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(r, s)-CONVERGENT NETS

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Abstract

We introduce the notions of (r, s)-adherent point, (r, s)-accumulation point, (r, s)-cluster point, (r, s)-limit point and (r, s)-derived set in an intuitionistic fuzzy topological spaces and investigate some of their properties. Also, we define (r, s)-convergent nets and investigate some of their properties.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy topology, (r, s)-adherent point, (r, s)-accumulation point, (r, s)-cluster point, (r, s)-limit point, (r, s)-derived set, (r, s)-convergent net.

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1. Introduction and preliminaries

Pu and Liu [19] introduced the notions of Q-neighborhood and fuzzy net with respect to Q-neighborhoods and established the convergence theory in fuzzy topological spaces. Chen and Cheng [6] introduced the concepts of fuzzy cluster and fuzzy limit point in fuzzy topological spaces with respect to R-neighborhoods instead of Q-neighborhoods. The convergence theory in fuzzy topological spaces has been developed in many directions [6,7,11,24].

Kubiak [15] and Šostak [21] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [3], in the sense that not only the objects are fuzzified, but also the axiomatics. In [22,23], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay *et al.* [4] have redefined the same concept under the name gradation of openness. A general approach to the study of topological type structures on fuzzy power sets was developed in [12-16].

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy set was introduced by Atanassov [2]. By using intuitionistic fuzzy sets, Çoker and his coworker [8,9] defined the topology of intuitionistic fuzzy sets. Recently, Samanta and Mondal [20], introduced

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the notion of intuitionistic fuzzy gradation of openness of fuzzy sets, where to each fuzzy subsets there is a definite grade of openness and there is a grade of non-openness. Thus, the concept of intuitionistic fuzzy gradation of openness is a generalization of the concept of gradation of openness and the topology of intuitionistic fuzzy sets.

In this paper, we introduce the notions of (r, s)-adherent point, (r, s)-accumulation point, (r, s)-cluster point, (r, s)-limit point and (r, s)-derived set in an intuitionistic fuzzy topological spaces and investigate some of their properties. Also, we define (r, s)-convergent nets and investigate some of their properties.

Throughout this paper, let X be a nonempty set, $I = [0, 1]$, $I_0 = (0, 1]$ and $I_1 = [0, 1)$. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A *fuzzy point* x_t for $t \in I_0$ is an element of I^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $P_t(X)$. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. A fuzzy set λ is quasi-coincident with μ, denoted by $\lambda\ q\ \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If λ is not quasi-coincident with μ, we write $\lambda\ \overline{q}\ \mu$.

1.1. Definition. [20] An *intuitionistic fuzzy gradation of openness* (IFGO, for short) on X is an ordered pair (τ, τ^*) of functions from I^X to I such that

(IGO1) $\tau(\lambda) + \tau^*(\lambda) \leq 1$, $\forall \lambda \in I^X$,
(IGO2) $\tau(\underline{0}) = \tau(\underline{1}) = 1$, $\tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$,
(IGO3) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \leq \tau^*(\lambda_1) \vee \tau^*(\lambda_2)$, for each $\lambda_1, \lambda_2 \in I^X$,
(IGO4) $\tau(\bigvee_{i\in\Delta} \lambda_i) \geq \bigwedge_{i\in\Delta} \tau(\lambda_i)$ and $\tau^*(\bigvee_{i\in\Delta} \lambda_i) \leq \bigvee_{i\in\Delta} \tau^*(\lambda_i)$, for each $\lambda_i \in I^X$, $i \in \Delta$.

The triplet (X, τ, τ^*) is called an *intuitionistic fuzzy topological space* (ifts, for short). τ and τ^* may be interpreted as *fuzzy gradation of openness* and *fuzzy gradation of nonopenness*, respectively.

1.2. Theorem. [1,17] *Let (X, τ, τ^*) be an ifts. For each $r \in I_0$, $s \in I_1$, $\lambda \in I^X$, we define an operator $\mathcal{C} : I^X \times I_0 \times I_1 \to I^X$ as follows:*

$$\mathcal{C}(\lambda, r, s) = \bigwedge\{\mu \mid \mu \geq \lambda, \tau(\underline{1} - \mu) \geq r, \tau^*(\underline{1} - \mu) \leq s\}.$$

Then it satisfies the following properties:

(1) $\mathcal{C}(\underline{0}, r, s) = \underline{0}, \mathcal{C}(\underline{1}, r, s) = \underline{1}$, *for all* $r \in I_0$, $s \in I_1$.
(2) $\mathcal{C}(\lambda, r, s) \geq \lambda$.
(3) $\mathcal{C}(\lambda_1, r, s) \leq \mathcal{C}(\lambda_2, r, s)$, *if* $\lambda_1 \leq \lambda_2$.
(4) $\mathcal{C}(\lambda \vee \mu, r, s) = \mathcal{C}(\lambda, r, s) \vee \mathcal{C}(\mu, r, s)$, *for all* $r \in I_0$, $s \in I_1$.
(5) $\mathcal{C}(\lambda, r, s) \leq \mathcal{C}(\lambda, r', s')$, *if* $r \leq r'$, $s \geq s'$, *where* $r, r' \in I_0$, $s, s' \in I_1$.
(6) $\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) = \mathcal{C}(\lambda, r, s)$. □

Let (X, τ_1, τ_1^*) and (Y, τ_2, τ_2^*) be intuitionistic fuzzy topological spaces. A function $f : X \to Y$ is called IF continuous if $\tau_2(\mu) \leq \tau_1(f^{-1}(\mu))$ and $\tau_2^*(\mu) \geq \tau_1^*(f^{-1}(\mu))$ for all $\mu \in I^Y$.

1.3. Definition. [19] Let $\lambda, \mu \in I^X$. Define the *fuzzy quasi-difference* of λ and μ, denoted by $\lambda \backslash \mu$, as

$$(\lambda\backslash\mu)(x) = \begin{cases} \lambda(x), & \text{if } \mu(x) = 0, \\ 0, & \text{if } \lambda(x) \geq \mu(x) > 0, \\ \lambda(x), & \text{if } \lambda(x) < \mu(x). \end{cases}$$

1.4. Lemma. [19] *For $\lambda, \mu \in I^X$ and $x_t \in P_t(X)$, the following properties hold:*

(1) $\lambda\backslash\mu \leq \lambda$ *and* $\lambda\backslash\underline{0} = \lambda$.
(2) *If* $x_t \notin \lambda$, *then* $\lambda\backslash x_t = \lambda$. *If* $x_t \in \lambda$, *then, for each* $y \in X$,

$$(\lambda\backslash x_t)(y) = \begin{cases} \lambda(y), & \text{if } y \neq x, \\ 0, & \text{if } y = x. \end{cases}$$

(3) $(\lambda \vee \mu)\backslash x_t \leq (\lambda\backslash x_t) \vee (\mu\backslash x_t)$.
(4) *If* $f : X \to Y$ *is injective, then* $f(\lambda\backslash\mu) = f(\lambda)\backslash f(\mu)$.

1.5. Definition. [10] Let (X, τ, τ^*) be an ifts, $\mu \in I^X$, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. Then μ is called an (r, s)-*open* Q-*neighborhood of* x_t if $x_t q\mu$ with $\tau(\mu) \geq r$ and $\tau^*(\mu) \leq s$. We write

$$\mathcal{N}(x_t, r, s) = \{\mu \mid \mu \in I^X, x_t q\mu, \quad \tau(\mu) \geq r \text{ and } \tau^*(\mu) \leq s\}.$$

1.6. Definition. [19] Let D be a directed set and $\lambda \in I^X$. A function $\mathcal{S} : D \to P_t(X)$ is called a *fuzzy net*. We say $\mathcal{S}$ is a *fuzzy net in* λ if $\mathcal{S}(n) \in \lambda$ for every $n \in D$. A fuzzy net $\mathcal{S}$ is *increasing* (resp. *decreasing*) if $\mathcal{S}(m) \leq \mathcal{S}(n)$ (resp. $\mathcal{S}(n) \leq \mathcal{S}(m)$) for every $m \leq n$ with $m, n \in D$.

1.7. Definition. [19] Let $\mathcal{S} : D \to P_t(X)$ and $\mathcal{W} : E \to P_t(X)$ be two fuzzy nets. Then, $\mathcal{W}$ is called a *subnet* of $\mathcal{S}$ if there exists a function $N : E \to D$, called by a *cofinal selection on* $\mathcal{S}$, such that

(1) $\mathcal{W} = \mathcal{S} \circ N$,
(2) For every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$ for $m \geq m_0$.

2. (r, s)-derived sets in intuitionistic fuzzy topological spaces

2.1. Definition. Let (X, τ, τ^*) be an ifts, $\lambda \in I^X$, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. Then:

(1) x_t is called an (r, s)-*adherent point of* λ if for every $x_t q\mu$ with $\tau(\mu) \geq r$ and $\tau^*(\mu) \leq s$, we have $\mu q\lambda$.
(2) x_t is called an (r, s)-*accumulation point of* λ if for every $x_t q\mu$ with $\tau(\mu) \geq r$ and $\tau^*(\mu) \leq s$, we have $\mu q(\lambda\backslash x_t)$.

