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FINITELY COATOMIC MODULES

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Abstract

In this paper, we introduce finitely coatomic modules. We prove that any direct sum of finitely coatomic modules is finitely coatomic. Also we prove, if M is a module with finitely generated Jacobson radical, then M is f-semilocal if and only if M is finitely weakly supplemented and f-coatomic.

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1. Introduction

Throughout the text, R will be an associative ring with unity and all modules will be unitary left R-modules. $\operatorname{Rad} M$ will denote the Jacobson radical of M.

In [6], Zöschinger defined and investigated coatomic modules over commutative noetherian rings. Also Harmancı and Güngöroğlu (see [2] and [3]) obtained more results on coatomic modules. A module M is called coatomic, if every proper submodule of M is contained in a maximal submodule of M; or equivalently, for a submodule N of M, whenever $\operatorname{Rad}(\frac{M}{N}) = \frac{M}{N}$ then $M = N$. Finitely generated and semisimple modules are coatomic.

In this paper, we will define and study finitely coatomic modules and give some properties and also some relationships with supplemented modules.

We say that a module M is finitely coatomic (or simply f-coatomic) if every proper finitely generated submodule of M is contained in a maximal submodule of M. As for coatomic modules, it is not difficult to see that, M is f-coatomic if and only if for a finitely generated submodule L of M, $\operatorname{Rad}(\frac{M}{L}) = \frac{M}{L}$ implies $M = L$.

We now give an example of a module which is f-coatomic but not coatomic.

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1.1. Example. Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of positive integers, integers and rational integers respectively. Let us consider the $\mathbb{Z}$-module $K = \bigoplus_{n\in\mathbb{N}} M_n$, where $M_n = \mathbb{Z}$ for all $n \in \mathbb{N}$. Since there is an epimorphism from K to $\mathbb{Q}$, and $\mathbb{Q}$ has no maximal submodules, K is not coatomic, as shown in [3, Example 2.8]. Now let L be a finitely generated submodule of K. Then $L \subseteq \bigoplus_{i\in J} M_i$ for some finite subset J of $\mathbb{N}$. Thus L is contained in a maximal submodule P of $\bigoplus_{i\in J} M_i$ because $\mathbb{Z}$ is coatomic and finite direct sums of coatomic modules are coatomic by [2, Corollary 5]. Then $P \oplus (\bigoplus_{i\in\mathbb{N}-J} M_i)$ is a maximal submodule of M containing L. Therefore K is f-coatomic.

2. Properties of f-coatomic modules

2.1. Proposition. *Any extension of a f-coatomic module by a small submodule is f-coatomic.*

Proof. Suppose that $\frac{M}{N}$ is f-coatomic, where N is a small submodule of M. Let K be a proper, finitely generated submodule of M. If $N \subseteq K$ then because of the assumption, $\frac{K}{N}$ and hence K is contained in a maximal submodule of M. If not, consider the factor module $\frac{K+N}{N}$. It is finitely generated and a proper submodule of $\frac{M}{N}$ since N is small in M. Then by assumption, $\frac{K+N}{N}$ is contained in a maximal submodule of $\frac{M}{N}$. Thus $K+N$ and hence K is contained in a maximal submodule of M. Thus, M is f-coatomic. □

Let us note that, not every extension of a f-coatomic module by an arbitrary submodule is f-coatomic. Here is an example:

2.2. Example. We consider the $\mathbb{Z}$-module K of Example 1.1. Let $M = K \oplus \mathbb{Z}(p^\infty)$ where $\mathbb{Z}(p^\infty)$ is the Prüfer p-group for any prime p. Then M is still f-coatomic, because any proper finitely generated submodule of M can be embedded in a submodule of M of the form, $T \oplus Z(p^\infty)$, where T is a finitely generated submodule of K, as was done in Example 1. But the factor module $\frac{M}{K}$ is not f-coatomic.

2.3. Proposition. *Let M be an f-coatomic module and N be a submodule of M. The factor module $\frac{M}{N}$ is f-coatomic if $N \subseteq \operatorname{Rad}(M)$ or N is finitely generated.*

Proof. Let M be an f-coatomic module and let N be a submodule of M such that $N \subseteq \operatorname{Rad}(M)$. Let us take any finitely generated proper submodule $\frac{K}{N}$ of $\frac{M}{N}$. Then $Rk_1 + Rk_2 + \cdots + Rk_s + N = K$ for some $s \in \mathbb{Z}^+$ and $k_1, \ldots, k_s \in K$. By assumption, $Rk_1 + Rk_2 + \cdots + Rk_s \subseteq L$ for some maximal submodule L of M. Since $N \subseteq \operatorname{Rad}(M) \subseteq L$, we also have $K \subseteq L$ and hence $\frac{K}{N} \subseteq \frac{L}{N}$, where $\frac{L}{N}$ is a maximal submodule of $\frac{M}{N}$.

