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A NOTE ON $(\in, \in\vee q)$-FUZZY EQUIVALENCE RELATIONS AND INDISTINGUISHABILITY OPERATORS

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Abstract

In this paper $(\in, \in\vee q)$-fuzzy equivalence relations are defined and some of their properties are studied. The partition of an $(\in, \in\vee q)$-fuzzy equivalence relation is studied. It is shown that under a semibalanced mapping the preimage of an $(\in, \in\vee q)$-fuzzy equivalence relation is an $(\in, \in\vee q)$-fuzzy equivalence relation, whereas the image of an $(\in, \in\vee q)$-fuzzy equivalence relation under a balanced mapping is an $(\in, \in\vee q)$-fuzzy equivalence relation. To conclude, $(\in, \in\vee q)$-fuzzy indistinguishability relations are defined and some of their properties are studied.

Keywords: Fuzzy set, Fuzzy equivalence relation, Fuzzy G-equivalence relation, Fuzzy partition, Fuzzy indistinguishability operator.

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1. Introduction

The theory of fuzzy relations is a generalization of that of crisp relations of a set. Zadeh [38] introduced the concept of fuzzy relations, he also introduced the concept of fuzzy similarity relations in [39]. This provided an impetus for research in this very important area. Many authors like Chakraborty and Das, Murali, Nemitz [10, 11, 27, 28] studied fuzzy equivalence relations.

As the research progressed it soon became clear that any given relation may or may not satisfy a particular requirement for the fuzzy equivalence/similarity relation defined

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by Zadeh in a precise manner as for the crisp cases. If someone is dealing with samples of slightly distorted crisp equivalence relations, where distortion took place due to unstable behavior or by a small noise interference, then the precision required for them to be a fuzzy equivalence relation in the sense of Zadeh may evaporate. Such objections can be found in [3, 4, 9, 15, 19].

As pointed out by Gupta and Gupta [16], the condition $\mu(x,x) = 1$ for all $x \in X$ is too strong for defining a fuzzy reflexive relation μ on a set X (see also Yeh [36]). So new types of fuzzy reflexive relations were introduced by some researchers. Yeh [36] introduced the concept of ϵ-reflexive fuzzy relations and weakly reflexive fuzzy relations. Gupta and Gupta [16] defined G-reflexive fuzzy relations. This concept of G-reflexive fuzzy relation is a generalization of reflexive fuzzy relations introduced by Zadeh [39]. With the introduction of G-reflexive fuzzy relations, a more generalized concept of G-equivalence fuzzy relation appeared as a natural consequence.

When considering fuzzy symmetric relations, authors have been quite contented with the condition $\mu(x,y) = \mu(y,x)$ for a fuzzy relation μ. In many papers [10, 11, 13, 16, 27, 28, 39] this condition has been considered.

Different approaches have been adopted while discussing fuzzy transitive relations. The first type of transitivity is that introduced by Zadeh [39], and the second type of transitivity is the T-transitivity of fuzzy relations. T-transitivity has been defined with the help of the T-norm. Demirci and Recasens [13] studied such type of fuzzy T-transitivity. Another type of transitivity has been introduced by Beg and Samina in [2]. They called it $\in$-fuzzy transitivity. With the help of this transitivity they introduced the concept of an $\in$-fuzzy dissimilarity relation.

The mentioned list of authors is by no means complete, however it gives us a slight idea about the importance of the concept of fuzzy equivalence relations in different contexts.

Ming and Ming [26] introduced the concept of a fuzzy point in a fuzzy subset. This idea of a fuzzy point, its belongingness to and quasi-coincidence with, proved vital for the inception of (α, β)-fuzzy algebraic structures. Among (α, β)-fuzzy algebraic structures $(\in, \in\!\vee q)$-fuzzy algebraic structures are the most important. This type of fuzzy structure is a generalization of classical fuzzy algebraic structures. Many authors [8, 12, 18, 33] studied these algebraic structures in different contexts.

This paper is arranged in the following manner. In Section 2, we give some basic definitions which are required in the sequel. In Section 3, we introduce the concept of (α, β)-fuzzy reflexive relations. This is a generalization of fuzzy reflexive relation as well as of fuzzy G-reflexive relations. It is shown that a fuzzy G-reflexive relation is a special case of (α, β)-fuzzy reflexive relations, viz. $(\in, \in)$-fuzzy reflexive relations. A more general type of fuzzy symmetric relation, that is a (α, β)-fuzzy symmetric relation is defined and it is shown that a classical fuzzy symmetric relation is actually an $(\in, \in)$-fuzzy symmetric relation. (α, β)-fuzzy transitive relations are also studied and it has been shown that a fuzzy transitive relation is actually an $(\in, \in)$-fuzzy relation.

These concepts of (α, β)-fuzzy reflexive, symmetric and transitive relations naturally lead to the concept of (α, β)-fuzzy equivalence relations on a set. Some properties of (α, β)-fuzzy equivalence relations are studied. In Section 4, the $(\in, \in\!\vee q)$-fuzzy partition of a fuzzy relation is studied. Section 5 contins a study of the inverse image and image of an $(\in, \in\!\vee q)$-fuzzy equivalence relation under a semibalanced and balanced mapping. In section 6, $(\in, \in\!\vee q)$-fuzzy indistinguishability operators are defined and some of their properties are studied.

2. Preliminaries

In this section we give a brief account of some definitions about fuzzy relations. These definitions will be required in later sections.

Given a non-empty set X, a subset σ of $X \times X$ is called a binary relation on X. A binary relation σ on X is reflexive if $(x,x) \in \sigma, \forall x \in X$; σ is symmetric if $(x,y) \in \sigma$ implies $(y,x) \in \sigma$, $\forall x,y \in X$; σ is transitive if $(x,y) \in \sigma$ and $(y,z) \in \sigma$ imply $(x,z) \in \sigma$, $\forall x,y,z \in X$. A binary relation is called an equivalence relation if it is reflexive, symmetric and transitive.

Recall that a fuzzy subset μ of X is a mapping $\mu : X \to [0,1]$.

2.1. Definition. [27] A fuzzy binary relation on X and Y is a fuzzy subset μ of $X \times Y$. So a fuzzy binary relation on a set X is a fuzzy subset μ of $X \times X$.

As this paper is concerned with only binary relations, so in the sequel we shall say relation instead of binary relation.

2.2. Definition. If λ is a fuzzy relation on X, then the subset λ_0 of $X \times X$ defined as $\lambda_0 = \{(x,y) \in X \times X : \lambda(x,y) > 0\}$ is called the *support of* λ.

2.3. Definition. [30] Let f be a mapping from a set X into the set Y. If λ is a fuzzy subset of X, the *image* $f(\lambda)$ of λ is the fuzzy subset of Y defined by

$$f(\lambda)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

If μ is a fuzzy subset of Y, the *preimage* or *inverse image* $f^{-1}(\mu)$ of μ is the fuzzy subset of X defined by

$$f^{-1}(\mu)(x) = \mu f(x), \text{ for all } x \in X.$$

2.4. Definition. [27] A fuzzy relation μ on a set X is said to be *reflexive* if $\mu(x,x) = 1$, $\forall x \in X$, and *symmetric* if $\mu(x,y) = \mu(y,x), \forall x,y \in X$.

2.5. Definition. [25] Let μ be a fuzzy relation on the set X and $0 < \alpha \leq 1$; μ is α*-reflexive* on X if $\mu(a,a) = \alpha$ and $\mu(a,b) \leq \alpha$ for all $a,b \in X$.