Define the (r, s)-*derived set* of λ, denote by $\mathcal{D}(\lambda, r, s)$, as

$$\mathcal{D}(\lambda, r, s) = \bigvee\{x_t \mid x_t \in P_t(X) \text{ and } x_t \text{ is an } (r, s)\text{-accumulation point of } \lambda\}.$$

2.2. Theorem. *Let* (X, τ, τ^*) *be an ifts. For* $\lambda \in I^X$, $r \in I_0$ *and* $s \in I_1$, *we have*

$$\mathcal{C}(\lambda, r, s) = \bigvee\{x_t \mid x_t \in P_t(X) \text{ and } x_t \text{ is an } (r, s)\text{-adherent point of } \lambda\}.$$

Proof. Put $\rho = \bigvee\{x_t \in P_t(X) \mid x_t \text{ is an } (r, s)\text{-adherent point of } \lambda\}$. Suppose $\mathcal{C}(\lambda, r, s) \not\leq \rho$. Then there exist $x \in X$ and $t \in I_0$ such that

$$\mathcal{C}(\lambda, r, s)(x) \geq t > \rho(x).$$

Since $\rho(x) < t$, x_t is not an (r, s)-adherent point of λ. Hence there exists $\mu \in I^X$ with $x_t q\mu$, $\tau(\mu) \geq r$ and $\tau^*(\mu) \leq s$ such that $\lambda\overline{q}\mu$, that is, $\lambda \leq \underline{1} - \mu$. Then $\lambda \leq \mathcal{C}(\lambda, r, s) \leq \underline{1} - \mu$. Since $x_t q\mu$ and $\mu \leq \underline{1} - \mathcal{C}(\lambda, r, s)$, we have $x_t q(\underline{1} - \mathcal{C}(\lambda, r, s))$. This implies $t > \mathcal{C}(\lambda, r, s)(x)$, a contradiction. Hence, $\mathcal{C}(\lambda, r, s) \leq \rho$.

Suppose $\mathcal{C}(\lambda, r, s) \not\geq \rho$. Then there exists an (r, s)-adherent point $x_t \in P_t(X)$ such that

$$\mathcal{C}(\lambda, r, s)(x) < t \leq \rho(x).$$

Since $\mathcal{C}(\lambda, r, s)(x) < t$, then $x_t q(\underline{1} - \mathcal{C}(\lambda, r, s))$, $\tau(\underline{1} - \mathcal{C}(\lambda, r, s)) \geq r$ and $\tau^*(\underline{1} - \mathcal{C}(\lambda, r, s)) \leq s$. Moreover, since $\lambda \leq \mathcal{C}(\lambda, r, s)$ we have

$$\lambda\overline{q}(\underline{1} - \mathcal{C}(\lambda, r, s)).$$

So x_t is not an (r,s)-adherent point of λ. It is a contradiction. Hence, $\mathcal{C}(\lambda,r,s) \geq \rho$. $\square$

2.3. Theorem. *Let (X,τ,τ^*) be an ifts. For $\lambda,\mu \in I^X$, $r \in I_0$ and $s \in I_1$, the following properties hold:*

(1) $\mathcal{D}(\lambda,r,s) \leq \mathcal{C}(\lambda,r,s)$.
(2) $\mathcal{C}(\lambda,r,s) = \lambda \vee \mathcal{D}(\lambda,r,s)$.
(3) $\mathcal{C}(\lambda,r,s) = \lambda$ *iff* $\mathcal{D}(\lambda,r,s) \leq \lambda$.
(4) *If* $r_1 \geq r$ *and* $s_1 \leq s$*, then* $\mathcal{D}(\lambda,r,s) \leq \mathcal{D}(\lambda,r_1,s_1)$.
(5) $\mathcal{D}(\lambda \vee \mu,r,s) \leq \mathcal{D}(\lambda,r,s) \vee \mathcal{D}(\mu,r,s)$.

Proof. (1) Clear because every (r,s)-accumulation point of λ is an (r,s)-adherent point of λ.

(2) Since $\lambda \leq \mathcal{C}(\lambda,r,s)$ and $\mathcal{D}(\lambda,r,s) \leq \mathcal{C}(\lambda,r,s)$, we have

$$\lambda \vee \mathcal{D}(\lambda,r,s) \leq \mathcal{C}(\lambda,r,s).$$

Conversely, suppose $\mathcal{C}(\lambda,r,s) \not\leq \lambda \vee \mathcal{D}(\lambda,r,s)$. Then there exist $x \in X$ and $t \in I_0$ such that

$$\mathcal{C}(\lambda,r,s)(x) > t > \lambda(x) \vee \mathcal{D}(\lambda,r,s)(x).$$

Since $\lambda(x) \vee \mathcal{D}(\lambda,r,s)(x) < t$, then $x_t \notin \lambda$ and x_t is not an (r,s)-accumulation point of λ. Hence there exists $\mu \in I^X$ with $x_t q \mu$, $\tau(\mu) \geq r$ and $\tau^*(\mu) \leq s$ such that $\mu \overline{q} (\lambda \backslash x_t)$. Since $x_t \notin \lambda$, we have $(\lambda \backslash x_t) = \lambda$. Thus $\mu \overline{q} \lambda$ which implies $\lambda \leq \mathcal{C}(\lambda,r,s) \leq \underline{1} - \mu$. Since $x_t q \mu$, that is, $(\underline{1} - \mu)(x) < t$,

$$\mathcal{C}(\lambda,r,s)(x) \leq (\underline{1} - \mu)(x) < t.$$

It is a contradiction. Hence $\mathcal{C}(\lambda,r,s) \leq \lambda \vee \mathcal{D}(\lambda,r,s)$.

(3) Follows immediately from (2).

(4) Suppose $\mathcal{D}(\lambda,r,s) \not\leq \mathcal{D}(\lambda,r_1,s_1)$. Then there exists an (r,s)-accumulation point $x_t \in P_t(X)$ of λ such that

$$\mathcal{D}(\lambda,r,s)(x) \geq t > \mathcal{D}(\lambda,r_1,s_1)(x).$$

Since $\mathcal{D}(\lambda,r_1,s_1)(x) < t$, then x_t is not an (r_1,s_1)-accumulation point of λ. Hence there exists $\rho \in I^X$ with $x_t q \rho$, $\tau(\rho) \geq r_1$ and $\tau^*(\rho) \leq s_1$ such that $\rho \overline{q} (\lambda \backslash x_t)$. Since $\tau(\rho) \geq r_1 \geq r$ and $\tau^*(\rho) \leq s_1 \leq s$, then x_t is not an (r,s)-accumulation point of λ. It is a contradiction.

(5) Suppose $\mathcal{D}(\lambda \vee \mu,r,s) \not\leq \mathcal{D}(\lambda,r,s) \vee \mathcal{D}(\mu,r,s)$. Then there exists an (r,s)-accumulation point $x_t \in P_t(X)$ of $\lambda \vee \mu$ such that

$$\mathcal{D}(\lambda \vee \mu,r,s)(x) \geq t > \mathcal{D}(\lambda,r,s)(x) \vee \mathcal{D}(\mu,r,s)(x).$$

Since $\mathcal{D}(\lambda,r,s)(x) < t$ and $\mathcal{D}(\mu,r,s)(x) < t$, then x_t is not an (r,s)-accumulation point of either λ or μ. Hence there exist $\rho_1,\rho_2 \in I^X$ with $x_t q \rho_i$, $\tau(\rho_i) \geq r$ and $\tau^*(\rho_i) \leq s$, for $i = 1,2$, such that

$$\rho_1 \overline{q} (\lambda \backslash x_t) \text{ and } \rho_2 \overline{q} (\mu \backslash x_t).$$

Take $\rho = \rho_1 \wedge \rho_2$. Then $x_t q (\rho_1 \wedge \rho_2)$, $\tau(\rho_1 \wedge \rho_2) \geq r$ and $\tau^*(\rho_1 \wedge \rho_2) \leq s$. Moreover,

$$\begin{aligned}(\lambda \vee \mu)\backslash x_t &\leq (\lambda \backslash x_t) \vee (\mu \backslash x_t) \quad \text{by Lemma 1.4(3)} \\ &\leq (\underline{1} - \rho_1) \vee (\underline{1} - \rho_2) \\ &= \underline{1} - (\rho_1 \wedge \rho_2) \\ &= \underline{1} - \rho.\end{aligned}$$

Hence, $\rho\overline{q}((\lambda \vee \mu)\backslash x_t)$. Thus x_t is not an (r, s)-accumulation point of $\lambda \vee \mu$. It is a contradiction. Therefore, $\mathcal{D}(\lambda \vee \mu, r, s) \leq \mathcal{D}(\lambda, r, s) \vee \mathcal{D}(\mu, r, s)$. □

2.4. Theorem. *Let (X, τ_1, τ_1^*) and (Y, τ_2, τ_2^*) be ifts's and $f : X \to Y$ an injective function. Then the following statements are equivalent:*

(1) *f is IF continuous.*
(2) *$f(\mathcal{D}(\lambda, r, s)) \leq \mathcal{D}(f(\lambda), r, s)$, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$.*
(3) *$f(\mathcal{C}(\lambda, r, s)) \leq \mathcal{C}(f(\lambda), r, s)$, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$*

