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# Kinematic Applications of Hyper-Dual Numbers

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**ABSTRACT**

**Hyper-dual numbers are a new number system that is an extension of dual numbers. A hyper-dual number can be written uniquely as an ordered pair of dual numbers. In this paper, some basic algebraic properties of hyper-dual numbers are given using their ordered pair representaions of dual numbers. Moreover, the geometric interpretation of a unit hyper-dual vector is given in module as a dual line. And a geometric interpretation of a subset of unit hyper-dual sphere (the set of all unit hyper-dual vectors) is given as two intersecting perpendicular lines in 3-dimensional real vector space.**

***Keywords:*** *Dual numbers, E. Study mapping, hyper-dual numbers, hyper-dual angle.*
***AMS Subject Classification (2020):*** *Primary: 53A25 ; Secondary: 70B10; 70E15.*

## 1. Introductions

The algebra of dual numbers $\mathbb{D}$ was first introduced by W. Clifford in 1873 as an extension of real numbers $\mathbb{R}$ [2]. The set of all dual vectors constructs the $\mathbb{D}$-module (also denoted by $\mathbb{D}^3$). Motion of a rigid body can be represented by two vectors in 3-dimensional real vector space $\mathbb{R}^3$. E. Study [11] and A. P. Kotelnikov [10] applied dual numbers in mechanism for the first time by using a dual vector instead of two vectors. In the following years, dual numbers are used in the investigation of instantaneous screw axes with the help of dual transformations in $\mathbb{R}^3$ and in Minkowski space $\mathbb{E}^3_1$ [13-14].

Complex numbers have important advantages in derivative calculations. However, these advantages are lost in the calculations of the second derivative [7]. To overcome this problem, J. A. Fike introduced hyper-dual numbers $\tilde{\mathbb{D}}$ that can be used in the calculation of the first and second derivatives maintaining the advantages of the first derivative by complex numbers [6]. In the following years, J. A. Fike and J. J. Alonso developed this number system for derivative calculations [7, 8]. And it is shown that this number system is suitable for complex software, analysis and design airspace systems, and open kinematic chain robot manipulator [7, 4].

A. Cohen and M. Shoham used hyper-dual numbers in the field of kinematics and dynamics to simplify derivative equations of the motion of multi-body systems [3, 4]. They interpreted hyper-dual numbers in the sense of E. Study and A. P. Kotelnikov by using derivative calculations [3-5]. Moreover, they showed that a hyper-dual number can be constituted of two dual numbers [3].

In this paper, some basic concepts of hyper-dual numbers are given using their ordered pair representaions of dual numbers. To give the geometric interpretation of hyper-dual numbers, the concept "dual line" is defined in $\mathbb{D}^3$. Also; E. Study mapping is defined in $\mathbb{D}^3$, and it is shown that to each unit hyper-dual vector corresponds a dual line in $\mathbb{D}^3$. The geometric interpretation of a hyper-dual angle is given as an angle between any two dual lines. Moreover; a subset (denoted by $\tilde{\mathbb{S}}_1$) of unit hyper-dual sphere $\tilde{\mathbb{S}}$ (the set of all unit hyper-dual vectors) is defined, and it is observed that to each element of $\tilde{\mathbb{S}}_1$ corresponds any two intersecting perpendicular directed lines in $\mathbb{R}^3$.

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## 2. Preliminaries

In this section a brief summary of the concepts dual and hyper-dual numbers will be given to provide a background to understand the main idea and the results of this study.

### 2.1. Dual numbers

The set of all dual numbers is defined by

$$\mathbb{D} = \{A = a + \varepsilon a^* : a,\, a^* \in \mathbb{R}\}\,, \tag{2.1}$$

where $\varepsilon$ is the dual unit and satisfies

$$\varepsilon \neq 0, \varepsilon^2 = 0 \quad \text{and} \quad r\varepsilon = \varepsilon r \quad \text{for all } r \in \mathbb{R}. \tag{2.2}$$

Addition and multiplication of any dual numbers $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$ are defined, respectively, as

$$A + B = (a + b) + \varepsilon\,(a^* + b^*)\,, \tag{2.3}$$
$$AB = ab + \varepsilon\,(ab^* + a^* b)\,. \tag{2.4}$$

