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Research Article

Bifurcation analysis and chaos control of a discrete-time prey-predator model with Allee effect

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Abstract

In this study, a discrete-time prey-predator model based on the Allee effect is presented. We examine the parametric conditions for the local asymptotic stability of the fixed points of this model. Furthermore, with the use of the center manifold theorem and bifurcation theory, we analyze the existence and directions of period-doubling and Neimark-Sacker bifurcations. The plots of maximum Lyapunov exponents provide indications of complexity and chaotic behavior. The feedback control approach is presented to stabilize the unstable fixed point. Numerical simulations are performed to support the theoretical results.

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1. Introduction

Analysis of nonlinear mathematical models based on biological assumptions for interacting species provides insight into species behavior. Such models have different behavior patterns depending on their parameter values and typical properties. These models can exhibit stable or unstable behaviors as well as complex behaviors that have a special interest. In this case, one way to analyze such models is to examine the behavior of the system around the fixed point. In general, since complex dynamics result from bifurcation of fixed points, identifying parameters that make significant changes in equilibrium is an important consideration in the analysis of population dynamics. The number of parameters that must be changed in a system determines the codimensional bifurcation size. Systems can produce codimension-1 bifurcations such as period doubling and Neimark-Sacker; codimension-2 bifurcations such as resonance 1:1, 1:2, 1:3 and 1:4; and higher order codimension bifurcations [40–43, 45].

The Lotka-Volterra system is typically used to describe the dynamics of biological systems. The Lotka-Volterra system, also known as the prey-predator system, consists of a pair of first-order differential equations between two species, one of which acts as prey and the other as a predator. It was suggested for the first time by Lotka in the United States in 1925 [26] and then by Volterra in Italy in 1926 [38]. It is known that the generations of many species do not overlap. Therefore their populations change in discrete time periods.

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Such population dynamics are defined by discrete-time systems. The analyses of these systems which contain unpredictable and rich behaviors are important from both mathematical and biological points of view [1,5,9–12,15–18,21,25,28,29,32–37]. The desired dynamic behaviors can be obtained by controlling the chaos and bifurcations that occur [23,31].

In behavioral prey-predator interactions, predators are often considered as sources of risk to which the prey reacts. Evolving competition from strong or weak predators affect predation. A prey attacked by a predator may take advantage of its distance, age, and physical condition to escape from the predator. In this case, the number of prey consumed by the predator will change based on these assumptions.

The incorporation of the Allee effect to prey and predator populations is one of the most fascinating and interesting area of ongoing mathematical biology studies. This effect, which is the most basic phenomenon in the biological world, has been considered as an extremely important factor in ecology and population dynamics. The Allee effect was first described in 1930 by famous ecologist Allee [2]. In population dynamics, the per capita population growth rate and the population density have a positive correlation when the population density is very small [2,3]. If the population density is very small, then the population is heavily influenced by the Allee factor, and the effect of the Allee factor gradually decreases as the population density increases. Various natural species exhibit the Allee effect, including plants, insects, marine invertebrates, birds, and mammals. Recently, the effects of the Allee factor on system or equation dynamics have been studied more intensively [6,13,14,19,20,22,24,30,39,44]. Small numerical changes in population can cause large changes in the dynamics of such models. Considering this factor, it is possible to offer more realistic approaches.

Liu and Xiao [25] consider the following discrete-time prey-predator system

$$\begin{aligned} x_{n+1} &= x_n + \delta[rx_n(1-x_n) - bx_ny_n] \\ y_{n+1} &= y_n + \delta[bx_ny_n - dy_n] \end{aligned} \tag{1.1}$$

where δ is the step size.

In this article, our aim is to discuss the dynamics of our model developed by incorporating the Allee effect into the prey population of the discrete-time predator-prey model [25], and this model is given by

$$\begin{aligned} x_{n+1} &= x_n + \delta[rx_n(1-x_n) - bx_ny_n\left(\frac{x_n}{x_n+c}\right)] \\ y_{n+1} &= y_n + \delta[bx_ny_n - dy_n] \end{aligned} \tag{1.2}$$

where x_n and y_n denote the numbers of prey and predators in year (generation) n, respectively, and the parameters r, b, d, c, δ are all positive parameters. In this model, $\frac{x_n}{x_n+c}$ is the Allee effect, and c is the Allee constant such that $0 < c < 1$. Here, r is the per capita growth rate of the prey population which has a logistic growth rate; d is the per capita mortality rate of predators, and b is the predator rate.

The remainder of this article can be summed up as follows: Section 2 discusses the existence and stability of biologically possible fixed points. The bifurcation analysis and chaos control studies of system (1.2) are included in Section 3. In Section 4, significant numerical simulations are created to support our analytical findings.

2. The existence and stability of fixed points of the system (1.2)

The existence and stability analyses of the fixed points of system (1.2) are presented in this section. The modules of the eigenvalues of the characteristic equation of the Jacobian matrix evaluated at a fixed point determine the stability condition of that fixed point (see [8]).

It is clear that the fixed points of (1.2) provide the following equations:

$$\begin{aligned} x &= x + \delta[rx(1-x) - bxy\left(\frac{x}{x+c}\right)] \\ y &= y + \delta[bxy - dy]. \end{aligned}$$

When we examine the existence of all available fixed points of system (1.2), it is straightforward to obtain the following Lemma.

Lemma 2.1. **(i)** *For all positive parameter values, system (1.2) has two fixed points, $O(0,0)$ and $A(1,0)$;*

(ii) *If $b > d$, then the system (1.2) has also a unique positive fixed point, $B(x^*, y^*) = (\frac{b}{d}, \frac{r(b-d)(bc+d)}{b^2 d})$.*

Now, we study the stability of these fixed points by using Jacobian matrix of system (1.2) as follows:

$$J_{(x,y)} = \begin{pmatrix} 1 - \frac{bx(2c+x)y\delta}{(c+x)^2} + r\delta(1-2x) & -\frac{\delta b x^2}{(c+x)} \\ by\delta & 1 + (-d+bx)\delta \end{pmatrix}. \tag{2.1}$$

For the fixed point $O(0,0)$, the roots of the corresponding characteristic equation are $\lambda_1 = 1 + r\delta$ and $\lambda_2 = 1 - d\delta$. Similarly, for the fixed point $A(1,0)$, the characteristic equation's roots are $\lambda_1 = 1 - r\delta$, $\lambda_2 = 1 + (b-d)\delta$. Note that we have the following Jacobian matrices

$$J_{(0,0)} = \begin{pmatrix} 1 + r\delta & 0 \\ 0 & 1 - d\delta \end{pmatrix}, \tag{2.2}$$

and

$$J_{(1,0)} = \begin{pmatrix} 1 - r\delta & -\frac{b\delta}{1+c} \\ 0 & 1 + (b-d)\delta \end{pmatrix}. \tag{2.3}$$

Hence, the following propositions express the local dynamics of the fixed points $O(0,0)$ and $A(1,0)$, respectively.