2.6. Definition. [16] A fuzzy relation μ on a set X is called G*-reflexive* if

(1) $\mu(x,x) > 0$
(2) $\mu(x,y) \leq \min_{t \in X} \mu(t,t)$ for all $x \neq y$ in X.

It can be seen that a fuzzy reflexive relation is an α-reflexive relation for $\alpha = 1$. A fuzzy α-reflexive relation is a fuzzy G-reflexive relation. Therefore a fuzzy G-reflexive relation is a generalization of a fuzzy reflexive relation and of an α-reflexive relation.

2.7. Definition. [16] For two fuzzy relations λ and μ on a set X we define the composition $\lambda \circ \mu$ as

$$(\lambda \circ \mu)(x,z) = \sup_{y \in X} \{\min(\lambda(x,y), \mu(y,z))\}.$$

2.8. Definition. [39] A fuzzy relation μ on a set X is said to be *fuzzy transitive* if $\mu(x,z) \geq \sup_{y \in X} \{\min(\mu(x,y), \mu(y,z))\}$ for all $(x,y),\ (y,z) \in X \times X$.

A fuzzy relation μ on X is a fuzzy equivalence relation if it is a fuzzy reflexive, symmetric and transitive relation on X. A fuzzy relation μ on X is a fuzzy G-equivalence relation if it is a fuzzy G-reflexive, symmetric and transitive relation on X.

For any $\alpha \in (0,1]$ and $x \in X$, a fuzzy subset x_α of X defined below is called a *fuzzy point*:

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

for each $y \in X$. A fuzzy point is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy subset λ, written as $x_\alpha \in \lambda$ (resp. $x_\alpha q\lambda$) if $\lambda(x) \geq \alpha$, $(\lambda(x) + \alpha > 1)$. "$x_\alpha \in \lambda$ or $x_\alpha q\lambda$" will be denoted by "$x_\alpha \in \vee q\lambda$". Likewise, "$x_\alpha \in \lambda$ and $x_\alpha q\lambda$" will be denoted by "$x_\alpha \in \wedge q\lambda$."

On the other hand, $x_\alpha \overline{\in} \lambda$, $x_\alpha \overline{q} \lambda$, $x_\alpha \overline{\in \vee q} \lambda$ and $x_\alpha \overline{\in \wedge q} \lambda$ will mean $x_\alpha \in \lambda$, $x_\alpha q\lambda$, $x_\alpha \in \vee q\lambda$ and $x_\alpha \in \wedge q\lambda$, respectively, do not hold.

3. (α, β)-Fuzzy equivalence relations

In this section we define (α, β)-fuzzy reflexive, symmetric and transitive relations on a set X. These definitions pave the way for defining (α, β)-fuzzy equivalence relations on the set X. We study here different types of (α, β)-fuzzy relations, in particular $(\in, \in\vee q)$-fuzzy relations.

3.1. Definition. A fuzzy subset $(x,y)_r$ of $X \times Y$ of the form

$$(x,y)_r(a,b) = \begin{cases} r & \text{if } x = a,\ y = b \\ 0 & \text{otherwise} \end{cases}$$

is called a *fuzzy ordered pair*.

In the sequel we shall call a fuzzy ordered pair a *fuzzy pair*.

3.2. Definition. A fuzzy pair $(x,y)_r$ is said to *belong to* (resp. *be quasi coincident with*) a fuzzy relation λ, written as $(x,y)_r \in \lambda$ (resp. $(x,y)_r q\lambda$) if $\lambda(x,y) \geq r$ (resp. $\lambda(x,y) + r > 1$).

If $(x,y)_r \in \lambda$ or $(x,y)_r q\lambda$ then we write $(x,y)_r \in \vee q\lambda$.

3.3. Definition. A fuzzy relation λ on X is called an (α, β)-fuzzy reflexive relation on X if for $x, y \in X$, $(x,y)_r \alpha\lambda$ implies $(t,t)_r \beta\lambda$ for all $t \in X$ and $r \in (0,1]$.

Here α, β is any one of $\in$, q, $\in\vee q$, $\in\wedge q$, and $\alpha \neq \in\wedge q$ because if $\lambda(x,y) \leq 0.5$ for all $(x,y) \in X \times X$ and $r \in (0,1]$ is such that $(x,y)_r \in \wedge q\lambda$ then $\lambda(x,y) \geq r$ and $\lambda(x,y) + r > 1$. It follows that $1 < \lambda(x,y) + r \leq \lambda(x,y) + \lambda(x,y) = 2\lambda(x,y)$ so that $\lambda(x,y) > 0.5$. This means that $\{(x,y)_r : (x,y)_r \in \wedge q\lambda\} = \emptyset$. Therefore, the case $\alpha = \in\wedge q$ in the above definition is omitted.

In [16], Gupta and Gupta defined G-reflexive relations. In the following we show that G-reflexive relations are a special type of (α, β)-fuzzy reflexive relations viz $(\in, \in)$-fuzzy reflexive relations.

3.4. Lemma. *A fuzzy relation μ on a set X is a G-reflexive if and only if μ is an $(\in, \in)$-fuzzy reflexive relation on X.*

Proof. Suppose that μ is G-reflexive and $(x,y)_r \in \mu$. Then $\mu(x,y) \geq r$ since

$$\min_{t \in X} \mu(t,t) \geq \mu(x,y) \geq r > 0.$$

Therefore $(t,t)_r \in \mu$ for all $t \in X$.

Conversely, assume that μ is an $(\in, \in)$-fuzzy reflexive relation. Then $(x,y)_r \in \mu$ implies $(t,t)_r \in \mu$, where $r \in (0,1]$. If possible let, for some $x \neq y$ in X,

$$\min \mu(t,t) < \mu(x,y) \text{ for all } t \in X.$$

Then there exists some $r \in (0,1]$ such that $\min \mu(t,t) < r \leq \mu(x,y)$. This implies $(x,y)_r \in \mu$, but $(t,t)_r \overline{\in} \mu$ for some $t \in X$, which is a contradiction. Therefore we have $\min\limits_{t \in X} \mu(t,t) \geq \mu(x,y)$, where $x \neq y$. □

A fuzzy reflexive relation on a set X is an $(\in, \in)$-fuzzy reflexive relation, but the converse is not true. To show this consider the following.

3.5. Example. Let $X = \{a,b,c\}$, define $\lambda : X \times X \to [0,1]$ by $\lambda(a,a) = \lambda(b,b) = \lambda(c,c) = 0.5$, $\lambda(a,b) = \lambda(a,c) = 0.3$, $\lambda(b,a) = \lambda(b,c) = 0.4$, $\lambda(c,a) = \lambda(c,b) = 0.2$. Then λ is an $(\in, \in)$-fuzzy reflexive relation, but not a fuzzy reflexive relation.

3.6. Proposition. *A fuzzy relation λ on a set X is an $(\in, \in\vee q)$-fuzzy reflexive relation on X if and only if $\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\}$ for all $x,y,t \in X$.*

Proof. Suppose λ is an $(\in, \in\vee q)$-fuzzy reflexive relation on X and $x,y,t \in X$ are such that

$$\lambda(t,t) < \min\{\lambda(x,y), 0.5\}.$$

First we consider the case when $\lambda(x,y) < 0.5$. Then $\lambda(t,t) < \lambda(x,y)$. Thus there exists some $r \in (0,1]$ such that $\lambda(t,t) < r \leq \lambda(x,y)$. This implies $(x,y)_r \in \lambda$ but $(t,t)_r \overline{\in \vee q} \lambda$, which is a contradiction to our hypothesis.