If $a = 1$ and $a^* = 0$, then $A = 1 + \varepsilon 0 = 1$ is called a unit dual number.
The multiplicative-inverse of a dual number $A = a + \varepsilon a^*$ is

$$A^{-1} = \frac{1}{a} - \varepsilon\frac{a^*}{a^2}, \quad a \neq 0 \tag{2.5}$$

that means a dual number in the form $A = 0 + \varepsilon a^* = \varepsilon a^*$ does not have an multiplicative-inverse.
The square root of a dual number $A = a + \varepsilon a^*$ is defined only for the case $a > 0$ as

$$\sqrt{A} = \sqrt{a} + \varepsilon\frac{a^*}{2\sqrt{a}}. \tag{2.6}$$

Taylor series expansion of a dual function $f(x + \varepsilon x^*)$ about a point $x + \varepsilon x^* = a + \varepsilon a^* \in \mathbb{D}$ can be given as

$$f(a + \varepsilon a^*) = f(a) + \varepsilon a^* f'(a), \tag{2.7}$$

where the prime represents differentiation with respect to $x$, i.e.

$$f'\,(x) = f'\,(x + \varepsilon 0) = \frac{d}{dx}f(x), \tag{2.8}$$

see [12].
Dual numbers form the module

$$\mathbb{D}^3 = \left\{\hat{A} = \boldsymbol{a} + \varepsilon\boldsymbol{a}^* : \boldsymbol{a},\, \boldsymbol{a}^* \in \mathbb{R}^3\right\}, \tag{2.9}$$

which is a commutative and associative ring. Each element $\hat{A}$ of $\mathbb{D}^3$ is called a dual vector.
The scalar product of any dual vectors $\hat{A} = \boldsymbol{a} + \varepsilon\boldsymbol{a}^*$ and $\hat{B} = \boldsymbol{b} + \varepsilon\boldsymbol{b}^*$ is defined by

$$\left\langle \hat{A}, \hat{B}\right\rangle_D = \langle\boldsymbol{a}, \boldsymbol{b}\rangle + \varepsilon\,(\langle\boldsymbol{a}, \boldsymbol{b}^*\rangle + \langle\boldsymbol{a}^*, \boldsymbol{b}\rangle)\,, \tag{2.10}$$

where "$\langle , \rangle$" denotes the usual scalar product in $\mathbb{R}^3$. It is obvious that $\langle\boldsymbol{a}, \boldsymbol{b}\rangle$ and $\langle\boldsymbol{a}, \boldsymbol{b}^*\rangle + \langle\boldsymbol{a}^*, \boldsymbol{b}\rangle$ are real numbers, and thus $\left\langle \hat{A}, \hat{B}\right\rangle_D$ is a dual number.
The norm of a dual vector $\hat{A} = \boldsymbol{a} + \varepsilon\boldsymbol{a}^*$ is defined to be

$$N_{\hat{A}} = \left\langle \hat{A}, \hat{A}\right\rangle_D = |\boldsymbol{a}|^2 + 2\varepsilon\,\langle\boldsymbol{a}, \boldsymbol{a}^*\rangle \in \mathbb{D}, \tag{2.11}$$

where "$|,|$" denotes the usual modulus in $\mathbb{R}^3$. And the modulus (i.e., square root of the norm) of the dual vector $\hat{A} = \boldsymbol{a} + \varepsilon\boldsymbol{a}^*$ is defined to be

$$\left|\hat{A}\right|_D = \sqrt{\left\langle \hat{A}, \hat{A}\right\rangle_D} = |\boldsymbol{a}| + \varepsilon\frac{\langle\boldsymbol{a}, \boldsymbol{a}^*\rangle}{|\boldsymbol{a}|}, \text{ where } |\boldsymbol{a}| \neq 0. \tag{2.12}$$

If $\left|\hat{A}\right|_D = 1$ (i.e., $|\boldsymbol{a}| = 1$ and $\langle \boldsymbol{a}, \boldsymbol{a}^* \rangle = 0$), then $\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ is called a unit dual vector.

The vector product of any dual vectors $\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ and $\hat{B} = \boldsymbol{b} + \varepsilon \boldsymbol{b}^*$ is defined by

$$\hat{A} \times_D \hat{B} = \boldsymbol{a} \times \boldsymbol{b} + \varepsilon \left( \boldsymbol{a} \times \boldsymbol{b}^* + \boldsymbol{a}^* \times \boldsymbol{b} \right), \tag{2.13}$$

where "$\times$" denotes the usual vector product in $\mathbb{R}^3$. It is obvious that $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}^* + \boldsymbol{a}^* \times \boldsymbol{b}$ are real vectors, and thus $\hat{A} \times_D \hat{B}$ is a dual vector.