Proposition 2.2. *The fixed point $O(0,0)$ is a saddle point when $0 < \delta < \frac{2}{d}$, it is a source point when $\delta > \frac{2}{d}$, it is a non-hyperbolic point when $\delta = \frac{2}{d}$.*

Proposition 2.3. *The fixed point $A(1,0)$ is a sink point when ($\delta < \min\left\{\frac{2}{r}, \frac{2}{d-b}\right\}$ and $d > b$), it is a saddle point when ($d > b$ and ($\frac{2}{r} < \delta < \frac{2}{d-b}$ or $\frac{2}{d-b} < \delta < \frac{2}{r}$)) or ($d < b$ and $\delta < \frac{2}{r}$), it is a source point when ($b < d$ and $\delta > \max\left\{\frac{2}{r}, \frac{2}{d-b}\right\}$) or ($d < b$ and $\delta > \frac{2}{r}$) , and it is a non-hyperbolic point when $b = d$, $\delta = \frac{2}{r}$, $\delta = \frac{2}{d-b}$.*

The following proposition, which is simply demonstrated by the relationships between the quadratic equation's roots and coefficients (see [25]), is given to classify the fixed point $B(x^*, y^*)$ topologically.

Proposition 2.4. *Let $b > d$. For a unique positive fixed point $B(x^*, y^*)$ of system (1.2), the following statements are true:*

(i) *It is a sink point if one of the following conditions is satisfied:*

(i.1)

$$\begin{aligned} r &\geq \frac{4b(b-d)d(bc+d)^2}{(b^2c+d^2)^2} \text{ and } 0 < \delta < A_1, \\ A_1 &= \frac{b^2cr + d^2r - \sqrt{r(4bd(-b+d)(bc+d)^2 + (b^2c+d^2)^2 r)}}{(b-d)d(bc+d)r} \end{aligned}$$

(i.2)

$$0 < r < \frac{4b(b-d)d(bc+d)^2}{(b^2c+d^22)^2} \text{ and } 0 < \delta < \frac{1}{b-d} + \frac{1}{d} - \frac{1}{bc+d}.$$

(ii) *It is a source point if one of the following conditions is satisfied:*

(ii.1)

$$\begin{aligned} r &\geq \frac{4b(b-d)d(bc+d)^2}{(b^2c+d^2)^2} \text{ and } \delta > A_2 \\ A_2 &= \frac{b^2cr + d^2r + \sqrt{r(4bd(-b+d)(bc+d)^2 + (b^2c+d^2)^2r)}}{(b-d)d(bc+d)r}, \end{aligned}$$

such that $A_1 < A_2$.

(ii.2)

$$0 < r < \frac{4b(b-d)d(bc+d)^2}{(b^2c+d^2)^2} \text{ and } \delta > \frac{1}{b-d} + \frac{1}{d} - \frac{1}{bc+d}.$$

(iii) *It is a saddle point if the following condition is satisfied:*

$$r \geq \frac{4b(b-d)d(bc+d)^2}{(b^2c+d^2)^2} \text{ and } A_1 < \delta < A_2$$

(iv) *It is a non-hyperbolic point if one of the following conditions is satisfied:*

(iv.1)

$$r \geq \frac{4b(b-d)d(bc+d)^2}{(b^2c+d^2)^2} \text{and } \delta = A_1 \text{ or } \delta = A_2,$$

(iv.2)

$$0 < r < \frac{4b(b-d)d(bc+d)^2}{(b^2c+d^2)^2} \text{ and } \delta = \frac{1}{b-d} + \frac{1}{d} - \frac{1}{bc+d}.$$

3. Bifurcation analysis and chaos control

In this section, we focus on the flip and Neimark-Sacker bifurcation of the positive fixed point $B(x^*, y^*)$ by using the center manifold theorem and bifurcation theory that have been reported in previous studies [6,7,25]. For the analysis of the flip and Neimark-Sacker bifurcation of $B(x^*, y^*)$, we select δ as a bifurcation parameter.

3.1. Flip bifurcation

First, we investigate the flip (period-doubling) bifurcation of the positive fixed point $B(x^*, y^*)$ of system (1.2) with the variation of parameters in the small neighborhood of the following set:

$$FB_{B(x^*,y^*)} = \left\{ \begin{array}{c} (r, b, c, d, \delta) : \delta = \delta_1 = \frac{b^2cr+d^2r-\sqrt{r(4bd(-b+d)(bc+d)^2+(b^2c+d^2)^2r)}}{(b-d)d(bc+d)r} \\ b > d \text{ and } r > \frac{4b(b-d)d(bc+d)^2}{(b^2c+d^2)^2} \\ \delta \neq \frac{2b(bc+d)}{b^2c+d^2)r}, \ \delta \neq \frac{4b(bc+d)}{b^2c+d^2)r} \end{array} \right\}.$$

Clearly, there are two eigenvalues for the linearized system at the positive fixed point $B(x^*, y^*)$, one of which is -1 and the other is neither 1 nor -1. These conditions that cause the flip bifurcation occurring at the positive fixed point $B(x^*, y^*)$ are determined depending on the coefficients of the characteristic equation of the matrix (2.1) (see [25]).

For the parameters $(r_1, b_1, d_1, c_1, \delta_1)$, system (1.2) is expressed by

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + \delta_1[r_1x(1-x) - b_1xy\left(\frac{x}{x+c_1}\right)] \\ y + \delta_1[b_1xy - d_1y] \end{pmatrix}. \tag{3.1}$$

Assuming that δ^* is a small bifurcation parameter, the associated perturbed mapping of (3.1) is given as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + (\delta_1 + \delta^*)[r_1 x(1-x) - b_1 xy\left(\frac{x}{x+c_1}\right)] \\ y + (\delta_1 + \delta^*)[b_1 xy - d_1 y] \end{pmatrix} \tag{3.2}$$

such that $|\delta^*| \ll 1$.

Using the transformation $u = x - \frac{d}{b}$, $v = y - \frac{(b-d)(bc+d)r}{b^2 d}$, the fixed point $B(x^*, y^*)$ is shifted to the origin. So, map (3.2) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v,\delta^*) \\ g(u,v,\delta^*) \end{pmatrix} \tag{3.3}$$

where

$$\begin{aligned} f(u,v,\delta^*) &= a_{13}uv + a_{14}u^2 + a_{15}u^2 v + a_{16}u^3 \\ &\quad + b_0 u\delta^* + b_2 v\delta^* + b_3 uv\delta^* + b_4 u^2\delta^* + O\left((|x|+|y|+|\delta^*|)^4\right) \\ g(u,v,\delta^*) &= a_{23}uv + c_2 u\delta^* + c_3 uv\delta^* + O\left((|x|+|y|+|\delta^*|)^4\right) \end{aligned}$$

such that

$$\begin{aligned} a_{11} &= 1 + \left(\frac{x^{*2}y^* b_1}{(c_1+x^*)^2} - \frac{2x^* y^* b_1}{c_1+x^*} + r_1 - 2x^* r_1\right)\delta_1, \quad a_{12} = -\frac{x^{*2}\delta_1 b_1}{c_1+x^*} \\ a_{21} &= b_1\delta_1 y^*, \quad a_{22} = 1, \\ a_{13} &= \left(\frac{x^{*2}b_1}{(c_1+x^*)^2} - \frac{2x^* b_1}{c_1+x^*}\right)\delta_1, \ \ a_{14} = -\frac{\left[y^* b_1 c_1^2 + (x^*+c_1)^3 r_1\right]\delta_1}{(x^*+c_1)^3} \\ a_{15} &= -\frac{b_1 c_1^2 \delta_1}{(c_1+x^*)^3}, \quad a_{16} = \frac{b_1 c_1^2 \delta_1 y^*}{(c_1+x^*)^4}, \ b_0 = \frac{x^{*2}y^* b_1}{(c_1+x^*)^2} - \frac{2x^* y^* b_1}{c_1+x^*} + r_1 - 2x^* r_1 \\ b_2 &= -\frac{x^{*2}b_1}{c_1+x^*}, \quad b_3 = \frac{x^{*2}b_1}{(c_1+x^*)^2} - \frac{2x^* b_1}{c_1+x^*}, \quad b_4 = -\frac{\left[y^* b_1 c_1^2 + (x^*+c_1)^3 r_1\right]}{(x^*+c_1)^3} \\ c_2 &= b_1 y^*, \quad c_3 = b_1. \end{aligned} \tag{3.4}$$