Now we consider the case when $\lambda(x,y) \geq 0.5$. Then $\lambda(t,t) < 0.5$. Since $\lambda(x,y) \geq 0.5$, so $(x,y)_{0.5} \in \lambda$ but $(t,t)_{0.5} \overline{\in \vee q} \lambda$, which is again a contradiction to our hypothesis. Hence $\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\}$ for all $x,y,t \in X$.

Conversely, assume that $\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\}$ for all $x,y,t \in X$. Suppose $(x,y)_r \in \lambda$. Then $\lambda(x,y) \geq r$. If $r \leq 0.5$, then $\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\} \geq r$. This implies $(t,t)_r \in \lambda$.

If $r > 0.5$ then $\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\} = 0.5$, therefore $\lambda(t,t) + r > 0.5 + 0.5 = 1$. This implies $(t,t)_r q\lambda$. Hence we have $(t,t)_r \in \vee q\lambda$. □

3.7. Proposition. *Let λ be a non zero (α, β)-fuzzy reflexive relation on a set X. Then the support λ_0 of λ is a reflexive relation on X.*

Proof. Since λ is a non zero (α, β)-fuzzy reflexive relation on the set X, then $\lambda_0 \neq \emptyset$. Suppose $(x,y) \in \lambda_0$ and $t \in X$ are such that $\lambda(t,t) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $(x,y)_{\lambda(x,y)} \alpha\lambda$ but $\lambda(t,t) < \lambda(x,y)$, also $\lambda(t,t) + \lambda(x,y) \leq 0 + 1 = 1$. Therefore $(t,t)_{\lambda(x,y)} \overline{\beta}\lambda$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, which is a contradiction to the hypothesis. Also $(x,y)_1 q\lambda$ but $(t,t)_1 \overline{\beta}\lambda$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, which is again a contradiction.

Therefore $\lambda(t,t) > 0$, so $(t,t) \in \lambda_0$ for all $t \in X$. □

The following Proposition suggests a way to construct $(\alpha, \in\vee q)$-fuzzy reflexive relations.

3.8. Proposition. *If R is a reflexive relation on a set X, then a fuzzy subset λ of $X \times X$ satisfying $\lambda(x,y) \begin{cases} \geq 0.5 & \text{for } (x,y) \in R \\ = 0 & \text{otherwise} \end{cases}$ is an $(\alpha, \in\vee q)$-fuzzy reflexive relation on X.*

Proof. (a) Let $x,y \in X$ and $r \in (0,1]$ be such that $(x,y)_r \in \lambda$. Then $(x,y) \in R$. For all $t \in X$, $(t,t) \in R$, that is $\lambda(t,t) \geq 0.5$. If $r \leq 0.5$ then $\lambda(t,t) \geq 0.5 \geq r$. Hence $(t,t)_r \in \lambda$. If $r > 0.5$ then $\lambda(t,t) + r > 0.5 + 0.5 = 1$, and so $(t,t)_r q\lambda$. Thus $(t,t)_r \in \vee q\lambda$ for all $t \in X$.

(b) Let $x,y \in X$ and $r \in (0,1]$ be such that $(x,y)_r q\lambda$. Then $(x,y) \in R$ and $\lambda(x,y) + r > 1$. Since $(t,t) \in R$, for all $t \in X$, we have $\lambda(t,t) \geq 0.5$. If $r \leq 0.5$ then $\lambda(t,t) \geq$

$0.5 \geq r$. Hence $(t,t)_r \in \lambda$. If $r > 0.5$ then $\lambda(t,t) + r > 0.5 + 0.5 = 1$, and so $(t,t)_r\, q\lambda$. Thus $(t,t)_r \in \vee q\lambda$ for all $t \in X$.

(c) Let $x, y \in X$ and $r \in (0,1]$ be such that $(x,y)_r \in \vee q\lambda$. Then $(x,y)_r \in \lambda$ or $(x,y)_r\, q\lambda$, analogously as in (a) and (b) we obtain $(t,t)_r \in \vee q\lambda$ for all $t \in X$. □

In the literature a fuzzy relation λ on X is called a fuzzy symmetric relation on X if $\lambda(x,y) = \lambda(y,x)$. In the following we define (α,β)-fuzzy symmetric relations, which is a more general concept.

3.9. Definition. A fuzzy relation λ on X is called an (α,β)-fuzzy symmetric relation on X if $(x,y)_r\, \alpha\lambda$ implies $(y,x)_r\, \beta\lambda$ for all $x, y \in X$ and $r \in (0,1]$.

3.10. Lemma. *A fuzzy relation λ on a set X is fuzzy symmetric if and only if λ is $(\in,\in)$-fuzzy symmetric.*

Proof. Suppose that λ is fuzzy symmetric and $x, y \in X$. If $(x,y)_r \in \lambda$ then $\lambda(x,y) \geq r$. Since $\lambda(x,y) = \lambda(y,x)$, so $\lambda(y,x) \geq r$. Thus $(y,x)_r \in \lambda$. Hence λ is an $(\in,\in)$-fuzzy symmetric relation.

Conversely, assume that λ is an $(\in,\in)$-fuzzy symmetric relation. Let $x, y \in X$ be such that $\lambda(x,y) \neq \lambda(y,x)$. Also assume that $\lambda(x,y) > \lambda(y,x)$. Then there exists some $r \in (0,1]$ such that $\lambda(x,y) \geq r > \lambda(y,x)$. This implies $(x,y)_r \in \lambda$ but $(y,x)_r \notin \lambda$, which is a contradiction to our hypothesis. Hence $\lambda(x,y) = \lambda(y,x)$. □

The above Lemma shows that a fuzzy symmetric relation is a special case of an (α,β)-fuzzy symmetric relations, viz an $(\in,\in)$-fuzzy symmetric relation.

3.11. Proposition. *A fuzzy relation λ on a set X is an $(\in, \in\vee q)$-fuzzy symmetric relation on X if and only if $\lambda(y,x) \geq \min\{\lambda(x,y), 0.5\}$.*

Proof. Similar to Proposition 3.6. □

3.12. Proposition. *Let λ be a non zero (α,β)-fuzzy symmetric relation on a set X. Then the support λ_0 of λ is a symmetric relation on X.*

Proof. Similar to Proposition 3.7. □

3.13. Proposition. *If R is a symmetric relation on a set X, then a fuzzy subset λ of $X \times X$ satisfying $\lambda(x,y) \begin{cases} \geq 0.5 & \text{for } (x,y) \in R \\ = 0 & \text{otherwise} \end{cases}$ is an $(\alpha, \in \vee q)$-fuzzy symmetric relation on X.*

Proof. Similar to Proposition 3.8. □

Different types of fuzzy transitive relations have been defined. Here we define a more general type of fuzzy transitive relation, that is an (α,β)-fuzzy transitive relation.

3.14. Definition. A fuzzy relation λ on a set X is called an (α,β)-fuzzy transitive relation on X if $(x,y)_{r_1}, (y,z)_{r_2}\, \alpha\lambda$ implies $(x,z)_{\min\{r_1,r_2\}}\, \beta\lambda$, for all $x, y, z \in X$ and $r_1, r_2 \in (0,1]$.