This time, let N be a finitely generated submodule of M. Consider the factor module $\frac{M}{N}$. Let $\frac{K}{N}$ be a finitely generated proper submodule of $\frac{M}{N}$. Since N is finitely generated, K is also finitely generated. By assumption, $K \subseteq M_0$ for a maximal submodule M_0 of M. Hence $\frac{M_0}{N}$ is a suitable maximal submodule. □

By Proposition 2.1 and Proposition 2.3, we easily get the following result:

2.4. Corollary. *Let M be a module and N a small submodule of M. Then M is f-coatomic if and only if $\frac{M}{N}$ is f-coatomic.*

The module K of Example 1.1 shows that not every infinite direct sum of coatomic modules is coatomic, although it is true for a finite direct sum [2, Corollary 5]. We will prove that any direct sum of f-coatomic modules is f-coatomic over a noetherian ring. But first we need some Lemmas.

2.5. Lemma. *Let R be a noetherian ring and M an R-module. Let N be a submodule of M such that N and $\frac{M}{N}$ are f-coatomic. Then M is also f-coatomic.*

Proof. Let X be a proper, finitely generated submodule of M. If $X+N$ is a proper submodule of M, then $\frac{X+N}{N}$ is finitely generated and by assumption, there exists a maximal submodule $\frac{M_0}{N}$ of $\frac{M}{N}$ containing $\frac{X+N}{N}$. Then M_0 is a maximal submodule of M containing X.

If $X+N$ is not a proper submodule of M, then since $X \cap N$ is finitely generated and N is f-coatomic, there is a maximal submodule N_0 of N containing $X \cap N$. Now,

$$\frac{M}{X+N_0} = \frac{X+N}{X+N_0} \cong \frac{N}{N_0}$$

because $\frac{N}{N_0}$ is simple and there is an epimorphism from $\frac{N}{N_0}$ to $\frac{X+N}{X+N_0}$. Therefore $X+N_0$ is a maximal submodule containing X. □

In [2, Corollary 5], it is proved that any finite direct sum of coatomic modules is coatomic. We now show the analogue of this for f-coatomic modules, and then extend this to arbitrary direct sums:

2.6. Lemma. *Any finite direct sum of f-coatomic modules is f-coatomic.*

Proof. Clearly it is enough to prove that the direct sum of two f-coatomic modules is f-coatomic. Let M_1 and M_2 be f-coatomic modules and take $M = M_1 \bigoplus M_2$. If M is not f-coatomic, then there exists a finitely generated submodule N of M such that N is not contained in a maximal submodule of M. Note that $\frac{N+M_1}{M_1}$ is a finitely generated submodule of $\frac{M}{M_1} \cong M_2$. If $N+M_1 \neq M$, then $N+M_1$ and consequently N is contained in a maximal submodule of M, a contradiction. Therefore, $M = N + M_1$. Then

$$M_2 \cong \frac{M}{M_1} = \frac{N+M_1}{M_1} \cong \frac{N}{N \cap M_1}$$

is finitely generated. By a similar consideration, M_1 is finitely generated. Therefore M is finitely generated and hence f-coatomic, a contradiction. □

2.7. Lemma. *Any direct sum of f-coatomic modules is f-coatomic.*

Proof. Let $\{M_i\}_{i\in I}$ be any collection of f-coatomic modules over a ring R where I is any index set. Then let us consider $M = \bigoplus_{i\in I} M_i$. Let K be any proper finitely generated submodule of M, then $K \subseteq \bigoplus_{i\in F} M_i$, where F is a finite subset of I. By Lemma 2.6, there is a maximal submodule T of $\bigoplus_{i\in F} M_i$, containing K. Then clearly $T \bigoplus (\bigoplus_{i\in I-F} M_i)$ is a maximal submodule of M containing K. Hence M is f-coatomic. □

3. Relationships with other modules

A module M is called f-semisimple if every finitely generated submodule of M is a direct summand of M. For instance, a von Neumann regular ring when it is considered as an R-module over itself is f-semisimple.

3.1. Lemma. *Every submodule of a f-semisimple module is f-semisimple.*

Proof. Let M be a f-semisimple module and N a submodule of M. For a finitely generated submodule K of N, $M = K \bigoplus S$ for some submodule S of M. By using the modular law, we obtain $N = K \bigoplus (N \cap S)$. □

3.2. Lemma. *f-semisimple modules have zero radical.*

Proof. Let M be an f-semisimple module. Let us suppose there is a non-zero small submodule K of M. Take a non-zero element k of K and consider the cyclic submodule Rk of K. Since K is small in M, Rk is also small in M; but, by assumption, $M = Rk \bigoplus L$ for some proper submodule L of M, a contradiction. Therefore M has no non-zero small submodules. Hence $\operatorname{Rad} M = 0$. □

3.3. Proposition. *f-semisimple modules are f-coatomic.*

Proof. Let M be an f-semisimple module and N a finitely generated proper submodule of M. If $m \in M - N$, then $Rm + N$ is a direct summand of M and N is a proper submodule of $Rm + N$. Thus, N is contained in a maximal submodule of the finitely generated module $Rm + N$ and hence in a maximal submodule of M. □

The converse of Proposition 3.3 is not true in general, as the following example shows.