We translate the coefficient matrix in map (3.3) into the normal form by using the translation as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

where

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}$$

is an invertible matrix. So, map (3.3) can be written as

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} f^*(u,v,\delta^*) \\ g^*(u,v,\delta^*) \end{pmatrix} \tag{3.5}$$

such that

$$\begin{aligned} f^*(u,v,\delta^*) &= \frac{a_{16}(\lambda_2 - a_{11})}{a_{12}(\lambda_2+1)}u^3 + \frac{a_{14}(\lambda_2 - a_{11})}{a_{12}(\lambda_2+1)}u^2 + \frac{a_{15}(\lambda_2 - a_{11})}{a_{12}(\lambda_2+1)}u^2 v \\ &\quad + \frac{(a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23})}{a_{12}(\lambda_2+1)}uv + \frac{(b_3(\lambda_2 - a_{11}) - a_{12}c_3)}{a_{12}(\lambda_2+1)}uv\delta^* \\ &\quad + \frac{(b_0(\lambda_2 - a_{11}) - a_{12}c_2)}{a_{12}(\lambda_2+1)}u\delta^* + \frac{b_2(\lambda_2 - a_{11})}{a_{12}(\lambda_2+1)}v\delta^* + \frac{b_4(\lambda_2 - a_{11})}{a_{12}(\lambda_2+1)}u^2\delta^* \\ &\quad + O\left((|u|+|v|+|\delta^*|)^4\right) \end{aligned}$$

$$\begin{aligned} g^*(u,v,\delta^*) &= \frac{a_{16}(1+a_{11})}{a_{12}(\lambda_2+1)}u^3 + \frac{a_{14}(1+a_{11})}{a_{12}(\lambda_2+1)}u^2 + \frac{a_{15}(1+a_{11})}{a_{12}(\lambda_2+1)}u^2v \\ &+ \frac{(a_{13}(1+a_{11})+a_{12}a_{23})}{a_{12}(\lambda_2+1)}uv + \frac{(b_3(1+a_{11})+a_{12}c_3)}{a_{12}(\lambda_2+1)}uv\delta^* \\ &+ \frac{(b_0(1+a_{11})+a_{12}c_2)}{a_{12}(\lambda_2+1)}u\delta^* + \frac{b_2(1+a_{11})}{a_{12}(\lambda_2+1)}v\delta^* + \frac{b_4(1+a_{11})}{a_{12}(\lambda_2+1)}u^2\delta^* \\ &+ O\left((|u|+|v|+|\delta^*|)^4\right), \end{aligned}$$

$$\begin{aligned} u &= a_{12}\left(\tilde{x}+\tilde{y}\right) \\ v &= \left(-1-a_{11}\right)\tilde{x} + \left(\lambda_2-a_{11}\right)\tilde{y}. \end{aligned}$$

Then, the center manifold $Wc\ (0,0,0)$ of (3.5) is found at the fixed point $(0,0)$ in a small region around $\delta^*=0$. A center manifold $Wc\ (0,0,0)$ is known to exist according to the center manifold theorem, and it can be approximately expressed as follows:

$$Wc\ (0,0,0) = \left\{(\tilde{x},\tilde{y},\delta^*) : \tilde{y} = a_0\delta^* + a_1\tilde{x}^2 + a_2\tilde{x}\delta^* + a_3\delta^{*2} + O\left((|\tilde{x}|+|\delta^*|)^3\right)\right\}$$

and

$$\begin{aligned} a_0 &= 0, \quad a_1 = \frac{(1+a_{11})\left((1+a_{11})a_{13} - a_{12}a_{14} + a_{12}a_{23}\right)}{-1+\lambda_2^2}, \\ a_2 &= \frac{-(1+a_{11})a_{12}b_0 + (1+a_{11})^2 b_2 - a_{12}^2 c_2}{(1+\lambda_2)^2 a_{12}}, \quad a_3 = 0. \end{aligned}$$

Accordingly, we focus on the map that is limited to the center manifold $Wc\ (0,0,0)$:

$$f : \tilde{x} \to -\tilde{x} + m_1\tilde{x}^2 + m_2\tilde{x}\delta^* + m_3\tilde{x}^2\delta^* + m_4\tilde{x}\delta^{*2} + m_5\tilde{x}^3 + O\left((|\tilde{x}|+|\delta^*|)^4\right) \tag{3.6}$$

where

$$\begin{aligned} m_1 &= \frac{1}{1+\lambda_2}\left\{a_{11}^2 a_{13} + a_{11}(a_{12}(-a_{14}+a_{23}) - a_{13}(-1+\lambda_2)) - a_{13}\lambda_2 + a_{12}(a_{23}+a_{14}\lambda_2)\right\} \\ m_2 &= \frac{1}{a_{12}(1+\lambda_2)}\left\{a_{11}^2 b_2 - a_{11}^2 c_2 + a_{11}(-a_{12}b_0 - b_2(-1+\lambda_2)) + (a_{12}b_0 - b_2\lambda_2)\lambda_2\right\} \\ m_3 &= \frac{1}{1+\lambda_2}\left\{\begin{array}{c} a_2\left[a_{13}(a_{11}-\lambda_2)(1+2a_{11}-\lambda_2)\right. \\ \left.+a_{12}(2a_{11}(-a_{14}+a_{23}) - a_{23}(-1+\lambda_2) + 2a_{14}\lambda_2)\right] \\ +a_{11}a_{12}(-b_4+c_3) + b_3(a_{11}^2 - a_{11}(-1+\lambda_2) - \lambda_2) + a_{12}(c_3+b_4\lambda_2) \end{array}\right\} \\ &+ \frac{1}{a_{12}(1+\lambda_2)}a_1\left[-a_{11}a_{12}b_0 - a_{11}^2 c_2 + a_{12}b_0\lambda_2 + b_2(a_{11}^2 - 2a_{11}\lambda_2 + \lambda_2^2)\right] \\ m_4 &= \frac{1}{1+\lambda_2}\left\{a_2\left[-a_{12}c_2 + b_0(-a_{11}+\lambda_2)\right]\right\} + \frac{a_2 b_2(a_{11}-\lambda_2)^2}{a_{12}(1+\lambda_2)} \\ m_5 &= \frac{a_1}{1+\lambda_2}\left\{\begin{array}{c} \left[a_{13}\left(2a_{11}^2 + a_{11}(1-3\lambda_2) + (-1+\lambda_2)\lambda_2\right)\right. \\ \left.+2a_{11}a_{12}(-a_{14}+a_{23}) + a_{12}\left(a_{23} + (2a_{14}-a_{23})\lambda_2\right)\right] \end{array}\right\} \\ &+ \frac{a_{12}\left[(1+a_{11})a_{15} - a_{12}a_{16}\right](a_{11}-\lambda_2)}{1+\lambda_2}. \end{aligned}$$

We get two nonzero real number values α_1 and α_2, which are necessary to express a flip bifurcation, as

$$\alpha_1 = \left(\frac{\partial^2 f}{\partial\tilde{x}\delta^*} + \frac{1}{2}\frac{\partial f}{\partial\delta^*}\frac{\partial^2 f}{\partial\tilde{x}^2}\right)|_{(0,0)} = m_2 + m_1 m_2,$$

and

$$\alpha_2 = \left(\frac{1}{6} \frac{\partial^3 f}{\partial \tilde{x}^3} + \left(\frac{1}{2} \frac{\partial^2 f}{\partial \tilde{x}^2} \right) \right) |_{(0,0)} = m_5 + m_1^2.$$

Consequently, the flip bifurcation of system (1.2) yields the following theorem.