Proof. (1) $\Longrightarrow$ (2): Suppose there exist $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that

$$f(\mathcal{D}(\lambda, r, s)) \not\leq \mathcal{D}(f(\lambda), r, s).$$

Then there exists $y \in Y$ such that

$$f(\mathcal{D}(\lambda, r, s))(y) > \mathcal{D}(f(\lambda), r, s)(y).$$

Since f is injective, there exists a unique $x \in f^{-1}(\{y\})$ such that

$$f(\mathcal{D}(\lambda, r, s))(y) \geq \mathcal{D}(\lambda, r, s)(x) > \mathcal{D}(f(\lambda), r, s)(y).$$

There exists an (r, s)-accumulation point x_t of λ on (X, τ_1, τ_1^*) such that

$$\mathcal{D}(\lambda, r, s)(x) \geq t > \mathcal{D}(f(\lambda), r, s)(f(x)).$$

Therefore $f(x)_t = f(x_t)$ is not an (r, s)-accumulation point of $f(\lambda)$. Hence there exists $\rho \in I^Y$ with $f(x_t)q\rho$, $\tau_2(\rho) \geq r$ and $\tau_2^*(\rho) \leq s$ such that $\rho\overline{q}(f(\lambda)\backslash f(x_t))$. Since f is injective, by Lemma 1.4 (4), $f(\lambda)\backslash f(x_t) = f(\lambda\backslash x_t)$. By the IF continuity of f, we have $\tau_1(f^{-1}(\rho)) \geq \tau_2(\rho) \geq r$ and $\tau_1^*(f^{-1}(\rho)) \leq \tau_2^*(\rho) \leq s$. Then we have $f(x_t)q\rho \Longrightarrow x_t q f^{-1}(\rho)$, which implies $\rho\overline{q}f(\lambda\backslash x_t) \Longrightarrow f^{-1}(\rho)\overline{q}(\lambda\backslash x_t)$. Hence x_t is not an (r, s)-accumulation point of λ. It is a contradiction. Hence $f(\mathcal{D}(\lambda, r, s)) \leq \mathcal{D}(f(\lambda), r, s)$, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$.

(2) $\Longrightarrow$ (3): Easily proved from the following:

$$\begin{aligned} f(\mathcal{C}(\lambda, r, s)) &= f(\lambda \vee \mathcal{D}(\lambda, r, s)) \quad \text{(By Theorem 2.3(2))} \\ &= f(\lambda) \vee f(\mathcal{D}(\lambda, r, s)) \\ &\leq f(\lambda) \vee \mathcal{D}(f(\lambda), r, s) \quad \text{(by (2))} \\ &= \mathcal{C}(f(\lambda), r, s). \end{aligned}$$

(3) $\Longrightarrow$ (1): Easily proved. □

3. (r, s)-cluster points and (r, s)-limit points

3.1. Definition. Let (X, τ, τ^*) be an ifts, $\mu \in I^X$, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. Then:

(1) x_t is called an *(r, s)-cluster point of* S, denoted by $\mathcal{S} \overset{(r,s)}{\infty} x_t$, if for every $\mu \in \mathcal{N}(x_t, r, s)$, S is frequently quasi-coincident with μ, i.e., for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $\mathcal{S}(n_0)q\mu$.
(2) x_t is called an *(r, s)-limit point of* $\mathcal{S}$, denoted by $\mathcal{S} \overset{(r,s)}{\longrightarrow} x_t$, if for every $\mu \in \mathcal{N}(x_t, r, s)$, S is eventually quasi-coincident with μ, i.e., there exists $n_0 \in D$ such that for each $n \in D$ with $n \geq n_0$, we have $\mathcal{S}(n)q\mu$.

We write

$$\mathrm{clu}(\mathcal{S}, r, s) = \bigvee\{x_t \mid x_t \in P_t(X) \text{ and } x_t \text{ is an } (r, s)\text{-cluster point of } \mathcal{S}\},$$

$$\lim(\mathcal{S}, r, s) = \bigvee\{x_t \mid x_t \in P_t(X) \text{ and } x_t \text{ is an } (r, s)\text{-limit point of } \mathcal{S}\}.$$

3.2. Theorem. *Let (X, τ, τ^*) be an ifts, $\mathcal{S} : D \to P_t(X)$ a fuzzy net and $\mathcal{W} : E \to P_t(X)$ a subnet of $\mathcal{S}$. For $r, m \in I_0$ and $s \in I_1$, the following properties hold:*

(1) *If* $\mathcal{S} \overset{(r,s)}{\longrightarrow} x_t$, *then* $\mathcal{S} \overset{(r,s)}{\infty} x_t$.
(2) $\lim(\mathcal{S}, r, s) \leq \mathrm{clu}(\mathcal{S}, r, s)$.
(3) *If* $\mathcal{S} \overset{(r,s)}{\infty} x_t$ *and* $x_t \geq x_m$, *then* $\mathcal{S} \overset{(r,s)}{\infty} x_m$.
(4) *If* $\mathcal{S} \overset{(r,s)}{\longrightarrow} x_t$ *and* $x_t \geq x_m$, *then* $\mathcal{S} \overset{(r,s)}{\longrightarrow} x_m$.
(5) $\mathcal{S} \overset{(r,s)}{\infty} x_t$ *iff* $x_t \in \mathrm{clu}(\mathcal{S}, r, s)$.
(6) $\mathcal{S} \overset{(r,s)}{\longrightarrow} x_t$ *iff* $x_t \in \lim(\mathcal{S}, r, s)$.
(7) *If* $\mathcal{S} \overset{(r,s)}{\longrightarrow} x_t$, *then* $\mathcal{W} \overset{(r,s)}{\longrightarrow} x_t$.
(8) $\lim(\mathcal{S}, r, s) \leq \lim(\mathcal{W}, r, s)$.
(9) *If* $\mathcal{W} \overset{(r,s)}{\infty} x_t$, *then* $\mathcal{S} \overset{(r,s)}{\infty} x_t$.
(10) $\mathrm{clu}(\mathcal{W}, r, s) \leq \mathrm{clu}(\mathcal{S}, r, s)$.

Proof. (1) and (2) are clear.

(3) For every $\mu \in \mathcal{N}(x_m, r, s)$, since $x_m \leq x_t$ then $\mu \in \mathcal{N}(x_t, r, s)$. Since $\mathcal{S} \overset{(r,s)}{\infty} x_t$, for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $\mathcal{S}(n_0) q \mu$. Hence $\mathcal{S} \overset{(r,s)}{\infty} x_m$.

(4) Similar to (3).

(5) $\Longrightarrow$. Clear.

$\Longleftarrow$. Let $x_t \in \mathrm{clu}(\mathcal{S}, r, s)$ and $\mu \in \mathcal{N}(x_t, r, s)$. Since $x_t q \mu$ and $\mathrm{clu}(\mathcal{S}, r, s)(x) \geq t$, we have

$$\mu(x) + \mathrm{clu}(\mathcal{S}, r, s)(x) \geq \mu(x) + t > 1.$$

From the definition of $\mathrm{clu}(\mathcal{S}, r, s)$, there exists an (r, s)-cluster point $x_m \in P_t(X)$ of $\mathcal{S}$ such that

$$\mu(x) + \mathrm{clu}(\mathcal{S}, r, s)(x) \geq \mu(x) + m > 1.$$

Thus $\mu \in \mathcal{N}(x_m, r, s)$. Since x_m is an (r, s)-cluster point of $\mathcal{S}$, for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $\mathcal{S}(n_0) q \mu$. Hence $\mathcal{S} \overset{(r,s)}{\infty} x_t$.

(6) Similar to (5).

(7) For every $\mu \in \mathcal{N}(x_t, r, s)$, since $\mathcal{S} \overset{(r,s)}{\longrightarrow} x_t$, there exists $n_0 \in D$ such that for all $n \geq n_0$, $\mathcal{S}(n) q \mu$. Let $N : E \to D$ be a cofinal selection on $\mathcal{S}$. Then for $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$ for all $m \geq m_0$. Thus $\mathcal{W}(m) = \mathcal{S}(N(m)) q \mu$ for $m \geq m_0$. Therefore, $\mathcal{W} \overset{(r,s)}{\longrightarrow} x_t$.

(8) Clear from (7).

(9) Suppose that $\mathcal{W} \overset{(r,s)}{\infty} x_t$ and $n \in D$. If $N : E \to D$ is a cofinal selection on $\mathcal{S}$, then there exists $m \in E$ such that $N(k) \geq n$ for $k \geq m$. Since $\mathcal{W} \overset{(r,s)}{\infty} x_t$, for every $\mu \in \mathcal{N}(x_t, r, s)$, there exists $m_0 \in E$ such that $m_0 \geq m$ and $\mathcal{W}(m_0) q \mu$. We let $n_0 = N(m_0)$. Then $n_0 \geq n$, and since $\mathcal{S}(n_0) = \mathcal{W}(m_0)$ we have $\mathcal{S}(n_0) q \mu$.

