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ON P-VALENTLY CLOSE-TO-CONVEX, STARLIKE AND CONVEX FUNCTIONS

Erhan Deniz*

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Abstract

The main object of the present paper is to derive several sufficient conditions for close-to-convexity, starlikeness, and convexity of certain p-valent analytic functions in the unit disk. Some interesting consequences of the main results are also mentioned.

Keywords: P-valent analytic functions, Starlike functions, Close-to-convex functions, Convex functions.

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1. Introduction and definitions

Let $\mathcal{A}_p$ denote the family of functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \ (p \in \mathbb{N} = \{1, 2, \ldots\}) \tag{1.1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$.

Also let $S_p^*(\alpha)$, $K_p(\alpha)$ and $C_p(\alpha)$ denote the subclasses of $\mathcal{A}_p$ consisting of functions which are respectively, *p-valently starlike of order α*, *p-valently convex of order α* and *p-valently close-to-convex of order α* in $\mathcal{U}$ $(0 \le \alpha < p)$. Thus, we have (see, for details, [1, 2], see also [10]),

$$S_p^*(\alpha) = \left\{ f : \ f \in \mathcal{A}_p \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ (z \in \mathcal{U}; \ 0 \le \alpha < p) \right\}, \tag{1.2}$$

$$K_p(\alpha) = \left\{ f : \ f \in \mathcal{A}_p \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ (z \in \mathcal{U}; \ 0 \le \alpha < p) \right\}, \tag{1.3}$$

and

$$C_p(\alpha) = \left\{ f : \ f \in \mathcal{A}_p \text{ and } \Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \ (z \in \mathcal{U}; \ 0 \le \alpha < p; \ g \in S_p^*) \right\}, \tag{1.4}$$

*Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey. E-mail: edeniz36@gmail.com

where, for convenience,

$$S_p^* := S_p^*(0),\ K_p := K_p(0),\ C_p := C_p(0). \tag{1.5}$$

Since $g(z) = z^p$ belongs to the class S_p^*, we observe that the function $f(z) \in \mathcal{A}_p$ satisfying

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > \alpha,\ (z \in \mathcal{U};\ 0 \le \alpha < p) \tag{1.6}$$

is a member of the class $C_p(\alpha)$.

Next, with a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in $\mathcal{U}$. Then we say that the function f is subordinate to g if there exists a function w, analytic in $\mathcal{U}$, with

$$w(0) = 0 \text{ and } |w(z)| < 1,\ (z \in \mathcal{U})$$

such that

$$f(z) = g(w(z)),\ (z \in \mathcal{U}).$$

We denote this subordination by

$$f(z) \prec g(z). \tag{1.7}$$

In particular, if the function g is univalent in $\mathcal{U}$, the subordination (1.7) is equivalent to (cf. [1, p.190]),

$$f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$$

Many authors have dedicated a great part of their work on developing sufficient conditions for close-to-convexity, starlikeness and convexity of functions $f(z) \in \mathcal{A}_p$ (see [4, 5, 7]–[12]). The main object of this paper is to give sufficient conditions for the functions $f(z) \in \mathcal{A}_p$ to be close-to-convex, starlike and convex of given order in the open unit disk.

The following lemma (popularly known as Jack's lemma) will be required in our present investigation.

1.1. Lemma. *(See [3, 6]) Let the (nonconstant) function $w(z)$ be analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then*

$$z_0 w'(z_0) = k w(z_0),$$

where k is a real number and $k \ge 1$.

2. Sufficient conditions for close-to-convexity

Our first result (Theorem 2.1 below) provides a sufficient condition for close-to-convexity of functions $f(z) \in \mathcal{A}_p$.

2.1. Theorem. *Let the function $f(z) \in \mathcal{A}_p$ satisfy the inequality*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{(2p-1)(p+\alpha)+2\alpha}{2(p+\alpha)},\ (z \in \mathcal{U};\ 0 \le \alpha < p). \tag{2.1}$$

Then,

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > \frac{p+\alpha}{2},\ (z \in \mathcal{U};\ 0 \le \alpha < p) \tag{2.2}$$

or equivalently, $f(z) \in C_p(\frac{p+\alpha}{2})$.