Theorem 3.1. *If $\alpha_2 \neq 0$, then map (1.2) undergoes a flip bifurcation at the fixed point $B(x^*, y^*)$ when the parameter δ varies in the small neighborhood of δ_1. Moreover, if $\alpha_2 > 0$ (resp., $\alpha_2 < 0$), then the period-2 points that bifurcate from $B(x^*, y^*)$ are stable (resp., unstable).*

3.2. Neimark-Sacker bifurcation

Here, we examine the existence and direction of the Neimark-Sacker bifurcation of the system (1.2) for a unique positive fixed point $B(x^*, y^*)$. The Jacobian matrix evaluated at $B(x^*, y^*)$ has a pair of complex conjugate eigenvalues with a modulus of one provided that the following condition is met:

$$NSB_{B(x^*,y^*)} = \left\{ \begin{array}{c} (r, b, d, c, \delta) : 0 < r < \frac{(4b(b-d)d(bc+d)^2)}{(b^2c+d^2)^2} \text{ and} \\ \delta = \delta_2 = \frac{1}{b-d} + \frac{1}{d} - \frac{1}{bc+d}. \end{array} \right\}.$$

To investigate the Neimark-Sacker bifurcation at the positive fixed point $B(x^*, y^*)$ of system (1.2), we take δ as a bifurcation parameter. Next, we show that the variation of r, b, d, c and δ in the small neighborhood of δ_2 yields a Neimark–Sacker bifurcation. For $(r_2, b_2, d_2, c_2, \delta_2) \in NSB_{B(x^*,y^*)}$, system (1.2) can be denoted by the following map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + \delta_2[r_2x(1-x) - b_2xy\left(\frac{x}{x+c_2}\right)] \\ y + \delta_2[b_2xy - d_2y] \end{pmatrix}. \tag{3.7}$$

Now, we consider a perturbed mapping of (3.7) by selecting the bifurcation parameter $\widetilde{\delta}$ such that $\left|\widetilde{\delta}\right| \ll 1$ as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + (\delta_2 + \widetilde{\delta})[r_2x(1-x) - b_2xy\left(\frac{x}{x+c_2}\right)] \\ y + (\delta_2 + \widetilde{\delta})[b_2xy - d_2y] \end{pmatrix}. \tag{3.8}$$

Shifting the unique positive fixed point $B(x^*, y^*)$ to the origin, with the transformation $u = x - \frac{d_2}{b_2}$, $v = y - \frac{(b_2-d_2)(b_2c_2+d_2)r_2}{b_2^2 d_2}$, we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f^\circ(u,v) \\ g^\circ(u,v) \end{pmatrix}, \tag{3.9}$$

where

$$\begin{aligned} f^\circ(u,v) &= k_{13}uv + k_{14}u^2 + k_{15}u^2v + k_{16}u^3 + O\left((|u| + |v|)^4\right) \\ g^\circ(u,v) &= k_{23}uv + O\left((|u| + |v|)^4\right) \end{aligned}$$

such that

$$\begin{aligned}
k_{11} &= \frac{b_2d_2 - d_2^2 r_2(\delta_2+\widetilde{\delta}) - b_2^2c_2(-1+r_2(\delta_2+\widetilde{\delta}))}{b_2(b_2c_2+d_2)}, \quad k_{12} = -\frac{d_2^2(\delta_2+\widetilde{\delta})}{b_2c_2+d_2} \\
k_{21} &= \frac{(b_2-d_2)(b_2c_2+d_2)r_2(\delta_2+\widetilde{\delta})}{b_2d_2}, \quad k_{22}=1, \quad k_{13} = \frac{-b_2d_2(2b_2c_2+d_2)(\delta_2+\widetilde{\delta})}{(b_2c_2+d_2)^2} \\
k_{14} &= -\frac{(b_2^3c_2^2+2b_2c_2d_2^2+d_2^3)r_2(\delta_2+\widetilde{\delta})}{d_2(b_2c_2+d_2)^2}, \quad k_{15} = -\frac{b_2^4c_2^2(\delta_2+\widetilde{\delta})}{(b_2c_2+d_2)^3}, \\
k_{16} &= \frac{b_2^3c_2^2(b_2-d_2)r_2(\delta_2+\widetilde{\delta})}{d_2(b_2c_2+d_2)^3}, \quad k_{23} = b_2(\delta_2+\widetilde{\delta}).
\end{aligned}$$

The characteristic equation of the matrix $J_{B(x^*,y^*)}$ associated with the linearization in map (3.9) at $(0,0)$ is represented by

$$\lambda^2 + p(\widetilde{\delta})\lambda + q(\widetilde{\delta}) = 0 \tag{3.10}$$

where

$$p(\widetilde{\delta}) = -2 + \frac{(b_2^2c_2+d_2^2)r_2(\delta_2+\widetilde{\delta})}{b_2(b_2c_2+d_2)},$$

$$q(\widetilde{\delta}) = 1 - \frac{r_2(b_2^2c_2+d_2^2)(\delta_2+\widetilde{\delta})}{b_2(b_2c_2+d_2)} + \frac{r_2d_2(\delta_2+\widetilde{\delta})^2\left(b_2^2c_2 - b_2(-1+c_2)d_2 - d_2^2\right)}{b_2(b_2c_2+d_2)}.$$

The characteristic equation (3.10) has the following complex conjugate roots:

$$\begin{aligned}
\lambda_{1,2} &= \frac{-p(\widetilde{\delta}) \pm \sqrt{p(\widetilde{\delta})^2 - 4q(\widetilde{\delta})}}{2} \\
&= 1 + \frac{-(b_2^2c_2+d_2^2)r_2(\delta_2+\widetilde{\delta}) \pm (\delta_2+\widetilde{\delta})i\sqrt{r_2\left[4b_2d_2(b_2-d_2)(b_2c_2+d_2)^2 - (b_2^2c_2+d_2^2)^2r_2\right]}}{2b_2(b_2c_2+d_2)},
\end{aligned}$$

with

$$|\lambda_{1,2}| = \left(q(\widetilde{\delta})\right)^{1/2}, \quad l = \frac{d\,|\lambda_{1,2}|}{d\widetilde{\delta}}\,|_{\widetilde{\delta}=0} \neq 0.$$

Since $(r_2, b_2, d_2, c_2, \delta_2) \in NSB_{B(x^*,y^*)}$, we find that $-2 < p(0) < 2$. In other words, $\frac{4b_2(b_2c_2+d_2)}{(b_2^2c_2+d_2^2)r_2} > \frac{1}{b_2-d_2} + \frac{1}{d_2} - \frac{1}{b_2c_2+d_2}$. So, $p(0) \neq -2, 2$. Additionally, we need to have $p(0) \neq 0, 1$ such that $p(0) = -2 + \frac{r_2(b_2^2c_2+d_2^2)\delta_2}{b_2(b_2c_2+d_2)}$. This case leads to the requirement for

$$\delta_2 \neq \frac{2b_2(b_2c_2+d_2)}{(b_2^2c_2+d_2^2)r_2}, \frac{3b_2(b_2c_2+d_2)}{(b_2^2c_2+d_2^2)r_2}. \tag{3.11}$$