(10) Clear from (9). □

3.3. Theorem. *Let (X, τ, τ^*) be an ifts, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. For every fuzzy net $\mathcal{S}$, $\mathcal{S} \overset{(r,s)}{\longrightarrow} x_t$ iff $\mathcal{W} \overset{(r,s)}{\infty} x_t$ for every fuzzy subnet $\mathcal{W}$ of $\mathcal{S}$.*

Proof. $\Longrightarrow$. From Theorem 3.2 (7), $\mathcal{S}\xrightarrow{(r,s)}x_t$ implies $\mathcal{W}\xrightarrow{(r,s)}x_t$. From Theorem 3.2 (1), $\mathcal{W}\xrightarrow{(r,s)}x_t$ implies $\mathcal{W}\overset{(r,s)}{\infty}x_t$.

$\Longleftarrow$. Suppose x_t is not an (r,s)-limit point of $\mathcal{S}$. Then there exists $\mu \in \mathcal{N}(x_t,r,s)$ satisfying the following: for each $n \in D$, there exists $N(n) \in D$ such that $N(n) \geq n$ and $\mathcal{S}(N(n))\overline{q}\mu$. So we get a cofinal selection $N : D \to D$. Since N is a cofinal selection on $\mathcal{S}$, $\mathcal{W} = \mathcal{S} \circ N$ is a fuzzy subnet of $\mathcal{S}$. Since for $\mu \in \mathcal{N}(x_t,r,s)$ and for each $n \in D$, $\mathcal{W}(n) = \mathcal{S}(N(n))\overline{q}\mu$, x_t is not an (r,s)-cluster point of $\mathcal{W}$. □

3.4. Theorem. *Let (X,τ,τ^*) be an ifts, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. For every fuzzy net $\mathcal{S} : D \to P_t(X)$, we have $\mathcal{S}\overset{(r,s)}{\infty}x_t$ iff $\mathcal{S}$ has a fuzzy subnet $\mathcal{W}$ such that $\mathcal{W}\xrightarrow{(r,s)}x_t$.*

Proof. $\Longrightarrow$. Let $E = D \times \mathcal{N}(x_t,r,s) = \{(m,\lambda) \mid m \in D, \lambda \in \mathcal{N}(x_t,r,s)\}$. Define a relation on E by

$$\forall (m,\lambda),(n,\mu) \in E, \quad (m,\lambda) \leq (n,\mu) \iff m \leq n, \lambda \geq \mu.$$

For each $(m,\lambda),(n,\mu) \in E$, we have $\lambda,\mu \in \mathcal{N}(x_t,r,s) \implies \lambda \wedge \mu \in \mathcal{N}(x_t,r,s)$ and there exists $k \in D$ such that $m \leq k$ and $n \leq k$. Hence there exists $(k,\lambda \wedge \mu) \in E$ such that $(m,\lambda) \leq (k,\lambda \wedge \mu)$ and $(n,\mu) \leq (k,\lambda \wedge \mu)$. So, E is a directed set.

For each $(n,\mu) \in E$, since $\mathcal{S}\overset{(r,s)}{\infty}x_t$, there exists $N(n,\mu) \in D$ such that $N(n,\mu) \geq n$ and $\mathcal{S}(N(n,\mu))q\mu$. So, we can define $N : E \to D$. For each $n_0 \in D$, since $\mathcal{S}\overset{(r,s)}{\infty}x_t$, for $\mu_0 \in \mathcal{N}(x_t,r,s)$, there exists $(n_0,\mu_0) \in E$ such that $N(n_0,\mu_0) \geq n_0$. Hence for every $(n,\mu) \geq (n_0,\mu_0)$, since $n \geq n_0$, we have $N(n,\mu) \geq n \geq n_0$. Therefore N is a cofinal selection on $\mathcal{S}$. So, $\mathcal{W} = \mathcal{S} \circ N$ is a fuzzy subnet of $\mathcal{S}$.

Now we show that $\mathcal{W}\xrightarrow{(r,s)}x_t$. For each $\mu_0 \in \mathcal{N}(x_t,r,s)$, since $\mathcal{S}\overset{(r,s)}{\infty}x_t$, for $n_0 \in D$, there exists $N(n_0,\mu_0) \in D$ such that $\mathcal{S}(N(n_0,\mu_0))q\mu_0$. Hence for every $(n,\mu) \geq (n_0,\mu_0)$, $\mathcal{S}(N(n,\mu))q\mu$ implies $\mathcal{S}(N(n,\mu))q\mu_0$ because $\mu \leq \mu_0$. So, $\mathcal{W}\xrightarrow{(r,s)}x_t$.

$\Longleftarrow$. From Theorem 3.2(1), $\mathcal{W}\xrightarrow{(r,s)}x_t$ implies $\mathcal{W}\overset{(r,s)}{\infty}x_t$. From Theorem 3.2 (9), $\mathcal{W}\overset{(r,s)}{\infty}x_t$ implies $\mathcal{S}\overset{(r,s)}{\infty}x_t$. □

3.5. Theorem. *Let (X,τ,τ^*) be an ifts, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. Then the following statements are equivalent:*

(1) *$x_t \in \mathcal{C}(\lambda,r,s)$.*

(2) *There exists a fuzzy net $\mathcal{S}$ in λ such that $\mathcal{S}\overset{(r,s)}{\infty}x_t$.*

(3) *There exists a fuzzy net $\mathcal{S}$ in λ such that $\mathcal{S}\xrightarrow{(r,s)}x_t$.*

Proof. (1) $\Longrightarrow$ (2) Define a relation on $\mathcal{N}(x_t,r,s)$ by,

$$\nu \preceq \omega \text{ iff } \omega \leq \nu, \quad \forall \nu,\omega \in \mathcal{N}(x_t,r,s).$$

Then $(\mathcal{N}(x_t,r,s),\preceq)$ is a directed set. For each $\mu \in \mathcal{N}(x_t,r,s)$, since $x_t \in \mathcal{C}(\lambda,r,s)$ we have $\mathcal{C}(\lambda,r,s)(x)+\mu(x) \geq t+\mu(x) > 1$. From Theorem 2.2, there exists an (r,s)-adherent point x_m of λ such that

$$\mathcal{C}(\lambda,r,s)(x) + \mu(x) \geq m + \mu(x) > 1.$$

Since x_m is an (r,s)-adherent point of λ and $\mu \in \mathcal{N}(x_m,r,s)$, we have $\lambda q\mu$. Then there exist $y \in X$ and $n \in I_0$ such that

$$\lambda(y) + \mu(y) \geq n + \mu(y) > 1.$$

Hence $y_n \in \lambda$ and $\mu \in \mathcal{N}(y_n,r,s)$. For each $\mu \in \mathcal{N}(x_t,r,s)$, we can define a fuzzy net $\mathcal{S} : \mathcal{N}(x_t,r,s) \to P_t(X)$ by $\mathcal{S}(\mu) = y_n$. Then $\mathcal{S}(\mu)q\mu$ and $\mathcal{S}(\mu) \in \lambda$.

Now we will show that $\mathcal{S} \overset{(r,s)}{\infty} x_t$. Let $\mu \in \mathcal{N}(x_t, r, s)$. Then for every $\nu \in \mathcal{N}(x_t, r, s)$, we have $\mu \wedge \nu \in \mathcal{N}(x_t, r, s)$ and $\mathcal{S}(\mu \wedge \nu) q(\mu \wedge \nu)$. Thus $\nu \preceq \mu \wedge \nu$ and $\mathcal{S}(\mu \wedge \nu) q\mu$.

(2) $\Longrightarrow$ (1) Suppose there exists a fuzzy net $\mathcal{S}$ in λ such that $\mathcal{S} \overset{(r,s)}{\infty} x_t$, i.e., for each $\mu \in \mathcal{N}(x_t, r, s)$ and for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $\mathcal{S}(n_0) q\mu$ is satisfied for the fuzzy net $\mathcal{S}$ in λ. Then since $\mathcal{S}(n_0) \in \lambda$, $\mathcal{S}(n_0) q\mu$ implies $\lambda q\mu$. Hence x_t is an (r, s)-adherent point of λ, that is, $x_t \in \mathcal{C}(\lambda, r, s)$.

(2) $\Longrightarrow$ (3) Easily proved from Theorem 3.4.

(3) $\Longrightarrow$ (2) Easily proved from Theorem 3.2 (1). □

3.6. Theorem. *Let (X, τ, τ^*) be an ifts and $\mathcal{S} : D \to P_t(X)$ a fuzzy net. For $r \in I_0$ and $s \in I_1$, the following properties hold:*

(1) $\mathcal{C}(\text{clu}(\mathcal{S}, r, s), r, s) = \text{clu}(\mathcal{S}, r, s)$.
(2) $\text{clu}(\mathcal{S}, r, s) \leq \mathcal{C}(\bigvee_{n\in D} \mathcal{S}(n), r, s)$.

