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ON THE STABILITY OF COMPOSITE PLASMA IN POROUS MEDIUM

Pardeep KUMAR[1,*] , Hari MOHAN [1]

[1]Department of Mathematics, ICDEOL, Himachal Pradesh University, Summerhill, SHIMLA-5, INDIA

[*] E-mail: pkdureja@gmail.com (corresponding author)

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Abstract

Rayleigh-Taylor instability of a composite plasma in porous medium is considered to include the frictional effect of collisions of ionized with neutral atoms in the presence of a variable magnetic field. The system is found to be stable for stable density stratification. The magnetic field can stabilize a system which was unstable in its absence. The medium permeability has a decreasing or an increasing effect on the growth rates. With the increase in collisional frequency, the growth rates decrease but may have increasing influence in certain region.

Keywords: Composite Plasma, Porous Medium, Rayleigh-Taylor Instability, Variable Magnetic Field.

1. Introduction

A detailed treatment of Rayleigh-Taylor instability under varying assumptions of hydrodynamics and hydromagnetics, together with the possible extensions in various domains of interest has been given by Chandrasekhar [1]. The medium has been considered to be fully ionized. Quite often the plasma is not fully ionized and is, instead, partially ionized. In cosmic physics, there are several situations such as chromoshere, solar photosphere, and in cool interstellar cloud where the plasma are frequently not fully ionized but may instead be partially ionized. A partially-ionized plasma represents a state which often exists in the Universe and there are several situations when the interaction between the ionized and neutral gas components becomes important in cosmic physics. Ionized hydrogen is limited to certain rather sharply bounded regions in space surrounding, for example, O-type stars and clusters of such stars and that the gas outside these regions is essentially non-ionized has been reported by Stromgren [2]. Other examples of the existence of such situations are given by Alfven's [3] theory on the origin of the planetary system, in which a high-ionization rate is suggested to appear from collisions between a plasma and a neutral-gas cloud and by the absorption of plasma waves due to ion-neutral collisions such as in the solar photosphere and chromosphere and in cool interstellar clouds (Piddington [4]; Lehnert [5]). Hans [6] and Bhatia [7] has shown that the medium may be idealized as a composite mixture of a hydromagnetic (ionized) component and a neutral component, the two interacting through mutual collisional effects. A stabilizing effect of collisionals on Rayleigh-Taylor configuration has been shown by Hans [6] and Bhatia [7].

The medium has been considered to be non-porous in all the above studies. In recent years, the investigations of flow of fluids through porous media have become an important topic due to the recovery of crude oil from the pores of reservoir rocks. A great number of applications in geophysics may be found in the books by Phillips [8], Ingham and Pop [9], and Nield and Bejan [10]. When the fluid slowly permeates through the pores of a macroscopically homogeneous and isotropic porous medium, the gross effect is represented by Darcy's law according to which the usual viscous term in the equations of fluid motion is replaced by the resistance term $-\left(\frac{\mu}{k_1}\right)\vec{q}$, where μ is the viscosity of the fluid, k_1 is the medium permeability and $\vec{q}$ is the Darcian (filter) velocity of the fluid. Lapwood [11] has studied the stability of convective flow in hydrodynamics in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through porous medium has been considered by Wooding [12]. Generally, it is accepted that comets consists of a dusty 'snowball' of a mixture of frozen gases which in the process of their journey changes from solid to gas and vice versa. The physical properties of comets, meteorites, and interplanetary dust strongly suggest the importance of porosity in astrophysical context (Mcdonnel [13]).

Diaz et al. [14] studied the modification of the classical criterion for the linear onset and growth rate of the Rayleigh-Taylor instability (RTI) in a partially ionized (PI) plasma in the one-fluid description by considering a generalized induction equation. The occurrence conditions of the condensation instability in heat-releasing partially ionized plasma in an external magnetic field with an induction vector normal to the direction along which a perturbation occurs are considered by Molevich et al. [15].

The problem of the hydromagnetic stability of conducting fluid of variable density and variable magnetic field in porous medium may play an important role in astrophysics (stability of stellar atmosphere in magnetic field, heating of solar corona, theories, and sunspot magnetic field) and geophysics (stability of Earth's core and geothermal regions). Keeping in mind such astrophysical situations, a study has been made of the Raleigh-Taylor instability of a partially-ionized plasma in porous medium in presence of variable horizontal magnetic field, in the present paper.