From the conditions $(r_2, b_2, d_2, c_2, \delta_2) \in NSB_{B(x^*,y^*)}$ and (3.11), $\lambda_{1,2}^n \neq 1$, $n = 1, 2, 3, 4$ at $\widetilde{\delta} = 0$. Moreover, we say that the roots $\lambda_{1,2}$ of (3.10) are not located in the intersection of the unit circle with the coordinate axes when $\widetilde{\delta} = 0$. Using the assumption that

$$\begin{aligned}
\alpha &= 1 - \frac{r_2(b_2^2c_2+d_2^2)}{2b_2(b_2c_2+d_2)}\left[\frac{1}{b_2-d_2} + \frac{1}{d_2} - \frac{1}{b_2c_2+d_2}\right], \\
\beta &= \frac{\sqrt{r_2\left[4b_2d_2(b_2-d_2)((b_2c_2+d_2)^2) - (b_2^2c_2+d_2^2)^2r_2\right]}}{2b_2(b_2c_2+d_2)}\left[\frac{1}{b_2-d_2} + \frac{1}{d_2} - \frac{1}{b_2c_2+d_2}\right]
\end{aligned}$$

at $\widetilde{\delta} = 0$, we can obtain the normal form of (3.9). For this, let us take into account the following transformation:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} k_{11} & 0 \\ \alpha - k_{11} & -\beta \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}. \tag{3.12}$$

With transformation (3.12), the normal form of (3.9) becomes

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \widetilde{f}(\tilde{x}, \tilde{y}) \\ \widetilde{g}(\tilde{x}, \tilde{y}) \end{pmatrix}, \tag{3.13}$$

where

$$\begin{aligned} \widetilde{f}(\tilde{x}, \tilde{y}) &= \frac{k_{13}}{k_{12}} uv + \frac{k_{14}}{k_{12}} u^2 + \frac{k_{15}}{k_{12}} u^2 v + \frac{k_{16}}{k_{12}} u^3 + O\left((|\tilde{x}| + |\tilde{y}|)^4\right) \\ \widetilde{g}(\tilde{x}, \tilde{y}) &= \left(\frac{(\alpha - k_{11})k_{13}}{\beta k_{12}} - \frac{k_{23}}{\beta}\right) uv + \frac{(\alpha - k_{11})k_{14}}{\beta k_{12}} u^2 + \frac{(\alpha - k_{11})k_{15}}{\beta k_{12}} u^2 v \\ &\quad + \frac{(\alpha - k_{11})k_{16}}{\beta k_{12}} u^3 + O\left((|\tilde{x}| + |\tilde{y}|)^4\right), \end{aligned}$$

$$u = k_{12}\tilde{x}, \ v = (\alpha - k_{11})\tilde{x} - \beta\tilde{y}.$$

The Neimark-Sacker bifurcation of map (3.13) requires that the following real constant is not zero:

$$a = \left(-\mathfrak{Re}\left[\frac{(1 - 2\lambda_1)\lambda_2^2}{1 - \lambda_1} \xi_{11}\xi_{20}\right] - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + \mathfrak{Re}(\lambda_2 \xi_{21})\right)|_{\delta_2 = 0}, \tag{3.14}$$

where

$$\begin{aligned} \xi_{20} &= \frac{1}{8}\left[\widetilde{f}_{\tilde{x}\tilde{x}} - \widetilde{f}_{\tilde{y}\tilde{y}} + 2\widetilde{g}_{\tilde{x}\tilde{y}} + i\left(\widetilde{g}_{\tilde{x}\tilde{x}} - \widetilde{g}_{\tilde{y}\tilde{y}} - 2\widetilde{f}_{\tilde{x}\tilde{y}}\right)\right], \\ \xi_{11} &= \frac{1}{4}\left[\widetilde{f}_{\tilde{x}\tilde{x}} + \widetilde{f}_{\tilde{y}\tilde{y}} + i\left(\widetilde{g}_{\tilde{x}\tilde{x}} - \widetilde{g}_{\tilde{x}\tilde{y}}\right)\right], \\ \xi_{02} &= \frac{1}{8}\left[\widetilde{f}_{\tilde{x}\tilde{x}} - \widetilde{f}_{\tilde{y}\tilde{y}} + 2\widetilde{g}_{\tilde{x}\tilde{y}} + i\left(\widetilde{g}_{\tilde{x}\tilde{x}} - \widetilde{g}_{\tilde{y}\tilde{y}} + 2\widetilde{f}_{\tilde{x}\tilde{y}}\right)\right], \\ \xi_{21} &= \frac{1}{16}\left[\widetilde{f}_{\tilde{x}\tilde{x}\tilde{x}} + \widetilde{f}_{\tilde{x}\tilde{y}\tilde{y}} + \widetilde{g}_{\tilde{x}\tilde{x}\tilde{y}} + \widetilde{g}_{\tilde{y}\tilde{y}\tilde{y}} + i\left(\widetilde{g}_{\tilde{x}\tilde{x}\tilde{x}} + \widetilde{g}_{\tilde{x}\tilde{y}\tilde{y}} - \widetilde{f}_{\tilde{x}\tilde{x}\tilde{y}} - \widetilde{f}_{\tilde{y}\tilde{y}\tilde{y}}\right)\right]. \end{aligned}$$

Additionally, the partial derivatives of $\widetilde{f}$ and $\widetilde{g}$ evaluated at $\delta = 0$ are provided by:

$$\begin{aligned} \widetilde{f}_{\tilde{x}\tilde{x}} &= 2(\alpha - k_{11})k_{13} + 2k_{12}k_{14} \ , \\ \widetilde{f}_{\tilde{x}\tilde{x}\tilde{x}} &= 6k_{12}\left((\alpha - k_{11})k_{15} + k_{12}k_{16}\right) \\ \widetilde{f}_{\tilde{y}\tilde{y}} &= 0, \quad \widetilde{f}_{\tilde{x}\tilde{x}\tilde{y}} = -2\beta k_{12}k_{15}, \\ \widetilde{f}_{\tilde{x}\tilde{y}\tilde{y}} &= 0 \ , \widetilde{f}_{\tilde{x}\tilde{y}} = -\beta k_{13}, \quad \widetilde{f}_{\tilde{y}\tilde{y}\tilde{y}} = 0 \\ \widetilde{g}_{\tilde{x}\tilde{x}} &= \frac{2(\alpha - k_{11})\left((\alpha - k_{11})k_{13} + k_{12}(k_{14} - k_{23})\right)}{\beta}, \\ \widetilde{g}_{\tilde{x}\tilde{y}} &= (-\alpha + k_{11})k_{13} + k_{12}\ k_{23} \\ \widetilde{g}_{\tilde{x}\tilde{x}\tilde{x}} &= \frac{6(\alpha - k_{11})k_{12}\left((\alpha - k_{11})k_{15} + k_{12}k_{16}\right)}{\beta}, \\ \widetilde{g}_{\tilde{x}\tilde{x}\tilde{y}} &= -2(\alpha - k_{11})k_{12}k_{15} \\ \widetilde{g}_{\tilde{x}\tilde{y}\tilde{y}} &= 0, \quad \widetilde{g}_{\tilde{y}\tilde{y}} = 0, \ \widetilde{g}_{\tilde{y}\tilde{y}\tilde{y}} = 0. \end{aligned}$$

The following conclusion provides parametric requirements for the existence and direction of the Neimark- Sacker bifurcation for the positive fixed point of system (1.2).