Unit dual sphere $\mathbb{S}$, consisting of all unit dual vectors, is defined as

$$\mathbb{S} = \left\{ \hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^* : \left|\hat{A}\right|_D = 1, \hat{A} \in \mathbb{D}^3 \right\}. \tag{2.14}$$

**Theorem 1.** (**E. Study Mapping**) To each point on unit dual sphere $\mathbb{S}$ corresponds a directed line in $\mathbb{R}^3$. In other words, there is a one to one correspondence between the points of unit dual sphere $\mathbb{S}$ and the directed lines in $\mathbb{R}^3$ [11].

The geometric interpretation of E. Study mapping can be given as: Let $\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ be the unit dual vector corresponding to the directed line $d$ in $\mathbb{R}^3$. The unit real vector $\boldsymbol{a}$ is the direction vector of the line $d$, and the real vector $\boldsymbol{a}^*$ determines the position of $d$, see Figure 1.

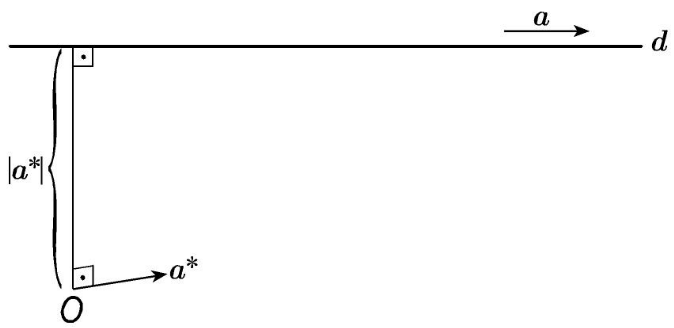


**Figure 1.** Geometric representation of E. Study mapping in $\mathbb{R}^3$

The scalar product of any unit dual vectors $\hat{A} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ and $\hat{B} = \boldsymbol{b} + \varepsilon \boldsymbol{b}^*$ is obtained as

$$\left\langle \hat{A}, \hat{B} \right\rangle_D = \cos\varphi = \cos\theta - \varepsilon\theta^* \sin\theta, \tag{2.15}$$

where $\varphi = \theta + \varepsilon\theta^*$ is a dual angle [11]. If $d_1$ and $d_2$ are the directed lines in $\mathbb{R}^3$ corresponding, respectively, to the unit dual vectors $\hat{A}$ and $\hat{B}$, then $\theta$ is the angle between the real vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\theta^*$ is the closest distance between $d_1$ and $d_2$, see Figure 2.

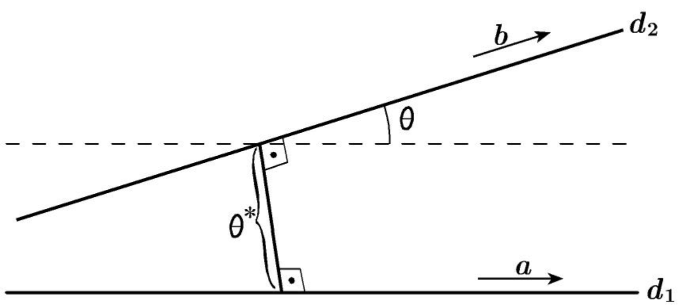


**Figure 2.** Geometric representation of dual angle between the directed lines $d_1$ and $d_2$ in $\mathbb{R}^3$

The following four cases can be given for a dual angle $\varphi$ satisfying $\cos\varphi = \cos\theta - \varepsilon\theta^* \sin\theta$:

1. If

$$\cos\theta = 0 \ \text{ and } \theta^* \neq 0, \tag{2.16}$$

then $\theta = \frac{\pi}{2}$ and $\left\langle \hat{A}, \hat{B} \right\rangle_D = \cos\varphi = -\varepsilon\theta^*$. Thus, lines $d_1$ and $d_2$ are perpendicular but not intersecting.