3.15. Lemma. *A fuzzy relation λ on a set X is fuzzy transitive if and only if it is an $(\in,\in)$-fuzzy transitive relation.*

Proof. Suppose that λ is fuzzy transitive and $x, y, z \in X, r_1, r_2 \in (0,1]$. If $(x,y)_{r_1}, (y,z)_{r_2} \in \lambda$ then $\lambda(x,y) \geq r_1$ and $\lambda(y,z) \geq r_2$. Since λ is fuzzy transitive, so

$$\lambda(x,z) \geq \sup_{y \in X} \{\min(\lambda(x,y), \lambda(y,z))\} \geq \min(\lambda(x,y), \lambda(y,z)) \geq \min(r_1, r_2).$$

Thus $(x,z)_{\min(r_1,r_2)} \in \lambda$. Hence λ is an $(\in, \in)$-fuzzy transitive relation.

Conversely, assume that λ is an $(\in, \in)$-fuzzy transitive relation. Let $x, z \in X$ be such that $\lambda(x,z) < \sup_{y \in X} \{\min(\lambda(x,y), \lambda(y,z))\}$. This implies that there exists some $y \in X$ such that $\lambda(x,z) < \min(\lambda(x,y), \lambda(y,z))$, so there exists some $r \in (0,1]$ such that $\lambda(x,z) < r \leq \min(\lambda(x,y), \lambda(y,z))$. This implies $(x,y)_r \in \lambda$, $(y,z)_r \in \lambda$, but $(x,z)_r \notin \lambda$, which is a contradiction to our hypothesis. Hence

$$\lambda(x,z) \geq \sup_{y \in X} \{\min(\lambda(x,y), \lambda(y,z))\}. \qquad \square$$

3.16. Proposition. *A fuzzy relation λ on a set X is an $(\in, \in\vee q)$-fuzzy transitive relation if and only if $\lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\}$ for all $x, y, z \in X$.*

Proof. Suppose λ is an $(\in, \in\vee q)$-fuzzy transitive relation on X and $x, y, z \in X$ is such that

$$\lambda(x,z) < \min\{\lambda(x,y), \lambda(y,z), 0.5\}.$$

First we consider the case when

$$\min\{\lambda(x,y), \lambda(y,z)\} < 0.5,$$

so

$$\lambda(x,z) < \min\{\lambda(x,y), \lambda(y,z)\}.$$

Thus there exists some $r \in (0,1]$ such that $\lambda(x,z) < r \leq \min\{\lambda(x,y), \lambda(y,z)\}$, which implies $(x,y)_r \in \lambda$ and $(y,z)_r \in \lambda$ but $(x,z)_{\min\{r,r\}} \overline{\in \vee q} \lambda$, which is a contradiction to our hypothesis.

Now we consider the case when

$$\min\{\lambda(x,y), \lambda(y,z)\} \geq 0.5,$$

so $\lambda(x,z) < 0.5$. Since $\lambda(x,y) \geq 0.5$ and $\lambda(y,z) \geq 0.5$, then $(x,y)_{0.5} \in \lambda$ and $(y,z)_{0.5} \in \lambda$, but $(x,z)_{0.5} \overline{\in \vee q} \lambda$, which is again a contradiction to our hypothesis. Hence $\lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\}$ for all $x, y, z \in X$.

Conversely, suppose that $\lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\}$ for all $x, y, z \in X$. Let $(x,y)_{r_1} \in \lambda$ and $(y,z)_{r_2} \in \lambda$. If $\min\{r_1, r_2\} \leq 0.5$ then

$$\lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\} \geq \min\{r_1, r_2\}.$$

So, $(x,z)_{\min\{r_1,r_2\}} \in \lambda$.

If $\min\{r_1, r_2\} > 0.5$ then $\lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\} = 0.5$, therefore $\lambda(x,z) + \min\{r_1, r_2\} > 0.5 + 0.5 = 1$. This implies $\lambda(x,z)_{\min\{r_1,r_2\}} q \lambda$. Hence we have $(x,z)_{\min\{r_1,r_2\}} \in \vee q \lambda$. $\square$

3.17. Proposition. *Let λ be a non zero (α, β)-fuzzy transitive relation on a set X. Then the support λ_0 of λ is a transitive relation on X.*

Proof. As λ is a non-zero (α, β)-fuzzy transitive relation on the set X, therefore $\lambda_0 \neq \emptyset$. Suppose for some $x, y, z \in X$ that $(x, y), (y, z) \in \lambda_0$ and $\lambda(x, z) = 0$.

If $\alpha \in \{\in, \in \vee q\}$ then $(x, y)_{\lambda(x,y)}, (y, z)_{\lambda(y,z)}\, \alpha\lambda$, but $\lambda(x, z) < \min\{\lambda(x, y), \lambda(y, z)\}$; also $\lambda(x, z) + \min\{\lambda(x, y), \lambda(y, z)\} \leq 0+1 = 1$. Therefore, $\lambda(x, z)_{\min\{\lambda(x,y),\lambda(y,z)\}}\, \overline{\beta}\lambda$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction to the hypothesis. Also $(x, y)_1\, q\lambda$ and $(y, z)_1\, q\lambda$ but $\lambda(x, z)_{\min\{\lambda(x,y),\lambda(y,z)\}}\, \overline{\beta}\lambda$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is again a contradiction to the hypothesis. Therefore $\lambda(x, z) > 0$, so $(x, z) \in \lambda_0$. □

3.18. Proposition. *If R is a transitive relation on a set X, then a fuzzy subset λ of $X \times X$ satisfying $\lambda(x, y) \begin{cases} \geq 0.5 & \text{for } (x, y) \in R \\ = 0 & \text{otherwise} \end{cases}$ is an $(\alpha, \in \vee q)$-fuzzy transitive relation on X.*

Proof. (a) Let $x, y, z \in X$ and $r_1, r_2 \in (0, 1]$ be such that $(x, y)_{r_1}, (y, z)_{r_2} \in \lambda$. Then $(x, y) \in R$ and $(y, z) \in R$. Since R is transitive, then $(x, z) \in R$. Thus $\lambda(x, z) \geq 0.5$. If $\min\{r_1, r_2\} \leq 0.5$ then $\lambda(x, z) \geq 0.5 \geq \min\{r_1, r_2\}$. Hence $(x, z)_{\min\{r_1,r_2\}} \in \lambda$. If $\min\{r_1, r_2\} > 0.5$ then $\lambda(x, z) + \min\{r_1, r_2\} > 0.5 + 0.5 = 1$ and so $(x, z)_{\min\{r_1,r_2\}} q\lambda$. Hence $(x, z)_{\min\{r_1,r_2\}} \in \vee q\lambda$.

(b) Let $x, y, z \in X$ and $r_1, r_2 \in (0, 1]$ be such that $(x, y)_{r_1}\, q\lambda$ and $(y, z)_{r_2}\, q\lambda$. Then $\lambda(x, y) + r_1 > 1$ and $\lambda(y, z) + r_2 > 1$, so $(x, y), (y, z) \in R$. Since R is transitive, so $(x, z) \in R$. Thus $\lambda(x, z) \geq 0.5$. If $\min\{r_1, r_2\} \leq 0.5$, then $\lambda(x, z) \geq 0.5 \geq \min\{r_1, r_2\}$. This implies $(x, z)_{\min\{r_1,r_2\}} \in \lambda$. If $\min\{r_1, r_2\} > 0.5$ then $\lambda(x, z) + \min\{r_1, r_2\} > 0.5 + 0.5 = 1$. Thus $(x, z)_{\min\{r_1,r_2\}} q\lambda$. Hence, $(x, z)_{\min\{r_1,r_2\}} \in \vee q\lambda$.