Theorem 3.2. *If the condition (3.11) holds and* $a \neq 0$, *then map (3.9) undergoes Neimark-Sacker bifurcations at the fixed point* $B(x^*, y^*)$ *when the parameter* $\delta_2 = \frac{1}{b_2-d_2} + \frac{1}{d_2} - \frac{1}{b_2c_2+d_2}$ *varies in the small neighborhood of* δ. *Moreover, if* $a < 0$ *(resp.,* $a > 0$*), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point for* $\delta > \delta_2$ *(resp.,* $\delta < \delta_2$*).*

3.3. Chaos control

Chaos is an unpredictable behavior that causes great sensitivity at small changes in nonlinear dynamical systems. Chemistry, physics, ecology, biology, chemical engineering, telecommunications, medicine, and other fields that study chaotic behavior use chaos management techniques to avoid chaos. Chaos consists of many periodic points and orbits that depend heavily on the initial state. Therefore, the outcome of a chaotic system is unpredictable, and a control tool is needed.

In this section, we perform the state feedback control method [4, 8, 9, 27] to stabilize the chaotic orbit at an unstable fixed point of system (1.2). We consider the following controlled system to apply this strategy to the model:

$$\begin{aligned} x_{n+1} &= x_n + \delta[rx_n(1-x_n) - bx_ny_n\left(\frac{x_n}{x_n+c}\right)] - U_n \\ y_{n+1} &= y_n + \delta[bx_ny_n - dy_n] \end{aligned} \tag{3.15}$$

where $U_n = k_1(x_n - x^*) + k_2(y_n - y^*)$ is the feedback controlling force, and k_1 and k_2 describe the feedback gains. The Jacobian matrix of the controlled system can be expressed as

$$F_J(x^*, y^*) = \begin{bmatrix} 1 - \frac{(b^2c+d^2)r\delta}{b(bc+d)} - k_1 & -\frac{d^2\delta}{(bc+d)} - k_2 \\ \frac{(b-d)(bc+d)r\delta}{bd} & 1 \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix $F_J(x^*, y^*)$ is given by

$$\mu^2 + \left(-2 + \frac{(b^2c+d^2)r\delta}{b(bc+d)} + k_1\right)\mu - k_1 + \frac{(b-d)(bc+d)r\delta k_2}{bd} + 1 - \frac{(b^2c+d^2)r\delta}{b(bc+d)} + \frac{(b-d)dr\delta^2}{b} = 0. \tag{3.16}$$

Let μ_1 and μ_2 be the roots of the characteristic equation of system (3.16), then we can write

$$\mu_1 + \mu_2 = 2 - \frac{(b^2c+d^2)r\delta}{b(bc+d)} - k_1, \tag{3.17}$$

$$\mu_1\mu_2 = -k_1 + \frac{(b-d)(bc+d)r\delta k_2}{bd} + 1 - \frac{(b^2c+d^2)r\delta}{b(bc+d)} + \frac{(b-d)dr\delta^2}{b}. \tag{3.18}$$

We must solve equations $\mu_1 = \mp 1$ and $\mu_1\mu_2 = 1$ to find the lines of marginal stability. These constraints guarantee that $|\mu_{1,2}| < 1$. Assume that $\mu_1\mu_2 = 1$, then (3.18) denotes that

$$L_1 = -k_1 + \frac{(b-d)(bc+d)r\delta k_2}{bd} - \frac{(b^2c+d^2)r\delta}{b(bc+d)} + \frac{(b-d)dr\delta^2}{b} = 0.$$

Now, we suppose that $\mu_1 = 1$, then Eq. (3.17) and Eq. (3.18) imply

$$L_2 = \frac{(b-d)dr\delta^2}{b} + \frac{(b-d)(bc+d)r\delta k_2}{bd} = 0.$$

Finally, if $\mu_1 = -1$, by using Eq. (3.17) and Eq. (3.18), we obtain

$$L_3 = \frac{(-b+d)(bc+d)r\delta k_2}{bd} + 2k_1 - 4 + \frac{2(b^2c+d^2)r\delta}{b(bc+d)} + \frac{(-b+d)dr\delta^2}{b} = 0.$$

For the given parametric values, it is clear that stable eigenvalues lie within the triangular region bounded by the lines L_1, L_2, and L_3.

4. Numerical simulations

Three numerical examples are given in this section to support the theoretical conclusion made in previous sections. For numerical simulations, we apply the Mathematica and MATLAB applications.

Example 4.1. System (1.2) exhibits a flip bifurcation as the bifurcation parameter varies in a small neighborhood of $\delta = 1.25149$ for $r = 2$, $b = 0.6$, $d = 0.5$, $c = 0.1$, and the initial condition $(x_0, y_0) = (0.83, 0.55)$. System (1.2) has a unique positive fixed point $(x^*, y^*) = (0.833333, 0.622222)$ at $(r, b, d, c, \delta) = (2, 0.6, 0.5, 0.1, 1.25149)$. The Jacobian matrix of system (1.2) has the following characteristic equation:

$$\lambda^2 + 0.130513\lambda - 0.869475 = 0, \tag{4.1}$$

and the eigenvalues of the characteristic equation are found as $\lambda_1 = -1$, $\lambda_2 = 0.869481$ such that $|\lambda_2| \neq 1$. In other words, $(r, b, d, c, \delta) = (2, 0.6, 0.5, 0.1, 1.25149) \in FB_{B(x^*,y^*)}$. The bifurcation diagram and maximum Lyapunov exponents (MLE) for system (1.2) can be seen in Figure 1. Figures 1-(a)-(b) show that the positive fixed point $(0.833333, 0.622222)$ of system (1.2) is stable for $\delta < 1.25149$ and loses stability through a period-doubling bifurcation for $\delta = 1.25149$ and $\delta > 1.25149$. There are oscillations that are nonperiodic when the parameter δ changes for increasing values. Figure 1-(c) shows the maximum Lyapunov exponents, which reveal the existence of periodic orbits and chaotic behavior. The chaotic oscillations in the nonlinear systems are supported by the positive values of the Lyapunov exponent.

Example 4.2. The positive fixed point of system (1.2) is calculated as $(0.571429, 0.431633)$ for the parameter values $r = 3, b = 3.5, d = 2, c = 0.1, \delta = 0.741135$, and the initial condition $(x_0, y_0) = (0.57143, 0.36735)$. Then, the system (1.2) exhibits a Neimark-Sacker bifurcation at $\delta \in (0.5, 0.95)$. It is possible to get the following characteristic equation of the Jacobian matrix for system (1.2):

$$\lambda^2 - 0.587563\lambda + 1 = 0. \tag{4.2}$$

So, the roots of the characteristic equation can be obtained as $\lambda_{1,2} = 0.293782 \mp 0.955873i$ such that $|\lambda_{1,2}| = 1$. Consequently, $(r, b, d, c, \delta) = (3, 3.5, 2, 0.1, 0.741135) \in NSB_{B(x^*,y^*)}$. Figure 2 shows the related bifurcation diagrams and maximum Lyapunov exponents (MLE). The unique positive fixed point $(0.571429, 0.431633)$ is stable for $\delta < 0.741135$, loses stability at $\delta = 0.741135$, and an attractive invariant curve arises if $\delta > 0.741135$, according to the bifurcation diagrams in Figures 2-(a)-(b). To determine the presence of a chaotic structure in Figure 2-(c), the maximum Lyapunov exponents are numerically calculated.