Proof. We begin by defining a function w by

$$\frac{f'(z)}{z^{p-1}} = \frac{p+\alpha w(z)}{1+w(z)}, \; (w(z) \neq -1; \; z \in \mathcal{U}; \; 0 \leq \alpha < p)\,. \tag{2.3}$$

Then, clearly, w is analytic in $\mathcal{U}$ with $w(0) = 0$. We easily find from (2.3) that

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{\alpha z w'(z)}{p+\alpha w(z)} - \frac{zw'(z)}{1+w(z)}. \tag{2.4}$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1, \text{ when } |z| < |z_0|\,.$$

Then, by applying Lemma 1.1, we have

$$z_0 w'(z_0) = k w(z_0), \; \left(k \geq 1; \; w(z_0) = e^{i\theta}; \; \theta \in \mathbb{R}\right). \tag{2.5}$$

Thus, we find from (2.4) and (2.5) that

$$\begin{aligned}\Re\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= p + \Re\left(\frac{\alpha k e^{i\theta}}{p+\alpha e^{i\theta}}\right) - \Re\left(\frac{ke^{i\theta}}{1+e^{i\theta}}\right)\\ &= p + \frac{\alpha k(\alpha + p\cos\theta)}{p^2+\alpha^2+2p\alpha\cos\theta} - \frac{k}{2}\\ &\leq \frac{2\alpha + (2p-1)(p+\alpha)}{2(p+\alpha)},\end{aligned}$$

which obviously contradicts our hypothesis (2.1).

Therefore, we see that there is no $z_0 \in \mathcal{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ $(z \in \mathcal{U})$. Thus, we conclude that

$$\left|\frac{\frac{f'(z)}{z^{p-1}} - p}{\frac{f'(z)}{z^{p-1}} - \alpha}\right| < 1, \; (z \in \mathcal{U}; \; 0 \leq \alpha < p)\,,$$

that is, that $f(z) \in C_p(\frac{p+\alpha}{2})$. This evidently completes the proof of Theorem 2.1. □

By setting $\alpha = 0$ in Theorem 2.1, we obtain the following criterion for p-valently close-to-convex of order $\frac{p}{2}$.

2.2. Corollary. *Let the function $f(z) \in \mathcal{A}_p$ satisfy the inequality*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{2p-1}{2}, \; (z \in \mathcal{U}).$$

Then,

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > \frac{p}{2}, \; (z \in \mathcal{U}),$$

or equivalently, $f(z) \in C_p(\frac{p}{2})$. □

2.3. Theorem. *Let the function $f(z) \in \mathcal{A}_p$ satisfy the inequality*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{p(2p+\alpha+1)+\alpha}{(2p+\alpha)}, \; (z \in \mathcal{U}; \; 0 \leq \alpha < p). \tag{2.6}$$

Then,

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p+\alpha, \; (z \in \mathcal{U}; \; 0 \leq \alpha < p). \tag{2.7}$$

Proof. Our proof of Theorem 2.3, also based upon Lemma 1.1, is similar to that of Theorem 2.1. Indeed, in place of definition (2.3), here we let the function w be given by

$$\frac{f'(z)}{z^{p-1}} = p + (p+\alpha)w(z), \ (z \in \mathcal{U};\ 0 \le \alpha < p).$$

The details are omitted. □

By setting $\alpha = 0$ in Theorem 2.3, we readily obtain the following criterion for $f(z) \in \mathcal{A}_p$ to be a p-valent close-to-convex function.

2.4. Corollary. *Let the function $f(z) \in \mathcal{A}_p$ satisfy the inequality*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{2p+1}{2}, \ (z \in \mathcal{U}). \tag{2.8}$$

Then,

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p, \ (z \in \mathcal{U}). \tag{2.9}$$

or equivalently, $f(z) \in C_p$. □

Next we prove the following theorem.

2.5. Theorem. *Let the function $f(z) \in \mathcal{A}_p$ satisfy the inequality*

$$\left|\frac{f'(z)}{z^{p-1}} - p\right|^{\lambda} \left|\frac{f''(z)}{z^{p-2}} - p(p-1)\right|^{\mu} < \frac{(p-\alpha)^{\lambda+\mu}}{2^{\lambda+2\mu}}, \ (z \in \mathcal{U};\ 0 \le \alpha < p;\ \lambda, \mu \ge 0). \tag{2.10}$$

Then,

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > \frac{p+\alpha}{2}, \ (z \in \mathcal{U};\ 0 \le \alpha < p), \tag{2.11}$$

or equivalently, $f(z) \in C_p(\frac{p+\alpha}{2})$.

Proof. We define the function w by (2.3). Then, clearly, w is analytic in $\mathcal{U}$ with $w(0) = 0$. We also find from (2.3) that

$$\begin{aligned} &\left|\frac{f'(z)}{z^{p-1}} - p\right|^{\lambda} \left|\frac{f''(z)}{z^{p-2}} - p(p-1)\right|^{\mu} \\ &\qquad = \frac{(p-\alpha)^{\lambda+\mu} \left|w(z)\right|^{\lambda} \left|(p-1)(1+w(z))w(z) + zw'(z)\right|^{\mu}}{\left|1+w(z)\right|^{\lambda+2\mu}}. \end{aligned} \tag{2.12}$$

Assume that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1, \text{ when } |z| < |z_0|.$$