(c) Let $x, y, z \in X$ and $r_1, r_2 \in (0, 1]$ be such that $(x, y)_{r_1} \in \lambda$ and $(y, z)_{r_2}\, q\lambda$. Then $\lambda(x, y) \geq r_1$ and $\lambda(y, z) + r_2 > 1$. Thus, $(x, y), (y, z) \in R$ and so $(x, z) \in R$, that is $\lambda(x, z) \geq 0.5$. Analogously as in (a) and (b), we obtain $(x, z)_{\min\{r_1,r_2\}} \in \lambda$ for $\min\{r_1, r_2\} \leq 0.5$, and $(x, z)_{\min\{r_1,r_2\}} q\lambda$ for $\min\{r_1, r_2\} > 0.5$. Thus $(x, z)_{\min\{r_1,r_2\}} \in \vee q\lambda$. □

3.19. Definition. A fuzzy relation λ on a set X is an (α, β)-fuzzy equivalence relation if

(1) λ is an (α, β)-fuzzy reflexive relation.
(2) λ is an (α, β)-fuzzy symmetric relation.
(3) λ is an (α, β)-fuzzy transitive relation.

3.20. Theorem. *A fuzzy relation λ on a set X is a G-fuzzy equivalence relation if and only if λ is an $(\in, \in)$-fuzzy equivalence relation.*

Proof. Follows from Lemmas 3.4, 3.10 and 3.15. □

3.21. Theorem. *A fuzzy relation λ on a set X is an $(\in, \in\vee q)$-fuzzy equivalence relation if and only if it satisfies the following conditions for all $t, x, y, z \in X$:*

(1) $\lambda(t, t) \geq \min\{\lambda(x, y), 0.5\}$.
(2) $\lambda(y, x) \geq \min\{\lambda(x, y), 0.5\}$.
(3) $\lambda(x, z) \geq \min\{\lambda(x, y), \lambda(y, z), 0.5\}$.

Proof. Follows from Proposition 3.6, 3.11 and 3.16. □

3.22. Theorem. *Let λ be an (α, β)-fuzzy equivalence relation on a set X. Then the support λ_0 of λ is an equivalence relation on X.*

Proof. Follows from Propositions 3.7, 3.12 and 3.17. □

3.23. Theorem. *If R is an equivalence relation on a set X, then a fuzzy subset λ of $X \times X$ satisfying* $\lambda(x,y) \begin{cases} \geq 0.5 & \text{for } (x,y) \in R \\ = 0 & \text{otherwise} \end{cases}$ *is an $(\in, \in\vee q)$-fuzzy equivalence relation on X.*

Proof. Follows from Lemmas 3.8, 3.13 and 3.18. □

Each fuzzy equivalence relation λ on X can be characterized by its level equivalence relations, that is by relations of the form $U(\lambda, r) = \{(x,y) \in X \times X : \lambda(x,y) \geq r\}$, where $r \in [0,1]$.

3.24. Theorem. *A fuzzy relation λ on X is an $(\in, \in\vee q)$-fuzzy equivalence relation on X if and only if $U(\lambda, r) \neq \emptyset$ is an equivalence relation on X for all $r \in (0, 0.5]$.*

Proof. Let λ be an $(\in, \in\vee q)$-fuzzy equivalence relation on X, and let $(x,y) \in U(\lambda, r)$ for some $0 < r \leq 0.5$, then $\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\} \geq \min\{r, 0.5\} = r$, which implies $(t,t) \in U(\lambda, r)$ for all $t \in X$. Hence $U(\lambda, r)$ is a reflexive relation.

With similar arguments it can be shown that $U(\lambda, r)$ is symmetric and transitive. Consequently, $U(\lambda, r)$ is an equivalence relation on X.

Conversely, let for every $r \in (0, 0.5]$, each non-empty $U(\lambda, r)$ be an equivalence relation on X. We claim

$$\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\} \text{ for all } t \in X.$$

If not then there exist some $x, x_0, y_0 \in X$ such that

$$\lambda(x,x) < \min\{\lambda(x_0, y_0), 0.5\}.$$

In such a situation we can find $r \in (0, 0.5]$, such that

$$\lambda(x,x) < r < \min\{\lambda(x_0, y_0), 0.5\}.$$

This implies $(x_0, y_0) \in U(\lambda, r)$ but $(x,x) \notin U(\lambda, r)$, which is a contradiction to our hypothesis. Therefore we have

$$\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\} \text{ for all } t \in X.$$

This shows that λ is an $(\in, \in\vee q)$-fuzzy reflexive relation on X. Similarly it can be shown that λ is an $(\in, \in\vee q)$-fuzzy symmetric and transitive relation on X. □

For any fuzzy relation λ on X and any $r \in (0,1]$ we consider two relations

$$Q(\lambda, r) = \left\{(x,y) \in X \times X : (x,y)_r \, q\lambda\right\}$$

and

$$[\lambda]_r = \left\{(x,y) \in X \times X : (x,y)_r \in \vee q\lambda\right\}.$$

It is clear that $[\lambda]_r = U(\lambda, r) \cup Q(\lambda, r)$.

In Theorem 3.24 it is shown that a fuzzy relation λ on X is an $(\in, \in\vee q)$-fuzzy equivalence relation on X if and only if $U(\lambda, r) \neq \emptyset$ is an equivalence relation on X for all $0 < r \leq 0.5$. Now similar types of result are shown for $Q(\lambda, r)$ and $[\lambda]_r$.

3.25. Theorem. *A fuzzy relation λ on X is an $(\in, \in\vee q)$-fuzzy equivalence relation on X if and only if $Q(\lambda, r) \neq \emptyset$ is an equivalence relation on X for all $r \in (0.5, 1]$.*

Proof. Similar to Theorem 3.24. □

3.26. Theorem. *A fuzzy relation λ on X is an $(\in, \in\vee q)$-fuzzy equivalence relation on X if and only if $[\lambda]_r \neq \emptyset$ is an equivalence relation on X for all $r \in (0,1]$.*

Proof. Let λ be an $(\in, \in\vee q)$-fuzzy equivalence relation on X and $(x,y) \in [\lambda]_r$ for some $r \in (0,1]$. Then $\lambda(x,y) \geq r$ or $\lambda(x,y) + r > 1$.

If $\lambda(x,y) \geq r$ then, since λ is an $(\in, \in\vee q)$-fuzzy reflexive relation on X therefore

$$\begin{aligned}\lambda(t,t) &\geq \min\{\lambda(x,y), 0.5\}\\ &\geq \min\{r, 0.5\}\\ &= \begin{cases} 0.5 & \text{if } r > 0.5\\ r & \text{if } r \leq 0.5.\end{cases}\end{aligned}$$

Therefore $\lambda(t,t) + r > 1$ or $\lambda(t,t) \geq r$. So

$$(t,t) \in U(\lambda, r) \cup Q(\lambda, r) = [\lambda]_r.$$

If $\lambda(x,y) + r > 1$ and $r > 0.5$, then

$$\begin{aligned}\lambda(t,t) &\geq \min\{\lambda(x,y), 0.5\}\\ &= \begin{cases} 0.5 > 1 - r & \text{if } \lambda(x,y) \geq 0.5\\ \lambda(x,y) > 1 - r & \text{if } \lambda(x,y) < 0.5.\end{cases}\end{aligned}$$

Thus, $(t,t) \in U(\lambda, r) \cup Q(\lambda, r) = [\lambda]_r$ for all $t \in X$.