Example 4.3. For system (1.2), we choose $(r, b, d, c, \delta) = (3, 3.5, 2, 0.1, 0.8)$ to apply the feedback control method. From there, we see that the fixed point $(0.571429, 0.431633)$ is an unstable point of system (1.2). Then, in accordance with these parametric values, we provide the following controlled system:

$$\begin{aligned} x_{n+1} &= x_n + 0.8[3x_n(1-x_n) - 3.5x_ny_n\left(\frac{x_n}{x_n+0.1}\right)] \\ &\quad -k_1(x_n - 0.571429) - k_2(y_n - 0.431633) \\ y_{t+1} &= y_n + 0.8[3.5x_ny_n - 2y_n]. \end{aligned} \tag{4.3}$$

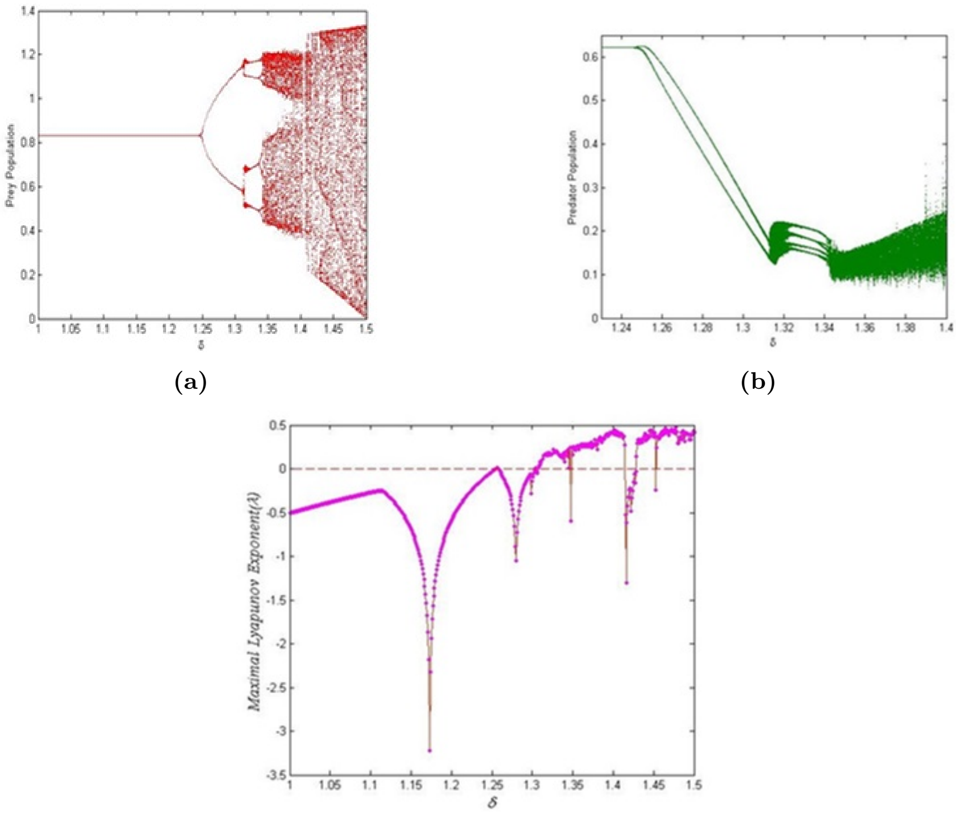
The Jacobian matrix of the controlled system is as follows:

$$F_J(x^*, y^*) = \begin{bmatrix} -0.52462 - k_1 & -1.3617 - k_2 \\ 1.20857 & 1 \end{bmatrix}.$$

Additionally, the lines L_1, L_2, and L_3 for marginal stability are provided by

$$\begin{aligned} L_1 &= 0.121094 - k_1 + 1.20857k_2 = 0, \\ L_2 &= -1.64571 - 1.20857k_2 = 0, \\ L_3 &= -2.59647 + 2k_1 - 1.20857k_2 = 0. \end{aligned}$$

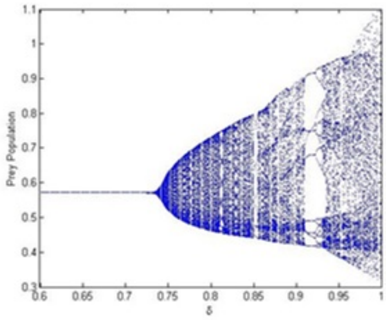
The stable triangular region for the controlled system (4.3) bounded by the L_1, L_2 and L_3 marginal lines is displayed in Figure 3.



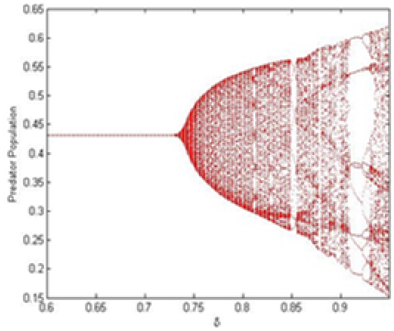
(a) **(b)**

(c)

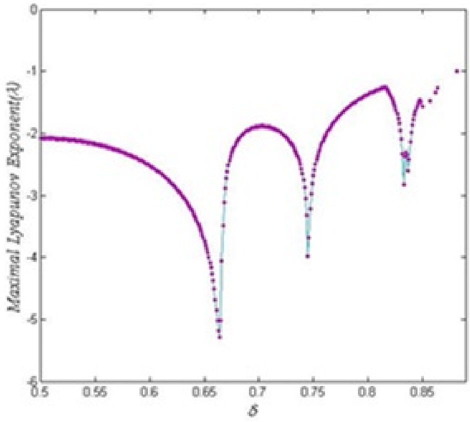
Figure 1. Bifurcation diagram and MLE for system (1.2) for $\delta \in (1, 1.5), r = 2, b = 0.6, d = 0.5, c = 0.1$, and the initial value $(0.83, 0.55)$. **(a)-(b)** Bifurcation diagrams **(c)** Maximum Lyapunov Exponent



(a)



(b)



(c)

Figure 2. Bifurcation diagram and MLE for system (1.2), for $r = 3, b = 3.5, d = 2, c = 0.1$, the initial value $(0.57143, 0.36735)$. **(a)-(b)** Bifurcation diagrams for $\delta \in (0.6, 1)$ **(c)** Maximum Lyapunov Exponent for $\delta \in (0.5, 0.9)$