Proof. (1) From Theorem 1.2 (2), we have

$$\mathcal{C}(\text{clu}(\mathcal{S}, r, s), r, s) \geq \text{clu}(\mathcal{S}, r, s).$$

Suppose $\mathcal{C}(\text{clu}(\mathcal{S}, r, s), r, s) \not\leq \text{clu}(\mathcal{S}, r, s)$. From Theorem 2.2, there exists an (r, s)-adherent point x_t of $\text{clu}(\mathcal{S}, r, s)$, such that

$$\mathcal{C}(\text{clu}(\mathcal{S}, r, s), r, s)(x) \geq t > \text{clu}(\mathcal{S}, r, s)(x).$$

Since x_t is an (r, s)-adherent point of $\text{clu}(\mathcal{S}, r, s)$, for each $\mu \in \mathcal{N}(x_t, r, s)$ we have $\mu q \text{clu}(\mathcal{S}, r, s)$. Since $\mu q \text{clu}(\mathcal{S}, r, s)$, there exists $y \in X$ such that

$$\mu(y) + \text{clu}(\mathcal{S}, r, s)(y) > 1.$$

From the definition of $\text{clu}(\mathcal{S}, r, s)$, there exists an (r, s)-cluster point y_p of $\mathcal{S}$ such that

$$\mu(y) + \text{clu}(\mathcal{S}, r, s)(y) \geq \mu(y) + p > 1.$$

Thus $\mu \in \mathcal{N}(y_p, r, s)$. Since $\mathcal{S} \overset{(r,s)}{\infty} y_p$ and $\mu \in \mathcal{N}(y_p, r, s)$, for each $n \in D$ there exists $n_0 \in D$ such that $n_0 \geq n$ and $\mathcal{S}(n_0) q\mu$. Hence x_t is an (r, s)-cluster point of $\mathcal{S}$. So, $\text{clu}(\mathcal{S}, r, s)(x) \geq t$. It is a contradiction. Hence $\mathcal{C}(\text{clu}(\mathcal{S}, r, s), r, s) \leq \text{clu}(\mathcal{S}, r, s)$.

(2) Suppose $\text{clu}(\mathcal{S}, r, s) \not\leq \mathcal{C}(\bigvee_{n\in D} \mathcal{S}(n), r, s)$. Then there exists an (r, s)-cluster point x_t of $\mathcal{S}$ such that

$$\text{(I)} \qquad \text{clu}(\mathcal{S}, r, s)(x) \geq t > \mathcal{C}\Big(\bigvee_{n\in D} \mathcal{S}(n), r, s\Big)(x).$$

Since x_t is an (r, s)-cluster point of $\mathcal{S}$, for each $\mu \in \mathcal{N}(x_t, r, s)$, for each $n \in D$, there exists $n_0 \geq n$ with $\mathcal{S}(n_0) q\mu$. Since $\mathcal{S}(n_0) \leq \bigvee_{n\in D} \mathcal{S}(n)$, we have $\bigvee_{n\in D} \mathcal{S}(n) q\mu$. Hence x_t is an (r, s)-adherent point of $\bigvee_{n\in D} \mathcal{S}(n)$. Thus $\mathcal{C}(\bigvee_{n\in D} \mathcal{S}(n), r, s)(x) \geq t$. It is a contradiction for (I). Hence $\text{clu}(\mathcal{S}, r, s) \leq \mathcal{C}(\bigvee_{n\in D} \mathcal{S}(n), r, s)$. □

3.7. Theorem. *Let (X, τ, τ^*) be an ifts and $\mathcal{S}, \mathcal{U} : D \to P_t(X)$ fuzzy nets such that $\mathcal{S}(n) \vee \mathcal{U}(n), \mathcal{S}(n) \wedge \mathcal{U}(n) \in P_t(X)$ for each $n \in D$. Define $\mathcal{S} \vee \mathcal{U}, \mathcal{S} \wedge \mathcal{U} : D \to P_t(X)$ by, for each $n \in D$,*

$$(\mathcal{S} \vee \mathcal{U})(n) = \mathcal{S}(n) \vee \mathcal{U}(n) \text{ and } (\mathcal{S} \wedge \mathcal{U})(n) = \mathcal{S}(n) \wedge \mathcal{U}(n).$$

For each $r \in I_0$ and $s \in I_1$, the following properties hold:

(1) *If $\mathcal{S}(n) \leq \mathcal{U}(n)$ for all $n \in D$, then*

$$\text{clu}(\mathcal{S}, r, s) \leq \text{clu}(\mathcal{U}, r, s) \text{ and } \lim(\mathcal{S}, r, s) \leq \lim(\mathcal{U}, r, s).$$

(2) $\text{clu}(\mathcal{S} \vee \mathcal{U}, r, s) = \text{clu}(\mathcal{S}, r, s) \vee \text{clu}(\mathcal{U}, r, s)$.
(3) $\text{clu}(\mathcal{S} \wedge \mathcal{U}, r, s) \leq \text{clu}(\mathcal{S}, r, s) \wedge \text{clu}(\mathcal{U}, r, s)$.

(4) $\lim(\mathcal{S} \vee \mathcal{U}, r, s) \geq \lim(\mathcal{S}, r, s) \vee \lim(\mathcal{U}, r, s)$.
(5) $\lim(\mathcal{S} \wedge \mathcal{U}, r, s) \leq \lim(\mathcal{S}, r, s) \wedge \lim(\mathcal{U}, r, s)$.

Proof. (1) Let x_t be an (r, s)-cluster point of $\mathcal{S}$. For each $\mu \in \mathcal{N}(x_t, r, s)$ and for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $\mathcal{S}(n_0)q\mu$. Since $\mathcal{S}(n) \leq \mathcal{U}(n)$ for all $n \in D$, $\mathcal{U}(n_0)q\mu$. Thus x_t is an (r, s)-cluster point of $\mathcal{U}$. Hence $\mathrm{clu}(\mathcal{S}, r, s) \leq \mathrm{clu}(\mathcal{U}, r, s)$.

Similarly, we have $\lim(\mathcal{S}, r, s) \leq \lim(\mathcal{U}, r, s)$.

(2) Since $\mathcal{S} \leq \mathcal{S} \vee \mathcal{U}$ and $\mathcal{U} \leq \mathcal{S} \vee \mathcal{U}$, by (1) we have $\mathrm{clu}(\mathcal{S} \vee \mathcal{U}, r, s) \geq \mathrm{clu}(\mathcal{S}, r, s) \vee \mathrm{clu}(\mathcal{U}, r, s)$. Suppose $\mathrm{clu}(\mathcal{S} \vee \mathcal{U}, r, s) \not\leq \mathrm{clu}(\mathcal{S}, r, s) \vee \mathrm{clu}(\mathcal{U}, r, s)$. Then there exists an (r, s)-cluster point x_t of $\mathcal{S} \vee \mathcal{U}$ such that

$$\mathrm{clu}(\mathcal{S} \vee \mathcal{U}, r, s)(x) \geq t > \mathrm{clu}(\mathcal{S}, r, s)(x) \vee \mathrm{clu}(\mathcal{U}, r, s)(x).$$

Hence $x_t \notin \mathrm{clu}(\mathcal{S}, r, s)$ and $x_t \notin \mathrm{clu}(\mathcal{U}, r, s)$.

Since x_t is not an (r, s)-cluster point of $\mathcal{S}$, there exist $\mu_1 \in \mathcal{N}(x_t, r, s)$ and $n_1 \in D$ such that $\mathcal{S}(n)\overline{q}\mu_1$ for every $n \in D$ with $n \geq n_1$.

Since x_t is not an (r, s)-cluster point of $\mathcal{U}$, there exist $\mu_2 \in \mathcal{N}(x_t, r, s)$ and $n_2 \in D$ such that $\mathcal{U}(n)\overline{q}\mu_2$ for every $n \in D$ with $n \geq n_2$.

Let $\mu = \mu_1 \wedge \mu_2$ and let $n_3 \in D$ be such that $n_3 \geq n_1$ and $n_3 \geq n_2$. Since $\mu_1 \leq \underline{1} - \mathcal{S}(n)$ and $\mu_2 \leq \underline{1} - \mathcal{U}(n)$ for $n \geq n_3$, we have $\mu_1 \wedge \mu_2 \leq \underline{1} - (\mathcal{S}(n) \vee \mathcal{U}(n))$. So, $\mu \in \mathcal{N}(x_t, r, s)$ and $n_3 \in D$ are such that $(\mathcal{S} \vee \mathcal{U})(n)\overline{q}\mu$ for every $n \in D$ with $n \geq n_3$. Thus x_t is not an (r, s)-cluster point of $\mathcal{S} \vee \mathcal{U}$. It is a contradiction. Hence we have $\mathrm{clu}(\mathcal{S} \vee \mathcal{U}, r, s) \leq \mathrm{clu}(\mathcal{S}, r, s) \vee \mathrm{clu}(\mathcal{U}, r, s)$.