2. Formulation of the Problem and Perturbation Equations

Here we consider an incompressible composite plasma layer consisting of an infinitely conducting hydromagnetic fluid of density ρ, permeated with neutrals of density ρ_d in porous medium, arranged in horizontal strata and acted on by the gravity force $\vec{g}(0,0,-g)$ and the variable horizontal magnetic field $\vec{H}\ (H_0\ (z), 0, 0)$. We assume that both the fluid and the neutral gas behave like continuum fluids and that effects on the neutral component resulting from the fields of gravity and pressure are neglected. The magnetic field interacts with the hydromagnetic component only.

Let $\vec{q}(u, v, w), \vec{h}(h_x, h_y, h_z), \delta\rho$ and δp denote, respectively, the perturbations in velocity, magnetic field $\vec{H}$, density ρ and pressure p; $\overrightarrow{q_d}, \rho_d, v_c, \mu, \mu_e, \varepsilon$ and k_1 denote the velocity of the neutral gas, density of neutral gas, mutual collisional (frictional) frequency between the two components of the composite medium, viscosity of the hydromagnetic fluid, magnetic permeability, medium porosity and medium permeability, respectively. Then the linearized perturbation equations governing the motion of the composite plasma are

$$\frac{\rho}{\varepsilon}\frac{\partial\vec{q}}{\partial t} = -\nabla\delta p + \frac{\mu_e}{4\pi}\left[(\nabla\times\vec{h})\times\vec{H} + (\nabla\times\vec{H})\times\vec{h}\right] + \vec{g}\,\delta\rho + \frac{\rho_d v_c}{\varepsilon}(\overrightarrow{q_d} - \vec{q}) - \frac{\mu}{k_1}\vec{q}, \tag{1}$$

$$\frac{\partial\overrightarrow{q_d}}{\partial t} = -v_c(\overrightarrow{q_d} - \vec{q}) \tag{2}$$

$$(\nabla.\vec{q}) = 0, \tag{3}$$

$$\varepsilon\frac{\partial}{\partial t}\delta\rho = -w\frac{d\rho}{dz}, \tag{4}$$

$$\nabla.\vec{h} = 0, \tag{5}$$

$$\varepsilon\frac{\partial\vec{h}}{\partial t} = (\vec{H}.\nabla)\vec{q} - (\vec{q}\,.\nabla)\vec{H}, \tag{6}$$

3. Dispersion relation

We seek solutions of the above equations, in terms of normal modes, whose dependence on space-time co-ordinates is of the form

$$f(z)exp\left[ik_x x + ik_y y + nt\right], \tag{7}$$

where $f(z)$ is some function of z only, k_x and k_y $(k^2 = k_x^2 + k_y^2)$ are horizontal wave numbers and n is the frequency of the harmonic disturbance.

Eliminating $\overrightarrow{q_d}$ between (1) and (2) and using (7), (1)-(6) yield

$$\left(\frac{n'\rho}{\varepsilon} + \frac{\mu}{k_1}\right)u = -ik_x\delta p + \frac{\mu_e}{4\pi}h_x(DH_0), \tag{8}$$

$$\left(\frac{n'\rho}{\varepsilon} + \frac{\mu}{k_1}\right)v = -ik_y\delta p + \frac{\mu_e H_0}{4\pi}(ik_x h_y - ik_y h_x), \tag{9}$$

$$\left(\frac{n'\rho}{\varepsilon} + \frac{\mu}{k_1}\right)w = -D\delta p + \frac{g}{n\varepsilon}(D\rho)w + \frac{\mu_e H_0}{4\pi}\left(ik_x h_z - Dh_x - h_x\frac{DH_0}{H_0}\right), \tag{10}$$

$$ik_x u + ik_y v + Dw = 0, \tag{11}$$

$$ik_x h_x + ik_y h_y + Dh_z = 0, \tag{12}$$

$$n\delta\rho = -wD\rho, \tag*{(12) **}$$

$$\varepsilon n h_x = ik_x H_0 u - wDH_0, \tag{13}$$

$$\varepsilon n h_y = i k_x H_0 v, \tag{14}$$

$$\varepsilon n h_z = i k_x H_0 w, \tag{15}$$

where

$$n' = n\left(1 + \frac{\alpha_0 v_c}{n + v_c}\right), \alpha_0 = \frac{\rho_d}{\rho} \ and \ D = \frac{d}{dz}.$$