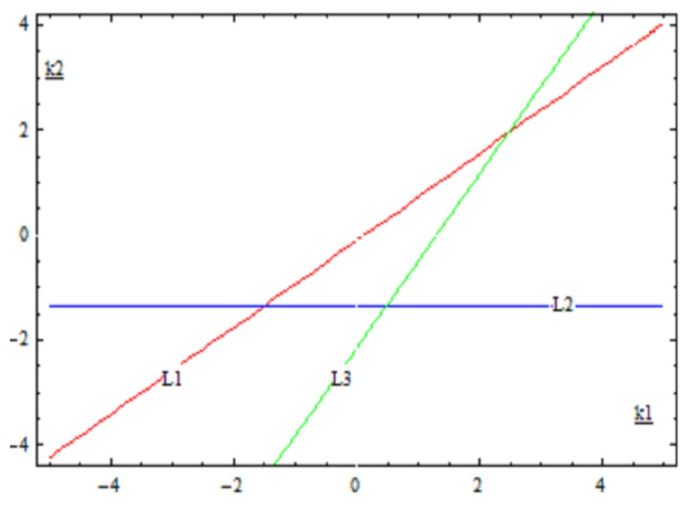
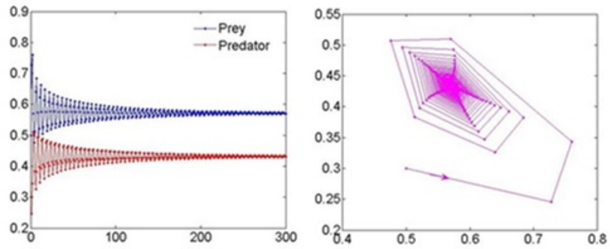
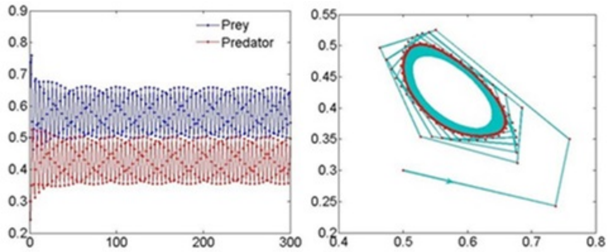


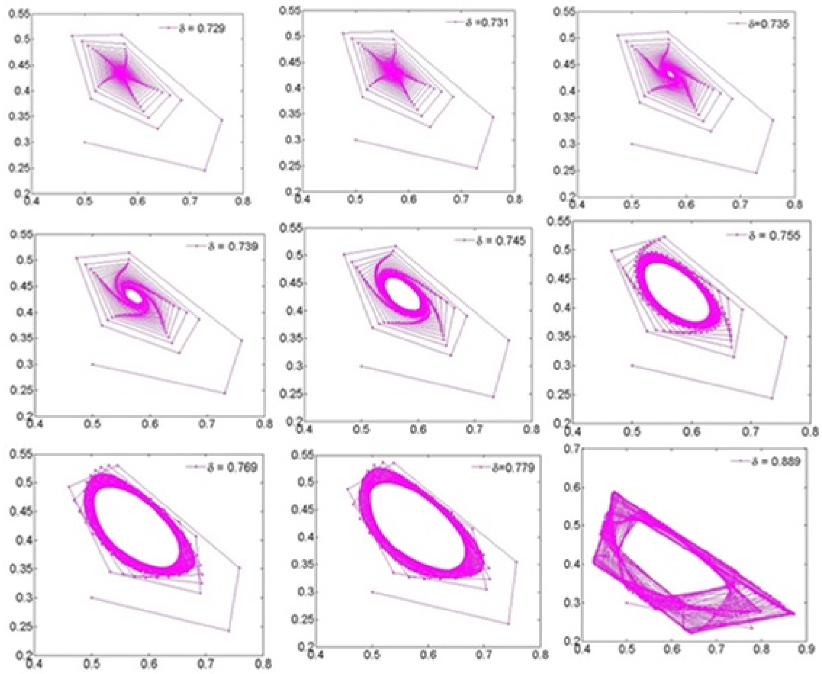
Figure 3. Stability region of system (4.3) for $r = 3, b = 3.5, d = 2, c = 0.1, \delta = 0.8$



(a)



(b)



(c)

Figure 4. The time series and phase plane diagrams of system (1.2) for $r = 3$, $b = 3.5$, $d = 2$, $c = 0.1$. **(a)** The time series and phase plane diagrams for $\delta = 0.731$ **(b)** The time series and phase plane diagrams for $\delta = 0.755$. **(c)** Phase plane diagrams according to the changing values δ.

Additionally, we provide more different dynamics for system (1.2). The figures above give the time series and its phase diagram plots for system (1.2). Phase diagrams are drawn according to values δ . Here, for $r = 3$, $b = 3.5$, $d = 2$, $c = 0.1$, and $\delta = 0.731$, the prey and predator populations approach the fixed point in a finite time and the path spirals towards the fixed point, which indicates stability. For $r = 3$, $b = 3.5$, $d = 2$, $c = 0.1$, and $\delta = 0.755$, the time series and phase plane diagrams provide a glimpse of an unstable behavior of the system with uniform oscillation. The trajectory spirals inwards but does not approach a point. The trajectory finally settles down as a limit cycle.

5. Conclusions

In this study, the dynamics of a discrete-time predator-prey model modified by adding the Allee effect to the prey population are discussed. For system (1.2), we investigated the existence of fixed points, as well as their local asymptotic stability, the presence of period-doubling and Neimark-Sacker bifurcation. The topological classification of the three fixed points of system (1.2) was obtained by using the linearization technique. If $b > d$, we show that system (1.2) has only a unique positive fixed point. With the help of the center manifold theorem and bifurcation theory, it is proven that system (1.2) is subject to both flip and Neimark-Sacker bifurcation. The theoretical results obtained are shown by drawing bifurcation diagrams and maximum Lyapunov exponents. Additionally, time series and phase plots are provided. Under the influence of the Neimark-Sacker bifurcation, system (1.2) produces unstable invariant closed curves. It is well-known that the presence or absence of chaotic solutions for a dynamic system is determined by calculating the Lyapunov exponent. In general, a positive Lyapunov exponent is considered as one of the properties that imply the presence of chaos. Lyapunov exponent values greater than zero confirm the existence of chaos with periodic orbits in the chaotic region. The bifurcation and chaos of system (1.2) are controlled using the chaos control strategy.

The parametric values selected in the examples are taken from a previous study[25]. In Figure 1-(a)-(b) and Figure 4.1-(a) derived from the aforementioned previous study [25], we see that the system goes into a flip bifurcation earlier with the Allee effect for the obtained parameter values. In Figure 2-(a)-(b) and Figure 4.3-(a) [25], we see that the Allee effect delays the Neimark-Sacker bifurcation. The Allee effect appears to have a different influence on different bifurcations. To clarify this situation, let's obtain the bifurcation values of the system with and without the Allee effect by taking the Allee constants $c = 0.005$ and $c = 0$, respectively, and compare the behaviors of these two systems. According to these values, we get the flip bifurcation values of the system as $\delta = 1.28045$ and $\delta = 1.2822$, respectively. It is clear that the system exposed to the Allee effect has undergone bifurcation before. Therefore, the Allee factor has a destabilizing role on the flip bifurcation behavior of the system. Similarly, the Neimark-Sacker bifurcation points for Allee constants $c = 0.005$ and $c = 0$ are obtained as $\delta = 0.671004$ and $\delta = 0.666667$, respectively. This case highlights the late entry of the system into the Neimark-Sacker bifurcation under the Allee factor. In other words, the Allee factor plays a stabilizing role in the Neimark-Sacker bifurcation behavior of the system. As a result, the Allee factor has the effect of accelerating the flip bifurcation of the system, while the Neimark-Sacker has a retarding effect on bifurcation behavior.

When the literature is examined, it is seen that there is no comprehensive study on higher codimension bifurcations, especially for discrete-time dynamical systems. The larger the codimension bifurcation size, the greater the number of conditions that must be met at the bifurcation point. Therefore, there are very few specific systems that satisfy the conditions for a codimension-3 bifurcation and greater than 3. In this case, it can be difficult to understand the dynamics of these systems. For the future study, codimension-2 bifurcation conditions such as resonance 1:1, 1:2, 1:3, and 1:4 of the relevant model can be investigated.

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