If $r \leq 0.5$ then

$$\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\} = 0.5 \geq r.$$

So, $(t,t) \in U(\lambda, r) \subseteq [\lambda]_r$ for all $t \in X$. Hence $[\lambda]_r$ is reflexive.

Similarly, it can be shown that $[\lambda]_r$ is symmetric.

For transitivity let (x,y), $(y,z) \in [\lambda]_r$. Then $\lambda(x,y) \geq r$, or $\lambda(x,y) + r > 1$ and $\lambda(y,z) \geq r$ or $\lambda(y,z) + r > 1$. Therefore we have the following four cases

(i) $\lambda(x,y) \geq r$ and $\lambda(y,z) \geq r$.
(ii) $\lambda(x,y) \geq r$ and $\lambda(y,z) + r > 1$.
(iii) $\lambda(x,y) + r > 1$ and $\lambda(y,z) \geq r$.
(iv) $\lambda(x,y) + r > 1$ and $\lambda(y,z) + r > 1$.

Now by Theorem 3.21, we have

$$\lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\}.$$

Now for Case (i) we have

$$\lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\} \geq \begin{cases} 0.5 & \text{if } r > 0.5,\\ r & \text{if } r \leq 0.5.\end{cases}$$

That is, $(x,z) \in U(\lambda, r) \cup Q(\lambda, r) = [\lambda]_r$.

For Case (ii), if we assume $r > 0.5$ then $1 - r < 0.5$.

If $\min\{\lambda(y,z), 0.5\} \leq \lambda(x,y)$, then $\lambda(x,z) \geq \min\{\lambda(y,z), 0.5\} > 1 - r$, so $(x,z) \in Q(\lambda, r)$. If $\min\{\lambda(y,z), 0.5\} \geq \lambda(x,y)$, then $\lambda(x,z) \geq \lambda(x,y)$, so $(x,z) \in U(\lambda, r)$. Therefore $(x,z) \in U(\lambda, r) \cup Q(\lambda, r) = [\lambda]_r$.

If $r \leq 0.5$ then $1 - r \geq 0.5$. Hence if

$$\min\{\lambda(x,y), 0.5\} \leq \lambda(y,z) \text{ then } \lambda(x,z) \geq \min\{\lambda(x,y), 0.5\} > r,$$

and if

$$\min\{\lambda(x,y), 0.5\} \geq \lambda(y,z) \text{ then } \lambda(x,z) \geq \lambda(y,z) > 1 - r,$$

so $(x,z) \in U(\lambda, r) \cup Q(\lambda, r) = [\lambda]_r$.

A similar result is obtained for Case (iii).

Now for Case (iv), if $r > 0.5$ then $1 - r < 0.5$. Hence

$$\lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\}$$
$$= \begin{cases} 0.5 > 1 - r & \text{if } \min\{\lambda(x,y), \lambda(y,z)\} > 0.5, \\ \min\{\lambda(x,y), \lambda(y,z)\} > 1 - r & \text{if } \min\{\lambda(x,y), \lambda(y,z)\} \leq 0.5. \end{cases}$$

Therefore, $(x,z) \in Q(\lambda, r) \subseteq [\lambda]_r$.

If $r \leq 0.5$ then $1 - r \geq 0.5$, thus

$$\lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\}$$
$$= \begin{cases} 0.5 > r & \text{if } \min\{\lambda(x,y), \lambda(y,z)\} \geq 0.5, \\ \min\{\lambda(x,y), \lambda(y,z)\} > 1 - r & \text{if } \min\{\lambda(x,y), \lambda(y,z)\} < 0.5. \end{cases}$$

This implies $(x,z) \in U(\lambda, r) \cup Q(\lambda, r) = [\lambda]_r$. Consequently, $[\lambda]_r$ is an equivalence relation on X.

Conversely, let λ be a fuzzy relation on X and $[\lambda]_r$ an equivalence relation on X for all $r \in (0,1]$. If

$$\lambda(t,t) < r < \min\{\lambda(x,y), 0.5\} \text{ for some } x, y, t \in X,$$

then $\lambda(x,y) \geq r$ or $\lambda(x,y) + 1 > r$, but $(t,t)_r \overline{\in \vee q} \lambda$, which is a contradiction, therefore we have $\lambda(t,t) \geq \min\{\lambda(x,y), 0.5\}$ for all $x, y, t \in X$.

Similarly it can be shown that

$$\lambda(y,x) \geq \min\{\lambda(x,y), 0.5\} \text{ and } \lambda(x,z) \geq \min\{\lambda(x,y), \lambda(y,z), 0.5\}.$$

Hence λ is an $(\in, \in\vee q)$-fuzzy equivalence relation on X. □

In this section so far we have studied ways to construct $(\in, \in\vee q)$-fuzzy equivalence relations. Now we study some of the basic properties of $(\in, \in\vee q)$-fuzzy equivalence relations.

3.27. Theorem. *If λ is an $(\in, \in\vee q)$-fuzzy equivalence relation on X, then $\lambda \circ \lambda$ is also an $(\in, \in\vee q)$-fuzzy equivalence relation on X.*

Proof. For any $x, y, z, t \in X$, consider

$$\begin{aligned} (\lambda \circ \lambda)(x,z) \wedge 0.5 &= \bigvee_{y \in X} \{\lambda(x,y) \wedge \lambda(y,z)\} \wedge 0.5 \\ &= \bigvee_{y \in X} \{\lambda(x,y) \wedge \lambda(y,z) \wedge 0.5\} \\ &= \bigvee_{y \in X} \{\lambda(x,y) \wedge 0.5 \wedge \lambda(y,z) \wedge 0.5\} \\ &\leq \bigvee_{t \in X} \{\lambda(t,t) \wedge \lambda(t,t)\} \\ &= (\lambda \circ \lambda)(t,t) \end{aligned}$$

This shows that $\lambda \circ \lambda$ is an $(\in, \in\vee q)$-fuzzy reflexive relation.

Now

$$\begin{aligned}(\lambda \circ \lambda)(x,y) \wedge 0.5 &= \bigvee_{z\in X} \{\lambda(x,z) \wedge \lambda(z,y)\} \wedge 0.5 \\ &= \bigvee_{z\in X} \{\lambda(x,z) \wedge \lambda(z,y) \wedge 0.5\} \\ &= \bigvee_{z\in X} \{\lambda(x,z) \wedge 0.5 \wedge \lambda(z,y) \wedge 0.5\} \\ &\leq \bigvee_{z\in X} \{\lambda(z,x) \wedge \lambda(y,z)\} \\ &= \bigvee_{z\in X} \{\lambda(y,z) \wedge \lambda(z,x)\} \\ &= (\lambda \circ \lambda)(y,x).\end{aligned}$$

Furthermore,

$$\begin{aligned}&(\lambda \circ \lambda)(x,y) \wedge (\lambda \circ \lambda)(y,z) \wedge 0.5 \\ &\quad= \bigvee_{y_1\in X} \{\lambda(x,y_1) \wedge \lambda(y_1,y)\} \wedge \bigvee_{y_2\in X} \{\lambda(y,y_2) \wedge \lambda(y_2,z)\} \wedge 0.5 \\ &\quad= \bigvee_{y_1,y_2\in X} \{\lambda(x,y_1) \wedge \lambda(y_1,y) \wedge \lambda(y,y_2) \wedge \lambda(y_2,z)\} \wedge 0.5 \\ &\quad= \bigvee_{y_1,y_2\in X} \{\lambda(x,y_1) \wedge \lambda(y_1,y) \wedge 0.5 \wedge \lambda(y,y_2) \wedge \lambda(y_2,z) \wedge 0.5\} \\ &\quad\leq \lambda(x,y) \wedge \lambda(y,z) \\ &\quad\leq \bigvee_{y\in X} \{\lambda(x,y) \wedge \lambda(y,z)\} \\ &\quad= (\lambda \circ \lambda)(x,z)\end{aligned}$$