If we apply Lemma 1.1 just as we did in the proof of Theorem 2.1, we shall obtain

$$\begin{aligned} \left|\frac{f'(z)}{z^{p-1}} - p\right|^{\lambda} \left|\frac{f''(z)}{z^{p-2}} - p(p-1)\right|^{\mu} &= \frac{(p-\alpha)^{\lambda+\mu} \left|(p-1)(1+e^{i\theta})e^{i\theta} + ke^{i\theta}\right|^{\mu}}{\left|1+e^{i\theta}\right|^{\lambda+2\mu}} \\ &= \frac{(p-\alpha)^{\lambda+\mu} \left|p+k-1+(p-1)e^{i\theta}\right|^{\mu}}{\left|1+e^{i\theta}\right|^{\lambda+2\mu}} \\ &\ge \frac{(p-\alpha)^{\lambda+\mu}}{2^{\lambda+2\mu}}, \end{aligned}$$

which obviously contradicts our hypothesis (2.10). Thus we have

$$|w(z)| < 1, \ (z \in \mathcal{U}),$$

which implies that

$$\left|\frac{\frac{f'(z)}{z^{p-1}}-p}{\frac{f'(z)}{z^{p-1}}-\alpha}\right|<1,\ (z\in\mathcal{U};\ 0\leq\alpha<p)\,,$$

that is, that $f(z)\in C_p(\frac{p+\alpha}{2})$. This evidently completes the proof of Theorem 2.5. □

By setting $\lambda=\mu-1=0$ in Theorem 2.5, we readily obtain the following result.

2.6. Corollary. *Let the function $f(z)\in\mathcal{A}_p$ satisfy the inequality*

$$\left|\frac{f''(z)}{z^{p-2}}-p(p-1)\right|<\frac{p-\alpha}{4},\ (z\in\mathcal{U};\ 0\leq\alpha<p).$$

Then $f(z)\in C_p(\frac{p+\alpha}{2})$. □

By setting $\lambda=\mu=1$ and $\alpha=0$ in Theorem 2.5, we obtain the following criterion for p-valently close-to-convex of order $\frac{p}{2}$.

2.7. Corollary. *Let the function $f(z)\in\mathcal{A}_p$ satisfy the inequality*

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\left|\frac{f''(z)}{z^{p-2}}-p(p-1)\right|<\frac{p^2}{8},\ (z\in\mathcal{U}).$$

Then,

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right)>\frac{p}{2},\ (z\in\mathcal{U}),$$

or equivalently, $f(z)\in C_p(\frac{p}{2})$. □

3. Starlikeness and Convexity

In this section, we first prove the following result (Theorem 3.1 below), which involves the already introduced principle of subordination between analytic functions (see Section 1).

3.1. Theorem. *Let the function $f(z)\in\mathcal{A}_p$ satisfy the inequality*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)<\begin{cases}\frac{p(5\beta+1)-2}{2(\beta+1)}; & 1<\beta\leq\frac{p+1}{p}\\ \frac{p(\beta-1)+2}{2(\beta-1)}; & \frac{p+1}{p}\leq\beta<\frac{p+2}{p}\end{cases},\ (z\in\mathcal{U})\tag{3.1}$$

for some $\beta\left(1<\beta<\frac{p+2}{p}\right)$. Then

$$\frac{zf'(z)}{f(z)}\prec\frac{p\beta(1-z)}{\beta-z},\ (z\in\mathcal{U}).\tag{3.2}$$

The result is sharp for the function f given by

$$f(z)=z^p\left(1-\frac{z}{\beta}\right)^{p(\beta-1)},\ (z\in\mathcal{U}).\tag{3.3}$$

Proof. Let us define the function w by

$$\frac{zf'(z)}{f(z)}=\frac{p\beta(1-w(z))}{\beta-w(z)},\ \left(w(z)\neq\beta;\ z\in\mathcal{U};\ 1<\beta<\frac{p+2}{p}\right).\tag{3.4}$$

Then, clearly, w is analytic in $\mathcal{U}$ with $w(0)=0$. By logarithmic differentiation of both sides of (3.4), we also find that

$$1+\frac{zf''(z)}{f'(z)}=\frac{p\beta(1-w(z))}{\beta-w(z)}+\frac{zw'(z)}{\beta-w(z)}-\frac{pzw'(z)}{1-w(z)}.\tag{3.5}$$

We assume that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1, \text{ when } |z| < |z_0|,$$

then Lemma 1.1 gives us that

$$z_0 w'(z_0) = k w(z_0), \ \left(k \geq 1; \ w(z_0) = e^{i\theta}; \ \theta \in \mathbb{R}\right).$$