Eliminating u, v, h_x, h_y, h_z and δp between (8) - (10) and using (11) - (15), after a certain amount of algebra, we get

$$\frac{1}{\varepsilon}\left[D\left(n'\rho Dw\right) - n'k^2 \rho w\right] + \frac{gk^2(D\rho)w}{n\varepsilon} + \frac{\mu}{k_1}(D^2 - k^2)w + \frac{(D\mu)(Dw)}{k_1} +$$

$$+ \frac{\mu_e k_x^2}{4\pi n\varepsilon}\left[H_0^2(D^2 - k^2)w + D(H_0^2)Dw\right] = 0\,. \tag{16}$$

Assume the stratifications in density, viscosity and magnetic field of the form

$$\rho = \rho_0 exp[\beta z], \ \ \mu = \mu_0 exp[\beta z], \ \rho_d = \rho_{d_0} exp[\beta z], \ H_0^2 = H_1^2 exp[\beta z], \tag{17}$$

where $\rho_0, \rho_{d_0}, \mu_0, H_1$ and β are constants. Equations (17) imply that the kinematic viscosity $\nu(= \mu/\rho = \mu_0/\rho_0)$ and the Alfven velocity V_A $(= \sqrt{\mu_e H_0^2/4\pi\rho} = \sqrt{\mu_e H_1^2/4\pi\rho_0}$ are constant everywhere.

Here we consider the case of two free boundaries. Let us assume that $\beta d \ll 1$, i.e. the variation of density at two neighbouring points in the velocity field which is much less than the average density has a negligible effect on the inertia of the fluid. The boundary conditions for the case of two free surfaces are

$$w = D^2 w = 0 \ \text{ at } z = 0 \text{ and } z = d \tag{18}$$

The solution of (16) satisfying the boundary conditions (18) is

$$w = A \sin\frac{m\pi z}{d}\ , \tag{19}$$

where A is a constant and m is any integer. Substituting (19) in (16) and neglecting the effect of heterogeneity on the inertia, we get

$$\left[\left(\frac{m\pi}{d}\right)^2 + k^2\right] - \frac{gk^2\beta}{n\left[\frac{k_x^2 V_A^2}{n} + n' + \frac{\nu_0 \varepsilon}{k_1}\right]} = 0\,. \tag{20}$$

Equation (20), on simplification, gives

$$n^3 + n^2\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + n\left[\left(k_x^2 V_A^2 - \frac{g\beta k^2}{L}\right) + \frac{\varepsilon}{k_1} v_c\right] + v_c\left[k_x^2 V_A^2 - \frac{g\beta k^2}{L}\right] = 0, \tag{21}$$

where $V_A^2 = \frac{\mu_e H_1^2}{4\pi\rho_0}$ and $L = (\frac{m\pi}{d})^2 + k^2$.

4. Discussion

Theorem 1: System is stable for stable density stratification and for unstable density stratification the system is stable or unstable under a condition.

Proof: For the **stable density stratification** ($\boldsymbol{\beta} < 0$) , (21) does not have any positive root and this implies the stability of the system. For **unstable density stratification** ($\boldsymbol{\beta} > 0$), the system is stable or unstable according as

$$k_x^2 V_A^2 \gtrless \frac{g\beta k^2}{L} \quad . \tag{22}$$

Theorem 2: For unstable density stratification, in the absence of magnetic field, the system is unstable.

Proof: The system is clearly unstable in the absence of a magnetic field as can be seen from (21). However, the system can be completely stabilized by a large enough magnetic field as can be seen from (21), if

$$V_A^2 > \frac{g\beta k^2}{k_x^2 L} \ .$$

Theorem 3: The medium permeability has decreasing or increasing effect on the growth rates of the Taylor instability of a partially-ionized plasma in porous medium in the presence of a variable horizontal magnetic field.