2. If
$$\theta^* = 0, \tag{2.17}$$
then $\left\langle \hat{A}, \hat{B} \right\rangle_D = \cos\varphi = \cos\theta$. Thus, lines $d_1$ and $d_2$ are intersecting.

3. If
$$\left\langle \hat{A}, \hat{B} \right\rangle_D = \cos\varphi = 0, \tag{2.18}$$
then $\theta = \frac{\pi}{2}$ and $\theta^* = 0$. Thus, lines $d_1$ and $d_2$ are perpendicular and intersecting.

4. If
$$\left\langle \hat{A}, \hat{B} \right\rangle_D = \cos\varphi = 1, \tag{2.19}$$
then $\theta = 0$. Thus, lines $d_1$ and $d_2$ are parallel.

The modulus of the vector product of any unit dual vectors $\hat{A}$ and $\hat{B}$ is obtained as
$$\left| \hat{A} \times_D \hat{B} \right|_D = \sin\varphi = \sin\theta + \varepsilon\theta^* \cos\theta. \tag{2.20}$$
For further information about dual numbers, see [2, 12, 1].

### 2.2. *Hyper-dual numbers*

The set of all hyper-dual numbers is defined by
$$\tilde{\mathbb{D}} = \{\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 : a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R}\}, \tag{2.21}$$
where the dual units $\varepsilon_1$ and $\varepsilon_2$ satisfy
$$\varepsilon_1^2 = \varepsilon_2^2 = (\varepsilon_1\varepsilon_2)^2 = 0 \quad \text{and} \quad \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2 \neq 0. \tag{2.22}$$
Addition and multiplication of any hyper-dual numbers $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3$ and $\mathbb{B} = b_0 + \varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_1\varepsilon_2 b_3$ are defined, respectively, as
$$\mathbb{A} + \mathbb{B} = (a_0 + b_0) + \varepsilon_1 (a_1 + b_1) + \varepsilon_2 (a_2 + b_2) + \varepsilon_1\varepsilon_2 (a_3 + b_3), \tag{2.23}$$
$$\begin{aligned} \mathbb{A}\mathbb{B} &= (a_0 b_0) + \varepsilon_1 (a_0 b_1 + a_1 b_0) + \varepsilon_2 (a_0 b_2 + a_2 b_0) \\ &\quad + \varepsilon_1\varepsilon_2 (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0), \end{aligned} \tag{2.24}$$
The multiplicative-inverse of a hyper-dual number $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3$ is
$$\mathbb{A}^{-1} = \frac{1}{\mathbb{A}} = \frac{1}{a_0} - \varepsilon_1 \frac{a_1}{a_0^2} - \varepsilon_2 \frac{a_2}{a_0^2} + \varepsilon_1\varepsilon_2 \left( -\frac{a_3}{a_0^2} + \frac{2a_1 a_2}{a_0^3} \right), \quad a_0 \neq 0 \tag{2.25}$$
that means a hyper-dual number in the form $\mathbb{A} = 0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 = \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3$ does not have an multiplicative-inverse.

Taylor series expansion of a hyper-dual function $f(x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3)$ about a point $x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3 = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 \in \tilde{\mathbb{D}}$ can be given as
$$\begin{aligned} f(a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3) &= f(a_0) + \varepsilon_1 a_1 f'(a_0) + \varepsilon_2 a_2 f'(a_0) \\ &\quad + \varepsilon_1\varepsilon_2 (a_3 f'(a_0) + a_1 a_2 f''(a_0)), \end{aligned} \tag{2.26}$$
where the prime represents differentiation with respect to $x_0$, i.e.
$$f'(x_0) = f'(x_0 + \varepsilon_1 0 + \varepsilon_2 0 + \varepsilon_1\varepsilon_2 0) = \frac{d}{dx_0} f(x_0), \tag{2.27}$$
see [6-9].

A hyper-dual number $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3$ can be given in terms of two dual numbers as

$$\begin{aligned}\mathbb{A} &= a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3\\ &= (a_0 + \varepsilon_1 a_1) + \varepsilon_2 (a_2 + \varepsilon_1 a_3)\\ &= (a_0 + \varepsilon a_1) + \varepsilon^* (a_2 + \varepsilon a_3)\\ &= A + \varepsilon^* A^*,\end{aligned} \tag{2.28}$$

where $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^*$ and $A = a_0 + \varepsilon a_1$, $A^* = a_2 + \varepsilon a_3 \in \mathbb{D}$.