This shows that $\lambda\circ\lambda$ is an $(\in, \in\vee q)$-fuzzy transitive relation. Hence, $\lambda\circ\lambda$ is an $(\in, \in\vee q)$-fuzzy equivalence relation. □

3.28. Theorem. *If $\{\lambda_i\}_{i\in I}$ is a family of $(\in, \in\vee q)$-fuzzy equivalence relations on X, then $\lambda = \bigcap_{i\in I} \{\lambda_i\}$ is also an $(\in, \in\vee q)$-fuzzy equivalence relation on X.*

Proof. For any $x, y, t \in X$ consider

$$\begin{aligned}\lambda(t,t) &= \left(\bigwedge_{i\in I} \lambda_i\right)(t,t) = \bigwedge_{i\in I} (\lambda_i(t,t)) \\ &\geq \bigwedge_{i\in I} (\lambda_i(x,y) \wedge 0.5) \\ &= \left(\bigwedge_{i\in I} \lambda_i(x,y)\right) \wedge 0.5.\end{aligned}$$

This shows that λ is $(\in, \in\vee q)$-fuzzy reflexive.

$$\begin{aligned}\lambda(y,x) &= \left(\bigwedge_{i\in I}\lambda_i\right)(y,x) = \bigwedge_{i\in I}\left(\lambda_i(y,x)\right)\\ &\geq \bigwedge_{i\in I}\left(\lambda_i(x,y)\wedge 0.5\right)\\ &= \left(\bigwedge_{i\in I}\lambda_i(x,y)\right)\wedge 0.5.\end{aligned}$$

That is, λ is $(\in, \in\vee q)$-fuzzy symmetric.

$$\begin{aligned}\lambda(x,z) &= \left(\bigwedge_{i\in I}\lambda_i\right)(x,z) = \bigwedge_{i\in I}\left(\lambda_i(x,z)\right)\\ &\geq \bigwedge_{i\in I}\left(\lambda_i(x,y)\wedge\lambda_i(y,z)\wedge 0.5\right)\\ &= \left(\bigwedge_{i\in I}\lambda_i(x,y)\wedge\lambda_i(y,z)\right)\wedge 0.5.\end{aligned}$$

So λ is $(\in, \in\vee q)$-fuzzy transitive. Hence $\lambda = \bigcap_{i\in I}\{\lambda_i\}$ is an $(\in, \in\vee q)$-fuzzy equivalence relation on X. □

4. $(\in, \in\vee q)$-fuzzy partitions

In [16] Gupta and Gupta defined the partition of a G-equivalence relation, here we extend this concept to $(\in, \in\vee q)$-fuzzy equivalence relations.

4.1. Definition. [16] Let λ be a non zero fuzzy subset of X. A family $\{\lambda_i\}$ of fuzzy subsets of X is a fuzzy partition of λ if

(1) λ_i and λ_j are disjoint, whenever $i \neq j$ and
(2) $\bigcup_i \lambda_i = \lambda$.

Let μ be an $(\in, \in\vee q)$-fuzzy equivalence relation on X. Then by Theorem 3.22 we know that μ_0, the support of μ, is an equivalence relation X, which give rise to partition of X into its equivalence classes. Let $[a]$ denotes the equivalence class containing an element a of X. For $a \in X$, define the fuzzy relation σ_a in X as follows

$$\sigma_a(x,y) = \begin{cases}\mu(x,y) & \text{if } (x,y)\in[a]\times[a]\\ 0 & \text{otherwise}\end{cases}$$

We may call $\{\sigma_a : a \in X\}$ the $(\in, \in\vee q)$ *decomposition of* μ.

4.2. Theorem. *The* $(\in, \in\vee q)$ *decomposition of an* $(\in, \in\vee q)$*-fuzzy equivalence relation* μ *is a partition of* μ.

Proof. Similar to [16, Theorem 4.2] for G-equivalence relations. □

It is interesting to see that if we have a fuzzy subset λ of X such that $\operatorname{supp}(\lambda) = X$, and if $\{\lambda_i\}$ is a partition of λ, then we can construct an $(\in, \in\vee q)$-fuzzy equivalence relation on X in the following manner.

4.3. Theorem. *If μ is a fuzzy subset defined on $X \times X$ such that*

$$\mu(x,x) = \lambda_{i_x}(x) \text{ and}$$
$$\mu(x,y) = \alpha \text{ for all } x \neq y,$$

where $x, y \in X$, $\alpha \in [0, \delta]$ is a fixed number and $\delta = \inf_{x \in X} \lambda_{i_x}(x)$, then μ is an $(\in, \in\vee q)$-fuzzy equivalence relation on X.

Proof. For any $x, y, t \in X$, we have

$$\mu(t,t) \geq \delta \geq \alpha \wedge 0.5 = \mu(x,y) \wedge 0.5,$$

which shows that μ is $(\in, \in\vee q)$-fuzzy reflexive.

Further, $\mu(y,x) \geq \alpha \wedge 0.5 = \mu(x,y) \wedge 0.5$, which shows that μ is $(\in, \in\vee q)$-fuzzy symmetric.

Now $\mu(x,z) \geq \alpha \wedge \alpha \wedge 0.5 = \mu(x,y) \wedge \mu(y,z) \wedge 0.5$. This shows that μ is $(\in, \in\vee q)$-fuzzy transitive.

Hence μ is an $(\in, \in\vee q)$-fuzzy equivalence relation on X. □

5. Image and pre-image of $(\in, \in\vee q)$-fuzzy equivalence relations

In [16, 17] it has been shown that the concept of G-equivalence is preserved by the image and preimage under a balanced mapping. In this section this result is generalized to $(\in, \in\vee q)$-fuzzy fuzzy equivalence relations.

5.1. Definition. [17] Let X and Y be two non empty sets. A mapping $f : X \times X \to Y \times Y$ is called a semibalanced mapping if

(1) Given $a \in X$, there exists $u \in Y$ such that $f(a,a) = (u,u)$;
(2) $f(a,a) = (u,u)$ and $f(b,b) = (v,v)$ implies $f(b,a) = (v,u)$, where $a, b \in X$ and $u, v \in Y$.

In [16] it is shown that if f is a balanced mapping from $X \times X$ to $Y \times Y$, and μ is a fuzzy G-equivalence relation on Y. Then $f^{-1}(\mu)$ is a G-equivalence relation on X. In the following we see that in case of an $(\in, \in\vee q)$-fuzzy equivalence relation for a semibalanced mapping f the inverse image $f^{-1}(\mu)$ is an $(\in, \in\vee q)$-fuzzy equivalence relation.