(3), (4) and (5) are easily proved. □

3.8. Theorem. *Let (X, τ, τ^*) be an ifts and $\mathcal{S} : D \to P_t(X)$ a fuzzy net. Then we have*

$$\mathrm{clu}(\mathcal{S}, r, s) = \bigwedge_{n_0 \in D} \mathcal{C}\Big(\bigvee_{n \geq n_0} \mathcal{S}(n), r, s\Big).$$

Proof. Let $x_t \in \mathrm{clu}(\mathcal{S}, r, s)$. From Theorem 3.2 (5), since x_t is an (r, s)-cluster point of $\mathcal{S}$, for each $\mu \in \mathcal{N}(x_t, r, s)$ and for each $n_0 \in D$, there exists $n \in D$ such that $n \geq n_0$ and $\mathcal{S}(n)q\mu$. Since $\mathcal{S}(n) \leq \bigvee_{n \geq n_0} \mathcal{S}(n)$, we have $\bigvee_{n \geq n_0} \mathcal{S}(n)q\mu$. Hence x_t is an (r, s)-adherent point of $\bigvee_{n \geq n_0} \mathcal{S}(n)$, for all $n_0 \in \mathcal{D}$, that is,

$$x_t \in \bigwedge_{n_0 \in D} \mathcal{C}\Big(\bigvee_{n \geq n_0} \mathcal{S}(n), r, s\Big).$$

Then we have

$$\mathrm{clu}(\mathcal{S}, r, s) \leq \bigwedge_{n_0 \in D} \mathcal{C}\Big(\bigvee_{n \geq n_0} \mathcal{S}(n), r, s\Big).$$

Suppose

$$\mathrm{clu}(\mathcal{S}, r, s) \not\geq \bigwedge_{n_0 \in D} \mathcal{C}\Big(\bigvee_{n \geq n_0} \mathcal{S}(n), r, s\Big).$$

There exists an (r, s)-adherent point x_t of $\bigvee_{n \geq n_0} \mathcal{S}(n)$, for all $n_0 \in D$, such that

$$\mathrm{clu}(\mathcal{S}, r, s) < t \leq \mathcal{C}\Big(\bigvee_{n \geq n_0} \mathcal{S}(n), r, s\Big).$$

Since x_t is an (r,s)-adherent point of $\bigvee_{n\geq n_0}\mathcal{S}(n)$, for each $n_0\in D$, for each $\mu\in\mathcal{N}(x_t,r,s)$, we have

$$\bigvee_{n\geq n_0}\mathcal{S}(n)q\mu.$$

Since $\bigvee_{n\geq n_0}\mathcal{S}(n)q\mu$, there exists $y\in X$ such that

$$\bigvee_{n\geq n_0}\mathcal{S}(n)(y)+\mu(y)>1.$$

Then there exists $n\in D$ such that $n\geq n_0$ and

$$\bigvee_{n\geq n_0}\mathcal{S}(n)(y)+\mu(y)\geq\mathcal{S}(n)(y)+\mu(y)>1.$$

It implies $\mathcal{S}(n)q\mu$. Hence x_t is an (r,s)-cluster point of $\mathcal{S}$, that is, $x_t\in\mathrm{clu}(\mathcal{S},r,s)$. It is a contradiction. Hence $\mathrm{clu}(\mathcal{S},r,s)\geq\bigwedge_{n_0\in D}\mathcal{C}\Big(\bigvee_{n\geq n_0}\mathcal{S}(n),r,s\Big)$. □

3.9. Theorem. *Let* (X,τ,τ^*) *be an ifts and* $\mathcal{S}:D\to P_t(X)$ *a fuzzy net. Then the following properties hold:*

(1) $\mathcal{C}(\lim(\mathcal{S},r,s),r,s)=\lim(\mathcal{S},r,s)$.
(2) $\bigwedge_{n\in D}\mathcal{S}(n)\leq\lim(\mathcal{S},r,s)$.
(3) $\bigvee_{n_0\in D}(\bigwedge_{n\geq n_0}\mathcal{S}(n))\leq\lim(\mathcal{S},r,s)$.

Proof. (1) Similar to that of Theorem 3.6 (1).

(2) Suppose $\bigwedge_{n\in D}\mathcal{S}(n)\not\leq\lim(\mathcal{S},r,s)$. Then there exist $x\in X$ and $t\in(0,1)$ such that

$$\bigwedge_{n\in D}\mathcal{S}(n)(x)>t>\lim(\mathcal{S},r,s)(x).$$

Since $t>\lim(\mathcal{S},r,s)(x)$, by Theorem 3.2 (6), x_t is not an (r,s)-limit point of $\mathcal{S}$. So, there exists $\mu\in\mathcal{N}(x_t,r,s)$ such that for each $n\in D$, there exists $n_0\in D$ satisfying $n_0\geq n$ and $\mu\overline{q}\mathcal{S}(n_0)$. Since $x_tq\mu$, we have

$$\mathcal{S}(n_0)(x)+1-t<\mathcal{S}(n_0)(x)+\mu(x)\leq 1.$$

Thus $\mathcal{S}(n_0)(x)<t$ implies $\bigwedge_{n\in D}\mathcal{S}(n)(x)<t$. It is a contradiction. Hence we have $\bigwedge_{n\in D}\mathcal{S}(n)\leq\lim(\mathcal{S},r,s)$.

(3) Suppose $\bigvee_{n_0\in D}(\bigwedge_{n\geq n_0}\mathcal{S}(n))\not\leq\lim(\mathcal{S},r,s)$. Then there exist $x\in X$ and $t\in(0,1)$ such that

$$\bigvee_{n_0\in D}\Big(\bigwedge_{n\geq n_0}\mathcal{S}(n)\Big)(x)>t>\lim(\mathcal{S},r,s)(x).$$

Since $t<\bigvee_{n_0\in D}\Big(\bigwedge_{n\geq n_0}\mathcal{S}(n)\Big)(x)$, there exists $n_0\in D$ such that $x_t\in\bigwedge_{n\geq n_0}\mathcal{S}(n)$. This implies $t\leq\mathcal{S}(n)(x)$ for all $n\geq n_0$. Hence for each $\mu\in\mathcal{N}(x_t,r,s)$, $t+\mu(x)>1$ implies $\mathcal{S}(n)(x)+\mu(x)>1$, for all $n\geq n_0$. So, x_t is an (r,s)-limit point of $\mathcal{S}$. It is a contradiction. Hence we have $\bigvee_{n_0\in D}(\bigwedge_{n\geq n_0}\mathcal{S}(n))\leq\lim(\mathcal{S},r,s)$. □

3.10. Theorem. *Let* (X,τ,τ^*) *be an ifts and* $\mathcal{S}:D\to P_t(X)$ *a decreasing fuzzy net. Then, for each* $r\in I_0$ *and* $s\in I_1$*, we have*

$$\mathrm{clu}(\mathcal{S},r,s)=\bigwedge_{n\in D}\mathcal{C}(\mathcal{S}(n),r,s).$$

Proof. Suppose

$$\mathrm{clu}(\mathcal{S}, r, s) \not\leq \bigwedge_{n \in D} \mathcal{C}(\mathcal{S}(n), r, s).$$

Then there exists an (r, s)-cluster point x_t of $\mathcal{S}$ such that

$$\mathrm{clu}(\mathcal{S}, r, s)(x) \geq t > \bigwedge_{n \in D} \mathcal{C}(\mathcal{S}(n), r, s)(x).$$

Since x_t is an (r, s)-cluster point of $\mathcal{S}$, for each $\mu \in \mathcal{N}(x_t, r, s)$ and $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $\mathcal{S}(n_0)q\mu$. Since $\mathcal{S}$ is a decreasing fuzzy net, for $n_0 \geq n$, $\mathcal{S}(n_0)q\mu$ implies $\mathcal{S}(n)q\mu$. Hence x_t is an (r, s)-adherent point of $\mathcal{S}(n)$, for each $n \in D$, that is,

$$x_t \in \bigwedge_{n \in D} \mathcal{C}(\mathcal{S}(n), r, s).$$

It is a contradiction. Hence $\mathrm{clu}(\mathcal{S}, r, s) \leq \bigwedge_{n \in D} \mathcal{C}(\mathcal{S}(n), r, s)$.

Suppose

$$\mathrm{clu}(\mathcal{S}, r, s) \not\geq \bigwedge_{n \in D} \mathcal{C}(\mathcal{S}(n), r, s).$$

Then there exists an (r, s)-adherent point x_t of $\mathcal{S}(n)$, for all $n \in D$, such that

$$\mathrm{clu}(\mathcal{S}, r, s)(x) < t \leq \Big(\bigwedge_{n \in D} \mathcal{C}(\mathcal{S}(n), r, s)\Big)(x).$$

Since x_t is an (r, s)-adherent point of $\mathcal{S}(n)$, for $n \in D$, for each $\mu \in \mathcal{N}(x_t, r, s)$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $\mathcal{S}(n_0)q\mu$. Hence x_t is an (r, s)-cluster point of $\mathcal{S}$, that is, $x_t \in \mathrm{clu}(\mathcal{S}, r, s)$. It is a contradiction. Hence $\mathrm{clu}(\mathcal{S}, r, s) \geq \bigwedge_{n \in D} \mathcal{C}(\mathcal{S}(n), r, s)$. □

3.11. Theorem. *Let (X, τ, τ^*) be an ifts and $\mathcal{S} : D \to P_t(X)$ an increasing fuzzy net. Then, for each $r \in I_0$ and $s \in I_1$, we have*

$$\lim(\mathcal{S}, r, s) = \mathcal{C}\Big(\bigvee_{n \in D} \mathcal{S}(n), r, s\Big).$$

Proof. Suppose

$$\lim(\mathcal{S}, r, s) \not\leq \mathcal{C}\Big(\bigvee_{n \in D} \mathcal{S}(n), r, s\Big).$$

Then there exists an (r, s)-limit point x_t of $\mathcal{S}$ such that

$$\lim(\mathcal{S}, r, s)(x) \geq t > \mathcal{C}\Big(\bigvee_{n \in D} \mathcal{S}(n), r, s\Big)(x).$$

Since x_t is an (r, s)-limit point of $\mathcal{S}$, for each $\mu \in \mathcal{N}(x_t, r, s)$, there exists $n_0 \in D$ such that for all $n \geq n_0$, $\mathcal{S}(n)q\mu$. It implies $\bigvee_{n \in D} \mathcal{S}(n)q\mu$. Hence x_t is an (r, s)-adherent point of $\bigvee_{n \in D} \mathcal{S}(n)$. It is a contradiction. Hence $\lim(\mathcal{S}, r, s) \leq \mathcal{C}\Big(\bigvee_{n \in D} \mathcal{S}(n), r, s\Big)$.