If we extend the real vectors $\boldsymbol{a}$ and $\boldsymbol{p} \times \boldsymbol{a}$ in a dual vector $\hat{A} = \boldsymbol{a} + \varepsilon (\boldsymbol{p} \times \boldsymbol{a})$, respectively, to the dual vectors $\hat{A}$ and $\hat{P} \times_D \hat{A}$, then we obtain the hyper-dual vector

$$\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \left(\hat{P} \times_D \hat{A}\right). \tag{2.29}$$

Scalar and vector products of any hyper-dual vectors $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \left(\hat{P} \times_D \hat{A}\right)$ and $\widetilde{\mathbb{B}} = \hat{B} + \varepsilon^* \left(\hat{K} \times_D \hat{B}\right)$ can be given, respectively, as

$$\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = \left|\hat{A}\right|_D \left|\hat{B}\right|_D \cos\tilde{\varphi} \tag{2.30}$$

$$\widetilde{\mathbb{A}} \times_{HD} \widetilde{\mathbb{B}} = \left|\hat{A}\right|_D \left|\hat{B}\right|_D n \sin\tilde{\varphi}, \tag{2.31}$$

where $\tilde{\varphi}$ is a hyper-dual angle and $n$ is the direction vector of the common perpendicular between these two hyper-dual vectors. For further information about hyper-dual numbers, see [3-5].

## 3. Applications of Hyper-Dual Numbers in $\mathbb{R}^3$ and $\mathbb{D}^3$

In this section, we show that the basic and kinematic concepts of hyper-dual numbers can be given by using dual numbers. Using these concepts, E. Study mapping and hyper-dual angle are abtained in module $\mathbb{D}^3$. Furthermore, we have defined a subset (denoted by $\tilde{\mathbb{S}}_1$) of unit hyper-dual sphere $\tilde{\mathbb{S}}$ such that to each element of this subset corresponds two intersecting and perpendicular directed lines in $\mathbb{R}^3$.

From the definition of a hyper-dual number given by the Equation (2.28), alternative representations of addition (given by Equation (2.23)) and multiplication (given by Equation (2.24)) of any hyper-dual numbers $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 = A + \varepsilon^* A^*$ and $\mathbb{B} = b_0 + \varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_1\varepsilon_2 b_3 = B + \varepsilon^* B^*$ can be given, respectively, as

$$\mathbb{A} + \mathbb{B} = (A + B) + \varepsilon^* \left(A^* + B^*\right), \tag{3.1}$$

$$\mathbb{A}\mathbb{B} = AB + \varepsilon^* \left(AB^* + A^* B\right). \tag{3.2}$$

Moreover, an alternative representation of the multiplicative-inverse (given by Equation (2.25)) of a hyper-dual number $\mathbb{A} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 = A + \varepsilon^* A^*$ can be given as

$$\mathbb{A}^{-1} = \frac{1}{A} - \varepsilon^* \frac{A^*}{A^2},\ a_0 \neq 0 \tag{3.3}$$

that means a hyper-dual number $\mathbb{A} = A + \varepsilon^* A^*$ providing $A = 0 + \varepsilon a_1 = \varepsilon a_1$ does not have an multiplicative-inverse.

The square root of a hyper-dual number $\mathbb{A} = A + \varepsilon^* A^*$ can be defined by

$$\sqrt{\mathbb{A}} = \sqrt{A} + \varepsilon^* \frac{A^*}{2\sqrt{A}},\ a_0 > 0 \tag{3.4}$$

or

$$\sqrt{\mathbb{A}} = \sqrt{a_0} + \varepsilon_1 \frac{a_1}{2\sqrt{a_0}} + \varepsilon_2 \frac{a_2}{2\sqrt{a_0}} + \varepsilon_1\varepsilon_2 \left(\frac{a_3}{2\sqrt{a_0}} - \frac{a_1 a_2}{4a_0\sqrt{a_0}}\right),\ a_0 > 0. \tag{3.5}$$

An alternative representation of the Taylor series expension of a hyper-dual function given by Equation (2.26) can be given by the following theorem.