Proof: If $\beta > 0$ and $k_x^2 V_A^2 < g\beta k^2/L$, (21) has at least one positive root. Let n_0 denotes the positive root of (21). Then

$$n_0^3 + n_0^2\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + n_0\left[\left(k_x^2 V_A^2 - \frac{g\beta k^2}{L}\right) + \frac{\varepsilon}{k_1} v_c\right] + v_c\left[k_x^2 V_A^2 - \frac{g\beta k^2}{L}\right] = 0, \tag{23}$$

To find the role of medium permeability on the growth rate of unstable modes, we examine the nature of dn_0/dk_1 . Equation (23) yields

$$\frac{dn_0}{dk_1} = \frac{n_0\varepsilon(n_0 + v_c)}{k_1^2\left[3n_0^2 + 2n_0\left\{v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right\} + \left\{k_x^2 V_A^2 - \frac{g\beta k^2}{L} + \frac{\varepsilon}{k_1} v_c\right\}\right]} \ . \tag{24}$$

For

$$\left|k_x^2 V_A^2 - \frac{g\beta k^2}{L}\right| > 3n_0^2 + 2n_0\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + \frac{\varepsilon}{k_1} v_c \ ,$$

dn_0/dk_1 , is negative. The growth rates, therefore, decrease with the increase in medium permeability. However, for

$$\left|k_x^2 V_A^2 - \frac{g\beta k^2}{L}\right| < 3n_0^2 + 2n_0\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + \frac{\varepsilon}{k_1} v_c \ ,$$

dn_0/dk_1 is positive and the growth rates increase with the increase in medium permeability. **The medium permeability, thus, has decreasing or increasing effect on the growth rates of the Taylor instability of a partially-ionized plasma in porous medium in the presence of a variable horizontal magnetic field.**

Theorem 4: The growth rates decrease with the increase in collisional frequency and the growth rates increase with the increase in collisional frequency under a region.

Proof: To find the role of collisional frequency on the growth rate of unstable modes, we examine the nature of dn_0/dv_c .

Equation (23) gives

$$\frac{dn_0}{dv_c} = -\frac{n_0^2(\alpha_0+1) + n_0\frac{\varepsilon}{k_1} + (k_x^2V_A^2 - g\beta k^2/L)}{3n_0^2 + 2n_0\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + \left[k_x^2V_A^2 - g\beta k^2/L + \frac{\varepsilon}{k_1}v_c\right]} . \tag{25}$$

Therefore, if, in addition to $k^2 > k_x^2V_A^2L/g\beta$, which is a sufficient condition for instability, we have either of the condition

$$\left|k_x^2V_A^2 - \frac{g\beta k^2}{L}\right| > 3n_0^2 + 2n_0\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + \frac{\varepsilon}{k_1}v_c \ , \tag{26}$$

or

$$\left|k_x^2V_A^2 - \frac{g\beta k^2}{L}\right| < (1+\alpha_0)n_0^2 + \frac{\varepsilon}{k_1}n_0 \ , \tag{27}$$

dn_0/dv_c is always negative. **The growth rates, therefore, decrease with the increase in collisional frequency. However, the growth rates increase with the increase in collisional frequency for the region**

$$\boldsymbol{(1+\alpha_0)n_0^2 + \frac{\varepsilon}{k_1}n_0 < \left|\left(k_x^2V_A^2 - \frac{g\beta k^2}{L}\right)\right| < 3n_0^2 + 2n_0\left[v_c(1+\alpha_0) + \frac{\varepsilon}{k_1}\right] + \frac{\varepsilon}{k_1}v_c} \ , \tag{28}$$

5. Results

The main results of the present study are:

(i) The system is found to be stable for stable stratification.

(ii) For unstable density stratification, the system is stable or unstable according as

$$\boldsymbol{k_x^2V_A^2 \gtrless \frac{g\beta k^2}{L}} .$$

(iii) For unstable density stratification, in the absence of magnetic field, the system is unstable. However, the system can be completely stabilized by a large enough magnetic field if $\boldsymbol{V_A^2 > \frac{g\beta k^2}{k_x^2 L}}$.

(iv) The medium permeability has decreasing or increasing effect on the growth rates of the Taylor instability of a partially-ionized plasma in porous medium in the presence of a variable horizontal magnetic field.

(v) The growth rates decrease with the increase in collisional frequency and the growth rates increase with the increase in collisional frequency for the region

$$(1+\alpha_0)n_0^2+\frac{\varepsilon}{k_1}n_0<\left|\left(k_x^2V_A^2-\frac{g\beta k^2}{L}\right)\right|<3n_0^2+2n_0\left[v_c(1+\alpha_0)+\frac{\varepsilon}{k_1}\right]+\frac{\varepsilon}{k_1}v_c.$$

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