Suppose

$$\lim(\mathcal{S}, r, s) \not\geq \mathcal{C}\Big(\bigvee_{n \in D} \mathcal{S}(n), r, s\Big).$$

Then there exists an (r, s)-adherent point x_t of $\bigvee_{n \in D} \mathcal{S}(n)$ such that

$$\lim(\mathcal{S}, r, s)(x) < t \leq \mathcal{C}\Big(\bigvee_{n \in D} \mathcal{S}(n), r, s\Big)(x).$$

Since x_t is an (r,s)-adherent point of $\bigvee_{n\in D}\mathcal{S}(n)$, for each $\mu \in \mathcal{N}(x_t,r,s)$, we have $\bigvee_{n\in D}\mathcal{S}(n)q\mu$, then there exists $n_0 \in D$ such that $\mathcal{S}(n_0)q\mu$. Since $\mathcal{S}$ is an increasing fuzzy net, for $n \geq n_0$, $\mathcal{S}(n_0)q\mu$ implies $\mathcal{S}(n)q\mu$. Hence x_t is an (r,s)-limit point of $\mathcal{S}$, that is, $x_t \in \lim(\mathcal{S},r,s)$. It is a contradiction. Hence $\lim(\mathcal{S},r,s) \geq \mathcal{C}\Big(\bigvee_{n\in D}\mathcal{S}(n),r,s\Big)$. □

3.12. Theorem. *Let (X,τ_1,τ_1^*) and (Y,τ_2,τ_2^*) be ifts's. For every fuzzy net $\mathcal{S}: D \to P_t(X)$, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, the following statements are equivalent:*

(1) *$f:(X,\tau_1,\tau_1^*) \to (Y,\tau_2,\tau_2^*)$ is IF continuous.*

(2) *If $\mathcal{S}\overset{(r,s)}{\infty}x_t$, then $f(\mathcal{S})\overset{(r,s)}{\infty}f(x)_t$.*

(3) *If $\mathcal{S}\xrightarrow{(r,s)}x_t$, then $f(\mathcal{S})\xrightarrow{(r,s)}f(x)_t$.*

(4) *$f(\mathcal{C}(\lambda,r,s)) \leq \mathcal{C}(f(\lambda),r,s)$.*

Proof. (1) $\Longrightarrow$ (2) Let $\mu \in \mathcal{N}(f(x)_t,r,s)$. Since f is IF continuous, then $\tau_1(f^{-1}(\mu)) \geq \tau_2(\mu) \geq r$, $\tau_1^*(f^{-1}(\mu)) \leq \tau_2^*(\mu) \leq s$ and $f(x)_t q\mu$ implies $x_t q f^{-1}(\mu)$. Hence $f^{-1}(\mu) \in \mathcal{N}(x_t,r,s)$. Since $\mathcal{S}\overset{(r,s)}{\infty}x_t$, for $f^{-1}(\mu) \in \mathcal{N}(x_t,r,s)$ and for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $\mathcal{S}(n_0)qf^{-1}(\mu)$. This implies $f(\mathcal{S}(n_0))q\mu$. Hence $f(\mathcal{S})\overset{(r,s)}{\infty}f(x)_t$.

(2) $\Longrightarrow$ (3) Let $\mathcal{S}\xrightarrow{(r,s)}x_t$. For every subnet $\mathcal{U}: E \to P_t(Y)$ of $f(\mathcal{S})$, there exists a cofinal selection $N: E \to D$ such that $\mathcal{U} = f(\mathcal{S}) \circ N = f \circ (\mathcal{S} \circ N)$. Put $T = \mathcal{S} \circ N$. Then T is a subnet of $\mathcal{S}$. This follows from the following:

$$\begin{aligned}
\mathcal{S}\xrightarrow{(r,s)}x_t &\Longrightarrow T\xrightarrow{(r,s)}x_t && \text{(by Theorem 3.2(7))}\\
&\Longrightarrow T\overset{(r,s)}{\infty}x_t && \text{(by Theorem 3.2(1))}\\
&\Longrightarrow f(T) = \mathcal{U}\overset{(r,s)}{\infty}f(x)_t && \text{(by (2))}\\
&\Longrightarrow f(\mathcal{S})\xrightarrow{(r,s)}f(x)_t. && \text{(by Theorem 3.3)}
\end{aligned}$$

(3) $\Longrightarrow$ (4) Suppose there exist $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that

$$f(\mathcal{C}(\lambda,r,s)) \not\leq \mathcal{C}(f(\lambda),r,s).$$

Then there exists $y \in Y$ such that

$$\text{(II)} \qquad f(\mathcal{C}(\lambda,r,s))(y) > \mathcal{C}(f(\lambda),r,s)(y).$$

So, there exists $x \in f^{-1}(\{y\})$ such that

$$f(\mathcal{C}(\lambda,r,s))(y) \geq \mathcal{C}(\lambda,r,s)(x) > \mathcal{C}(f(\lambda),r,s))(y).$$

From Theorem 2.2, there exists an (r,s)-adherent point x_t of λ on (X,τ_1,τ_1^*) such that

$$\mathcal{C}(\lambda,r,s)(x) \geq t > \mathcal{C}(f(\lambda),r,s)(f(x)).$$

Since $x_t \in \mathcal{C}(\lambda,r,s)$, by Theorem 3.5, there exists a fuzzy net $\mathcal{S}$ in λ such that $\mathcal{S}\xrightarrow{(r,s)}x_t$. By (3), $f(\mathcal{S})\xrightarrow{(r,s)}f(x)_t$ with $f(\mathcal{S})$ in $f(\lambda)$. From Theorem 3.5, we have $f(x)_t = y_t \in \mathcal{C}(f(\lambda),r,s)$. It is a contradiction for (II). Hence, for all $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, we have $f(\mathcal{C}(\lambda,r,s) \leq \mathcal{C}(f(\lambda),r,s)$.

(4) $\Longrightarrow$ (1) Similar to Theorem 2.4. □

From Theorem 3.12, we can easily obtain the following corollary.

3.13. Corollary. *Let (X,τ_1,τ_1^*) and (Y,τ_2,τ_2^*) be ifts's. For every fuzzy net $\mathcal{S}: D \to P_t(X)$, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, the following statements are equivalent:*

(1) *$f:(X,\tau_1,\tau_1^*) \to (Y,\tau_2,\tau_2^*)$ is IF continuous.*

(2) $f(\text{clu}(\mathcal{S}, r, s)) \leq \text{clu}(f(\mathcal{S}), r, s)$.
(3) $f(\lim(\mathcal{S}, r, s)) \leq \lim(f(\mathcal{S}), r, s)$.
(4) $f(\mathcal{C}(\lambda, r, s)) \leq \mathcal{C}(f(\lambda), r, s)$. □

4. (r, s)-convergent nets

4.1. Definition. Let (X, τ, τ^*) be an ifts, $\mu \in I^X$, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. A fuzzy net $\mathcal{S}$ is said to be (r, s)-convergent to μ, denoted by $\text{con}(\mathcal{S}, r, s) = \mu$, if $\text{clu}(\mathcal{S}, r, s) = \lim(\mathcal{S}, r, s) = \mu$.

4.2. Theorem. *Let (X, τ, τ^*) be an ifts and $\mathcal{S}, \mathcal{U} : D \to P_t(X)$, (r, s)-convergent nets such that $\mathcal{S}(n) \vee \mathcal{U}(n) \in P_t(X)$ for each $n \in D$. Then*

$$\text{con}(\mathcal{S} \vee \mathcal{U}, r, s) = \text{con}(\mathcal{S}, r, s) \vee \text{con}(\mathcal{U}, r, s).$$

Proof. From Theorem 3.7, $\mathcal{S} \vee \mathcal{U}$ is a fuzzy net. This is easily proved by the following:

$$\begin{aligned} \text{clu}(\mathcal{S} \vee \mathcal{U}, r, s) &= \text{clu}(\mathcal{S}, r, s) \vee \text{clu}(\mathcal{U}, r, s) \qquad \text{(by Theorem 3.7(2))} \\ &= \lim(\mathcal{S}, r, s) \vee \lim(\mathcal{U}, r, s) \\ &\leq \lim(\mathcal{S} \vee \mathcal{U}, r, s) \qquad \text{(by Theorem 3.7(4))} \\ &\leq \text{clu}(\mathcal{S} \vee \mathcal{U}, r, s). \qquad \text{(by Theorem 3.2(2))} \end{aligned}$$

□

4.3. Theorem. *Let (X, τ, τ^*) be an ifts, $\mathcal{S}$ a fuzzy net and $\mathcal{H} = \{T \mid T$ is a subnet of $\mathcal{S}\}$. Then the following statements hold:*

(1) $\lim(\mathcal{S}, r, s) = \bigwedge_{T \in \mathcal{H}} \text{clu}(T, r, s)$.
(2) $\text{clu}(\mathcal{S}, r, s) = \bigvee_{T \in \mathcal{H}} \lim(T, r, s)$.
(3) *If* $\text{con}(\mathcal{S}, r, s) = \mu$, *then* $\text{con}(T, r, s) = \mu$ *for each* $T \in \mathcal{H}$.