**Theorem 2.** Let $\mathbb{A} = A + \varepsilon^* A^*$ be a hyper-dual number, where $A = a_0 + \varepsilon a_1$, $A^* = a_2 + \varepsilon a_3 \in \mathbb{D}$. Then, the Taylor series expansion of the hyper-dual function $f(x_0 + \varepsilon x_1 + \varepsilon^* x_2 + \varepsilon\varepsilon^* x_3)$ about a point $x_0 + \varepsilon x_1 + \varepsilon^* x_2 + \varepsilon\varepsilon^* x_3 = a_0 + \varepsilon a_1 + \varepsilon^* a_2 + \varepsilon\varepsilon^* a_3 \in \tilde{\mathbb{D}}$ can be given as

$$f(A + \varepsilon^* A^*) = f(A) + \varepsilon^* A^* f'(A), \tag{3.6}$$

where $f'(A) = f'(a_0 + \varepsilon a_1)$ is the first derivative of the dual function $f(x_0 + \varepsilon x_1)$ with respect to $x_0$ at the point $x_0 + \varepsilon x_1 = a_0 + \varepsilon a_1 \in \mathbb{D}$, i.e.

$$f'(x_0) = f'(x_0 + \varepsilon 0) = \frac{d}{dx_0} f(x_0). \tag{3.7}$$

*Proof.* From Equation (2.7), the Taylor series expansions of $f(A)$ and $f'(A)$ can be given, respectively, as

$$f(A) = f(a_0 + \varepsilon a_1) = f(a_0) + \varepsilon a_1 f'(a_0), \tag{3.8}$$
$$f'(A) = f'(a_0 + \varepsilon a_1) = f'(a_0) + \varepsilon a_1 f''(a_0), \tag{3.9}$$

where the prime represents differentiation with respect to $x_0$, i.e.

$$f'(x_0) = f'(x_0 + \varepsilon 0) = \frac{d}{dx_0} f(x_0), \tag{3.10}$$
$$f''(x_0) = f''(x_0 + \varepsilon 0) = \frac{d}{dx_0} f'(x_0). \tag{3.11}$$

Using the Equation (2.26), we get

$$\begin{aligned} f(\mathbb{A}) &= f(a_0) + \varepsilon a_1 f'(a_0) + \varepsilon^* a_2 f'(a_0) + \varepsilon\varepsilon^* (a_3 f'(a_0) + a_1 a_2 f''(a_0)) \\ &= (f(a_0) + \varepsilon a_1 f'(a_0)) + \varepsilon^* (a_2 f'(a_0) + \varepsilon(a_3 f'(a_0) + a_1 a_2 f''(a_0))) \\ &= (f(a_0) + \varepsilon a_1 f'(a_0)) + \varepsilon^* (a_2 f'(a_0) + \varepsilon a_1 a_2 f''(a_0) + \varepsilon a_3 f'(a_0)) \\ &= (f(a_0) + \varepsilon a_1 f'(a_0)) + \varepsilon^* (a_2 f'(a_0) + \varepsilon a_1 a_2 f''(a_0) + \varepsilon a_3 f'(a_0) \\ &\quad + \varepsilon^2 a_1 a_3 f''(a_0)) \\ &= (f(a_0) + \varepsilon a_1 f'(a_0)) + \varepsilon^* (a_2 (f'(a_0) + \varepsilon a_1 f''(a_0)) + \varepsilon a_3 (f'(a_0) \\ &\quad + \varepsilon a_1 f''(a_0))) \\ &= (f(a_0) + \varepsilon a_1 f'(a_0)) + \varepsilon^* (a_2 + \varepsilon a_3)(f'(a_0) + \varepsilon a_1 f''(a_0)). \end{aligned} \tag{3.12}$$

Inserting Equations (3.8) and (3.9) in the Equation (3.12), we also get

$$f(A + \varepsilon^* A^*) = f(A) + \varepsilon^* (a_2 + \varepsilon a_3) f'(A), \tag{3.13}$$

and using $A^* = a_2 + \varepsilon a_3$, we obtain

$$f(A + \varepsilon^* A^*) = f(A) + \varepsilon^* A^* f'(A). \tag{3.14}$$

□

We need to define the concept line in $\mathbb{D}^3$ to give the geometric interpretations of hyper-dual numbers in $\mathbb{D}^3$.