3.4. Example. For the module M of Example 2.2, M is f-coatomic, with $\operatorname{Rad}(M) = \mathbb{Z}(p^\infty)$. Thus $\operatorname{Rad}(M) \neq 0$ and so M is not f-semisimple by Lemma 3.2.

Let M be a module and U, V submodules of M. We say that V is a *weak supplement of U in M* if $M = U + V$ and $U \cap V \ll M$. Then M is called *weakly supplemented* if every submodule of M has a weak supplement in M, see [5, 6, 7]. More generally, M is called *finitely weakly supplemented* (simply, fws) if every finitely generated submodule of M has a weak supplement in M. These modules were defined and investigated in [1].

3.5. Lemma. *Let M be a module and N a finitely generated submodule of M. Then $\frac{M}{N}$ is f-semisimple if and only if, for every finitely generated submodule L of M, there exists a submodule K of M such that $M = L + K$ and $L \cap K \subseteq N$.*

Proof. First assume that $\frac{M}{N}$ is f-semisimple. Let L be a finitely generated submodule of M. Then $\frac{L+N}{N}$ is finitely generated. By assumption,

$$\frac{M}{N} = \frac{L+N}{N} \bigoplus \frac{K}{N}$$

for some $N \leq K \leq M$. Then $M = L + K$ and $L \cap K \subseteq N$.

Conversely, let $\frac{L}{N}$ be a finitely generated submodule of $\frac{M}{N}$. Since N is finitely generated, L is also finitely generated, and by assumption there is a submodule K of M such that $M = L + K$ and $L \cap K \subseteq N$. Since $L \cap (K + N) = L \cap K + N = N$, it follows that

$$\frac{L}{N} \bigoplus \frac{K+N}{N} = \frac{M}{N}.$$

□

In [4], Lomp defined a module M to be *semilocal* if $\frac{M}{\operatorname{Rad} M}$ is semisimple. We say M is *f-semilocal* if $\frac{M}{\operatorname{Rad} M}$ is f-semisimple.

3.6. Theorem. *Let M be a module with finitely generated radical. Then consider the following statements:*

(1) *M is f-semisimple.*
(2) *M is f-semilocal.*
(3) *M is fws and f-coatomic.*

Then (1) $\Longrightarrow$ (2), (1) $\Longrightarrow$ (3) *and* (2) $\Longleftrightarrow$ (3). *All three conditions are equivalent if* $\operatorname{Rad} M = 0$ *(for instance, if R is a V-ring)*

Proof. (1) $\Longrightarrow$ (2) Let $\sigma : M \to \frac{M}{\mathrm{Rad}\, M}$ be the canonical epimorphism. For any submodule X of M, we represent $\sigma(X)$ by $\overline{X}$. Let $\overline{K}$ be a finitely generated submodule of $\overline{M}$. Then K is finitely generated too and so by assumption, there is a submodule T of M such that $M = K \bigoplus T$. Then clearly $\overline{M} = \overline{K} \bigoplus \overline{T}$. Therefore M is f-semilocal.

(1) $\Longrightarrow$ (3) By definition, M is fws. Hence, M is f-coatomic by Proposition 3.3.

(2) $\Longleftrightarrow$ (3) Let M be f-semilocal. Then $\frac{M}{\mathrm{Rad}\, M}$ is f-semisimple and then fws, by [1, Corollary 3.6], M is fws. Now let N be a finitely generated submodule of M and let us suppose $\mathrm{Rad}\,(\frac{M}{N}) = \frac{M}{N}$. Since $\overline{N}$ is finitely generated, there exists a submodule K of M containing $\mathrm{Rad}\, M$ so that $\overline{M} = \overline{N} \bigoplus \overline{K}$. Then $M = N + K$ and $N \cap K \subseteq \mathrm{Rad}\, M$ implies

$$\frac{M}{N} = \mathrm{Rad}\,(\frac{M}{N}) \subseteq \mathrm{Rad}\,(\frac{K}{\mathrm{Rad}\, M}) = 0$$

because of the isomorphism $\frac{M}{N} \cong \frac{K}{N \cap K}$. Therefore M is f-coatomic.

Conversely, for every finitely generated submodule L of M, there is a weak supplement K of M such that $L + K = M$ and $L \cap K \subseteq \mathrm{Rad}\, M$. Then the result follows by Lemma 3.5. □

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