Therefore, we obtain

$$\begin{aligned}
\Re\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= \Re\left(\frac{p\beta(1-e^{i\theta})}{\beta - e^{i\theta}}\right) + \Re\left(\frac{ke^{i\theta}}{\beta - e^{i\theta}}\right) - \Re\left(\frac{pke^{i\theta}}{1-e^{i\theta}}\right) \\
&= \frac{p\beta(\beta+1)(1-\cos\theta)}{\beta^2+1-2\beta\cos\theta} + \frac{k(\beta\cos\theta - 1)}{\beta^2+1-2\beta\cos\theta} + \frac{pk}{2} \\
&\geq \frac{p\beta(\beta+1)(1-\cos\theta)}{\beta^2+1-2\beta\cos\theta} + \frac{\beta\cos\theta - 1}{\beta^2+1-2\beta\cos\theta} + \frac{p}{2}
\end{aligned}$$

which yields the inequality

$$\Re\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) \geq \begin{cases} \frac{p(5\beta+1)-2}{2(\beta+1)}; & 1 < \beta \leq \frac{p+1}{p}, \\ \frac{p(\beta-1)+2}{2(\beta-1)}; & \frac{p+1}{p} \leq \beta < \frac{p+2}{p}. \end{cases} \tag{3.6}$$

This contradicts our condition (3.1) of Theorem 3.1. Therefore, we conclude that

$$|w(z)| < 1, \ (z \in \mathcal{U}),$$

that is, that

$$\left|\frac{zf'(z)}{f(z)} - \frac{p\beta}{\beta+1}\right| < \frac{p\beta}{\beta+1}, \quad \left(z \in \mathcal{U}; \ 1 < \beta < \frac{p+2}{p}\right)$$

which implies the subordination (3.2) asserted by Theorem 3.1.

Finally, for the function f given by (3.3), we have

$$\frac{zf'(z)}{f(z)} = \frac{p\beta(1-z)}{\beta - z}, \ (z \in \mathcal{U}),$$

which evidently completes the proof of Theorem 3.1. □

Furthermore, since

$$f(z) \in K_p(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\alpha), \ (z \in \mathcal{U}; \ 0 \leq \alpha < p)$$

whose special case, when $p = 1$ and $\alpha = 0$, is the familiar Alexander theorem (cf., e.g., [1, p. 43, Theorem 2.12]), Theorem 3.1 can be applied in order to deduce the following result.

3.2. Corollary. *Let the function $f(z) \in \mathcal{A}_p$ satisfy the inequality*

$$\Re\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right) < \begin{cases} \frac{\beta(5p-2)+p-4}{2(\beta+1)}; & 1 < \beta \leq \frac{p+1}{p}, \\ \frac{\beta(p-2)-p+4}{2(\beta-1)}; & \frac{p+1}{p} \leq \beta < \frac{p+2}{p}, \end{cases} \quad (z \in \mathcal{U})$$

for some $\beta \left(1 < \beta < \frac{p+2}{p}\right)$, then

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{p\beta(1-z)}{\beta - z}, \ (z \in \mathcal{U}).$$

The result is sharp for the function f given by

$$f'(z) = pz^{p-1}\left(1 - \frac{z}{\beta}\right)^{p(\beta-1)}, \ (z \in \mathcal{U}).$$ □

By setting $\beta = \frac{p+1}{p}$ in Theorem 3.1, we obtain the following criterions for p-valent starlikeness and p-valent convexity, respectively.

3.3. Corollary. *Let the function $f(z) \in \mathcal{A}_p$ satisfy the inequality*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3p}{2}, \ (z \in \mathcal{U})$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p(p+1)(1-z)}{p+1-z}, \ (z \in \mathcal{U}).$$

and

$$\left|\frac{zf'(z)}{f(z)} - \frac{p(p+1)}{2p+1}\right| < \frac{p(p+1)}{2p+1}, \ (z \in \mathcal{U}).$$

This implies that $f \in S_p^*$. The result is sharp for the function f given by

$$f(z) = z^p - \frac{p}{p+1} z^{p+1}, \ (z \in \mathcal{U}). \qquad \square$$

3.4. Corollary. *Let the function $f(z) \in \mathcal{A}_p$ satisfy the inequality*

$$\Re\left(\frac{2zf''(z) + z^2 f'''(z)}{'(z) + zf''(z)}\right) < \frac{3p-2}{2}, \ (z \in \mathcal{U}),$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{p(p+1)(1-z)}{p+1-z}, \ (z \in \mathcal{U})$$

and

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{p(p+1)}{2p+1}\right| < \frac{p(p+1)}{2p+1}, \ (z \in \mathcal{U}).$$

This implies that $f \in K_p$. The result is sharp for the function f given by

$$f'(z) = pz^{p-1} - \frac{p^2}{p+1} z^p, \ (z \in \mathcal{U}). \qquad \square$$

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