**Definition 1.** (**Dual line**) Let $\hat{A}$ be a unit dual vector and $\hat{P}$ be a point in $\mathbb{D}^3$. Then, a line in $\mathbb{D}^3$ can be defined by

$$\hat{d} = \hat{P} + T\hat{A}, \tag{3.15}$$

where the parameter $T$ is a dual number, the unit dual vector $\hat{A}$ is the direction vector of $\hat{d}$, and $\hat{P}$ is a point on $\hat{d}$. We will call a line in $\mathbb{D}^3$ as dual line.

**Definition 2.** (**Hyper-dual vectors**) The set of all hyper-dual vectors is defined by

$$\tilde{\mathbb{D}}^3 = \left\{\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* : \hat{A},\ \hat{A}^* \in \mathbb{D}^3\right\} \tag{3.16}$$

$$= \left\{\widetilde{\mathbb{A}} = \boldsymbol{a}_0 + \varepsilon \boldsymbol{a}_1 + \varepsilon^* \boldsymbol{a}_2 + \varepsilon\varepsilon^* \boldsymbol{a}_3 : \boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in \mathbb{R}^3\right\}, \tag{3.17}$$

and each element $\widetilde{\mathbb{A}}$ of $\tilde{\mathbb{D}}^3$ is called a hyper-dual vector.

The scalar and vector products of any hyper-dual vectors $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* = \boldsymbol{a}_0 + \varepsilon \boldsymbol{a}_1 + \varepsilon^* \boldsymbol{a}_2 + \varepsilon\varepsilon^* \boldsymbol{a}_3$ and $\widetilde{\mathbb{B}} = \hat{B} + \varepsilon^* \hat{B}^* = \boldsymbol{b}_0 + \varepsilon \boldsymbol{b}_1 + \varepsilon^* \boldsymbol{b}_2 + \varepsilon\varepsilon^* \boldsymbol{b}_3$ are defined, respectively, by

$$\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = \left\langle \hat{A}, \hat{B} \right\rangle_D + \varepsilon^* \left( \left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D \right) \tag{3.18}$$

$$\begin{aligned} &= \langle \boldsymbol{a}_0, \boldsymbol{b}_0 \rangle + \varepsilon(\langle \boldsymbol{a}_0, \boldsymbol{b}_1 \rangle + \langle \boldsymbol{a}_1, \boldsymbol{b}_0 \rangle) + \varepsilon^*(\langle \boldsymbol{a}_0, \boldsymbol{b}_2 \rangle + \langle \boldsymbol{a}_2, \boldsymbol{b}_0 \rangle) \\ &\quad + \varepsilon\varepsilon^*(\langle \boldsymbol{a}_0, \boldsymbol{b}_3 \rangle + \langle \boldsymbol{a}_1, \boldsymbol{b}_2 \rangle + \langle \boldsymbol{a}_2, \boldsymbol{b}_1 \rangle + \langle \boldsymbol{a}_3, \boldsymbol{b}_0 \rangle), \end{aligned} \tag{3.19}$$

$$\widetilde{\mathbb{A}} \times_{HD} \widetilde{\mathbb{B}} = \hat{A} \times_D \hat{B} + \varepsilon^* \left( \hat{A} \times_D \hat{B}^* + \hat{A}^* \times_D \hat{B} \right). \tag{3.20}$$

$$\begin{aligned} &= \boldsymbol{a}_0 \times \boldsymbol{b}_0 + \varepsilon(\boldsymbol{a}_0 \times \boldsymbol{b}_1 + \boldsymbol{a}_1 \times \boldsymbol{b}_0) + \varepsilon^*(\boldsymbol{a}_0 \times \boldsymbol{b}_2 + \boldsymbol{a}_2 \times \boldsymbol{b}_0) \\ &\quad + \varepsilon\varepsilon^*(\boldsymbol{a}_0 \times \boldsymbol{b}_3 + \boldsymbol{a}_1 \times \boldsymbol{b}_2 + \boldsymbol{a}_2 \times \boldsymbol{b}_1 + \boldsymbol{a}_3 \times \boldsymbol{b}_0). \end{aligned} \tag{3.21}$$

Since $\left\langle \hat{A}, \hat{B} \right\rangle_D$ and $\left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D$ are dual numbers, $\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD}$ is a hyper-dual number. And since $\hat{A} \times_D \hat{B}$ and $\hat{A} \times_D \hat{B}^* + \hat{A}^* \times_D \hat{B}$ are dual vectors, $\widetilde{\mathbb{A}} \times_{HD} \widetilde{\mathbb{B}}$ is a hyper-dual vector.