5.2. Theorem. *If f is a semibalanced map from $X \times X$ to $Y \times Y$ and μ is an $(\in, \in\vee q)$-fuzzy equivalence relation on Y, then $f^{-1}(\mu)$ is an $(\in, \in\vee q)$-fuzzy equivalence relation on X.*

Proof. Let $a, b, c \in X$, $u, v, w \in Y$. Then

$$\begin{aligned} f^{-1}(\mu)(c,c) &= \mu(f(c,c)) = \mu(w,w) \geq \mu(u,v) \wedge 0.5 \\ &= \mu(f(a,b)) \wedge 0.5 = f^{-1}(\mu)(a,b) \wedge 0.5. \end{aligned}$$

Thus $f^{-1}(\mu)$ is $(\in, \in\vee q)$-fuzzy reflexive.

Now

$$\begin{aligned} f^{-1}(\mu)(b,a) &= \mu(f(b,a)) = \mu(v,u) \\ &\geq \mu(u,v) \wedge 0.5 \\ &= \mu(f(a,b)) \wedge 0.5 = f^{-1}(\mu)(a,b) \wedge 0.5. \end{aligned}$$

That is $f^{-1}(\mu)$ is $(\in, \in\vee q)$-fuzzy symmetric.

Further,

$$\begin{aligned} f^{-1}(\mu)(a,c) &= \mu(f(a,c)) \\ &= \mu(u,w) \\ &\geq \mu(u,v) \wedge \mu(v,w) \wedge 0.5 \\ &= \mu(f(a,b)) \wedge \mu(f(b,c)) \wedge 0.5 \\ &= f^{-1}(\mu)(a,b) \wedge f^{-1}(\mu)(b,c) \wedge 0.5. \end{aligned}$$

That is $f^{-1}(\mu)$ is an $(\in, \in\vee q)$-fuzzy transitive relation.

Hence $f^{-1}(\mu)$ is an $(\in, \in\vee q)$-fuzzy equivalence relation on X. □

5.3. Definition. [17] A mapping $f : X \times X \to Y \times Y$ is a *balanced mapping* if

(i) $f(a,b) = (u,u) \implies a = b$

(ii) $f(a,b) = (u,v) \implies f(b,a) = (v,u)$,

(iii) $f(a,a) = (u,u)$ and $f(b,b) = \mu(v,v) \implies f(a,b) = (u,v)$ for all $a, b \in X$, $u, v \in Y$.

5.4. Remark. [17] A mapping $f : X \times X \to Y \times Y$ is a balanced mapping if and only if it is one-to-one semibalanced mapping.

5.5. Theorem. *Let f be balanced mapping from $X \times X$ onto $Y \times Y$. If λ is an $(\in, \in\vee q)$-fuzzy equivalence relation on X then $f(\lambda)$ is an $(\in, \in\vee q)$-fuzzy equivalence relation on Y.*

Proof. Since f is one-to-one and onto, therefore

$$f(\lambda)(u,v) = \sup_{(c,d)\in f^{-1}(u,v)} \lambda(c,d) = \lambda(a,b).$$

Now

$$f(\lambda)(w,w) = \lambda(c,c) \geq \lambda(a,b) \wedge 0.5 = f(\lambda)(u,v) \wedge 0.5,$$

that is $f(\lambda)$ is $(\in, \in\vee q)$-fuzzy reflexive relation.

Moreover

$$f(\lambda)(v,u) = \lambda(b,a) \geq \lambda(a,b) \wedge 0.5 = f(\lambda)(u,v) \wedge 0.5,$$

thus $f(\lambda)$ is $(\in, \in\vee q)$-fuzzy symmetric.

Finally,

$$\begin{aligned} f(\lambda)(u,w) = \lambda(a,c) &\geq \lambda(a,b) \wedge \lambda(b,c) \wedge 0.5 \\ &= f(\lambda)(u,v) \wedge f(\lambda)(v,w) \wedge 0.5. \end{aligned}$$

That is, $f(\lambda)$ is $(\in, \in\vee q)$-fuzzy transitive.

Hence $f(\lambda)$ is an $(\in, \in\vee q)$-fuzzy equivalence relation on Y. □

6. $(\in, \in\vee q)$-fuzzy indistinguishability operators

In this section we introduce the concept of $(\in, \in\vee q)$-fuzzy indistinguishability operators.

6.1. Definition. Let X be a non empty set and T a continuous t-norm on $[0,1]$. A mapping $\lambda : X \times X \to [01]$ is called an $(\in, \in\vee q)$-*fuzzy indistinguishability operator* if

(1) λ is $(\in, \in\vee q)$-fuzzy reflexive.

(2) λ is $(\in, \in\vee q)$-fuzzy symmetric.

(3) λ is T-transitive. That is, $\lambda(x,z) \geq T(\lambda(x,y), \lambda(y,z))$.

6.2. Lemma. *Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in, \in\vee q)$-fuzzy indistinguishability operators. Then $\lambda = \bigcap_{i\in I} \{\lambda_i\}$ is an $(\in, \in\vee q)$-fuzzy indistinguishability operator.*

Proof. For any $x, y, t \in X$ we have

$$\begin{aligned}\lambda(t,t) &= \inf_{i\in I} \lambda_i(t,t) \geq \inf_{i\in I} \{\lambda_i(x,y) \wedge 0.5\} \\ &= \left\{\inf_{i\in I} \lambda_i(x,y) \wedge 0.5\right\} = \lambda(x,y) \wedge 0.5.\end{aligned}$$

This shows that λ is $(\in, \in\vee q)$-fuzzy reflexive.

Now

$$\begin{aligned}\lambda(y,x) &= \inf_{i\in I} \lambda_i(y,x) \geq \inf_{i\in I} \{\lambda_i(x,y) \wedge 0.5\} \\ &= \left\{\inf_{i\in I} \lambda_i(x,y) \wedge 0.5\right\} = \lambda(x,y) \wedge 0.5.\end{aligned}$$

This shows that λ is $(\in, \in\vee q)$-fuzzy symmetric. The T-transitivity of λ is shown in [34, Proposition 3.1].

Hence λ is an $(\in, \in\vee q)$-fuzzy indistinguishability operator. □

6.3. Theorem. *Let μ be an $(\in, \in\vee q)$-fuzzy reflexive and $(\in, \in\vee q)$-fuzzy transitive relation on X. Then T_μ, the set of all $(\in, \in\vee q)$-fuzzy indistinguishability operators λ on X such that $\mu \subseteq \lambda$ is non empty and*

$$\mu(x,y) = \inf_{\lambda\in T_\mu} \lambda(x,y) \text{ for all } x, y \in X$$

is the least $(\in, \in\vee q)$-fuzzy indistinguishability operator on X containing μ.

Proof. The fuzzy relation $1(x,y) = 1$ for all $x, y \in X$, is the greatest $(\in, \in\vee q)$-fuzzy indistinguishability operator on X containing μ, hence T_μ is non empty. The remaining part of the proof is clear from Lemma 6.2. □

7. Conclusion

In this study we have introduced $(\in, \in\vee q)$-fuzzy equivalence relations and investigated some of their properties. We have also discussed the partition of an $(\in, \in\vee q)$-fuzzy equivalence relation. We have shown that under a semibalanced mapping the preimage of an $(\in, \in\vee q)$-fuzzy equivalence relation is an $(\in, \in\vee q)$-fuzzy equivalence relation, whereas the image of an $(\in, \in\vee q)$-fuzzy equivalence relation under a balanced mapping is an $(\in, \in\vee q)$-fuzzy equivalence relation. Furthermore, $(\in, \in\vee q)$-fuzzy indistinguishability relations were studied.

As future work, the application of the newly defined notions to the study of some algebraic structures (such as semigroups and semirings) could be further explored.

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