation

$$\left(t^2\left(x(t) + \frac{\gamma}{t}x(\tau(t))\right)'\right)' + \beta\sigma(t)x(\sigma(t)) = 0,\ t \ge 1, \tag{3.1}$$

where $\gamma > 0$ and $\beta > 0$ are constants.

Let $t_0 = 1$, $\alpha = 1$, $r(t) = t^2$, $p(t) = \gamma/t$ and $q(t) = \beta\sigma(t)$. Then $\pi(t) = 1/t$ and

$$\int_1^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(v)}\left[\int_1^{v} q(u)\pi^{\alpha}(\sigma(u))\mathrm{d}u\right]^{\frac{1}{\alpha}}\mathrm{d}v = \infty.$$

Furthermore, let $\rho(t) = 1$. Then

$$\int_{t_0}^{\infty}\left[\rho(s)q(s)\big(1 - p(\sigma(s))\big)^{\alpha} - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(\sigma(s))((\rho'(s))_+)^{\alpha+1}}{\rho^{\alpha}(s)(\sigma'(s))^{\alpha}}\right]\mathrm{d}s \ge \int_{t_0}^{\infty}\beta(1-p_1)\sigma(s)\mathrm{d}s = \infty,$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Kq(s)\pi^{\alpha}(\sigma(s))\pi^{\alpha}(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(s)r^{\frac{1}{\alpha}}(s)}\right]\mathrm{d}s = \infty,$$

when $K\beta > 1/4$. Hence, by Theorems 2.1 and 2.3, equation (3.1) is oscillatory.

3.2. Remark. One can easily see that Theorem 2.1 and Theorem 2.3 complement the results given in [15]. It is also interesting to further study equation (1.1) for the case when (1.4), since there are unknown results, e.g., when $p(t) > k\pi(t)$, k is a positive constant.

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