Proof. (1) For each $T \in \mathcal{H}$, by Theorem 3.2 (2,8,10), we have

(III) $\qquad \lim(\mathcal{S}, r, s) \leq \lim(T, r, s) \leq \text{clu}(T, r, s) \leq \text{clu}(\mathcal{S}, r, s)$.

Hence

$$\lim(\mathcal{S}, r, s) \leq \bigwedge_{T \in \mathcal{H}} \text{clu}(T, r, s).$$

Suppose

$$\lim(\mathcal{S}, r, s) \not\geq \bigwedge_{T \in \mathcal{H}} \text{clu}(T, r, s).$$

Then there exist $x \in X$ and $t \in (0, 1)$ such that

(IV) $\qquad \lim(\mathcal{S}, r, s)(x) < t < \bigwedge_{T \in \mathcal{H}} \text{clu}(T, r, s)$.

Since $\lim(\mathcal{S}, r, s)(x) < t$, by Theorem 3.2 (6), x_t is not an (r, s)-limit point of $\mathcal{S}$, that is, there exists $\mu \in \mathcal{N}(x_t, r, s)$ such that for each $n \in D$ there exists $N(n) \in D$ with for $N(n) \geq n$ and $\mathcal{S}(N(n))\overline{q}\mu$. Hence there exists a cofinal selection $N : D \to D$ such that $T = \mathcal{S} \circ N$. Thus T is a subnet of $\mathcal{S}$. Moreover, x_t is not an (r, s)-cluster point of T. By Theorem 3.2 (5), $\text{clu}(T, r, s)(x) < t$. It is a contradiction for (IV). Hence $\lim(\mathcal{S}, r, s) = \bigwedge_{T \in \mathcal{H}} \text{clu}(T, r, s)$.

(2) From (III) of (1), we have

$$\bigvee_{T \in \mathcal{H}} \lim(T, r, s) \leq \text{clu}(\mathcal{S}, r, s).$$

Suppose

$$\bigvee_{T\in\mathcal{H}} \lim(T,r,s) \not\geq \mathrm{clu}(\mathcal{S},r,s).$$

Then there exist $x \in X$ and $t \in (0,1)$ such that

$$\text{(V)} \qquad \bigvee_{T\in\mathcal{H}} \lim(T,r,s)(x) < t < \mathrm{clu}(\mathcal{S},r,s)(x).$$

Since $x_t \in \mathrm{clu}(\mathcal{S},r,s)$, by Theorem 3.2(5), we have $\mathcal{S}\overset{(r,s)}{\infty}x_t$. By Theorem 3.4, there exists a subnet T of $\mathcal{S}$ such that $T\xrightarrow{(r,s)}x_t$. Thus

$$x_t \in \lim(T,r,s) \leq \bigvee_{T\in\mathcal{H}} \lim(T,r,s).$$

It is a contradiction for (V). Hence $\bigvee_{T\in\mathcal{H}} \lim(T,r,s) \geq \mathrm{clu}(\mathcal{S},r,s)$.

(3) Easily proved from (III) of (1). □

4.4. Theorem. *Let (X,τ,τ^*) be an ifts, $\mathcal{S}$ a fuzzy net. If every subnet of $\mathcal{S}$ has a subnet which is (r,s)-convergent to μ, then* $\mathrm{con}(\mathcal{S},r,s)=\mu$.

Proof. Let $\mathcal{H}=\{T \mid T \text{ is a subnet of } \mathcal{S}\}$. For each $T\in\mathcal{H}$, since T has a subnet K with $\mathrm{con}(K,r,s)=\mu$, by Theorem 3.2 (8), we have

$$\lim(T,r,s) \leq \lim(K,r,s) = \mathrm{clu}(K,r,s) = \mu.$$

Hence, by Theorem 4.3 (2),

$$\text{(VI)} \qquad \mathrm{clu}(\mathcal{S},r,s) = \bigvee_{T\in\mathcal{H}} \lim(T,r,s) \leq \mu.$$

Conversely, by Theorem 3.2 (10),

$$\mu = \lim(K,r,s) = \mathrm{clu}(K,r,s) \leq \mathrm{clu}(T,r,s).$$

Hence, by Theorem 4.3 (1),

$$\text{(VII)} \qquad \mu \leq \bigwedge_{T\in\mathcal{H}} \mathrm{clu}(T,r,s) = \lim(\mathcal{S},r,s).$$

By (VI) and (VII), $\mathrm{clu}(\mathcal{S},r,s) \leq \lim(\mathcal{S},r,s)$. Since $\lim(\mathcal{S},r,s) \leq \mathrm{clu}(\mathcal{S},r,s)$ from Theorem 3.2 (2), $\mathrm{clu}(\mathcal{S},r,s) = \lim(\mathcal{S},r,s)$, that is, $\mathrm{con}(\mathcal{S},r,s)=\mu$. □

4.5. Example. Let $X=\{a,b\}$ be a set, N the set of natural numbers and let $\mu \in I^X$ be defined by $\mu(a)=0.3$ and $\mu(b)=0.4$. We define the IFGO, (τ,τ^*) as follows:

$$\tau(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0} \text{ or } \underline{1},\\ \frac{1}{2}, & \text{if } \lambda=\mu,\\ 0, & \text{otherwise,}\end{cases} \qquad \tau^*(\lambda)=\begin{cases}0, & \text{if } \lambda=\underline{0} \text{ or } \underline{1},\\ \frac{1}{2}, & \text{if } \lambda=\mu,\\ 1, & \text{otherwise.}\end{cases}$$

Define a fuzzy net $\mathcal{S}: N \to P_t(X)$ by

$$\mathcal{S}(n)=x_{a_n}, \quad a_n = 0.6+(-1)^n 0.2.$$

We can show $\mathrm{clu}(\mathcal{S},\frac{1}{2},\frac{1}{2})=\underline{1}$, from (1) and (2):

(1) x_t for $t \leq 0.7$ or y_m for $m \leq 0.6$ is an $(\frac{1}{2},\frac{1}{2})$-cluster point of $\mathcal{S}$, because, for $\underline{1} \in \mathcal{N}(p,\frac{1}{2},\frac{1}{2})$ with $p=x_t$ or y_m and for all $n\in N$, we have $\mathcal{S}(n)q\underline{1}$.

(2) x_t for $t > 0.7$ or y_m for $m > 0.6$ is an $(\frac{1}{2},\frac{1}{2})$-cluster point of $\mathcal{S}$, because, for $\underline{1},\mu \in \mathcal{N}(p,\frac{1}{2},\frac{1}{2})$ with $p=x_t$ or y_m and for all $n\in N$, there exists $2n\in N$ such that $2n\geq n$, $\mathcal{S}(2n)=x_{0.8}q\mu$.

We can show $\lim(\mathcal{S},\frac{1}{2},\frac{1}{2})=\underline{1}-\mu$, from (3) and (4):

(3) x_t for $t \leq 0.7$ or y_m for $m \leq 0.6$ is an $(\frac{1}{2}, \frac{1}{2})$-limit point of $\mathcal{S}$, because, for $\underline{1} \in \mathcal{N}(p, \frac{1}{2}, \frac{1}{2})$ with $p = x_t$ or y_m and for all $n \in N$, we have $\mathcal{S}(n)q\underline{1}$.

(4) x_t for $t > 0.7$ or y_m for $m > 0.6$ is not an $(\frac{1}{2}, \frac{1}{2})$-limit point of $\mathcal{S}$, because, for $\mu \in \mathcal{N}(p, \frac{1}{2}, \frac{1}{2})$ such that for all $n \in N$, there exists $2n + 1 \in N$ such that $2n + 1 \geq n$, $\mathcal{S}(2n + 1) = x_{0.4}\overline{q}\mu$.

Since $\mathrm{clu}(\mathcal{S}, \frac{1}{2}, \frac{1}{2}) \neq \lim(\mathcal{S}, \frac{1}{2}, \frac{1}{2})$, $\mathcal{S}$ is not $(\frac{1}{2}, \frac{1}{2})$-convergent.

By a similar method, we show for $0 < r \leq \frac{1}{2}$ and $\frac{1}{2} \leq s < 1$,

$$\underline{1} = \mathrm{clu}(\mathcal{S}, r, s) \neq \lim(\mathcal{S}, r, s) = \underline{1} - \mu,$$

and for $r > \frac{1}{2}$ and $s \leq \frac{1}{2}$,

$$\underline{1} = \mathrm{clu}(\mathcal{S}, r, s) = \lim(\mathcal{S}, r, s).$$

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