The norm of a hyper-dual vector $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* = \boldsymbol{a}_0 + \varepsilon \boldsymbol{a}_1 + \varepsilon^* \boldsymbol{a}_2 + \varepsilon\varepsilon^* \boldsymbol{a}_3$ is defined to be

$$N_{\widetilde{\mathbb{A}}} = \left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{A}} \right\rangle_{HD} = \left| \hat{A} \right|_D^2 + 2\varepsilon^* \left\langle \hat{A}, \hat{A}^* \right\rangle_D \tag{3.22}$$

$$= |\boldsymbol{a}_0|^2 + 2\left(\varepsilon \langle \boldsymbol{a}_0, \boldsymbol{a}_1 \rangle + \varepsilon^* \langle \boldsymbol{a}_0, \boldsymbol{a}_2 \rangle + \varepsilon\varepsilon^*(\langle \boldsymbol{a}_0, \boldsymbol{a}_3 \rangle + \langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle)\right). \tag{3.23}$$

And the modulus (i.e., square root of the norm) of the hyper-dual vector $\widetilde{\mathbb{A}}$ is defined to be

$$\left| \widetilde{\mathbb{A}} \right|_{HD} = \sqrt{\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{A}} \right\rangle_{HD}} = \left| \hat{A} \right|_D + \varepsilon^* \frac{\left\langle \hat{A}, \hat{A}^* \right\rangle_D}{\left| \hat{A} \right|_D} \tag{3.24}$$

$$\begin{aligned} &= |\boldsymbol{a}_0| + \varepsilon \frac{\langle \boldsymbol{a}_0, \boldsymbol{a}_1 \rangle}{|\boldsymbol{a}_0|} + \varepsilon^* \frac{\langle \boldsymbol{a}_0, \boldsymbol{a}_2 \rangle}{|\boldsymbol{a}_0|} \\ &\quad + \varepsilon\varepsilon^* \left( \frac{\langle \boldsymbol{a}_0, \boldsymbol{a}_3 \rangle}{|\boldsymbol{a}_0|} + \frac{\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle}{|\boldsymbol{a}_0|} - \frac{\langle \boldsymbol{a}_0, \boldsymbol{a}_1 \rangle \langle \boldsymbol{a}_0, \boldsymbol{a}_2 \rangle}{|\boldsymbol{a}_0|^3} \right), \end{aligned} \tag{3.25}$$

where $|\boldsymbol{a}_0| \neq 0$.

If $\left| \widetilde{\mathbb{A}} \right|_{HD} = 1$ (i.e., $\left| \hat{A} \right|_D = 1$ and $\left\langle \hat{A}, \hat{A}^* \right\rangle_D = 0$), then $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ is called a unit hyper-dual vector.

**Definition 3.** (**Unit hyper-dual sphere**) Unit hyper-dual sphere $\tilde{\mathbb{S}}$, consisting of all unit hyper-dual vectors, can be defined as

$$\tilde{\mathbb{S}} = \left\{ \widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* : \left| \widetilde{\mathbb{A}} \right|_{HD} = 1;\ \ \hat{A},\ \hat{A}^* \in \mathbb{D}^3 \right\}. \tag{3.26}$$

**Theorem 3.** (**E. Study mapping for unit hyper-dual vectors**) To each point on unit hyper-dual sphere $\tilde{\mathbb{S}}$ corresponds a directed dual line $\hat{d}$ in $\mathbb{D}^3$. In other words, there is a one to one correspondence between the points of unit hyper-dual sphere $\tilde{\mathbb{S}}$ and the directed dual lines in $\mathbb{D}^3$.

*Proof.* A directed line in $\mathbb{D}^3$ (i.e., directed dual line) can be given by any two points $\hat{X}$ and $\hat{Y}$ on it. The parametric equation of this dual line is

$$\hat{Y} = \hat{X} + T\hat{A}, \tag{3.27}$$

where $T$ is a non-zero dual constant and $\hat{A}$ is a unit dual vector. The moment of the vector $\hat{A}$ with respect to the origin $\hat{O}$ is

$$\hat{A}^* = \hat{X} \times_D \hat{A} = \hat{Y} \times_D \hat{A}. \tag{3.28}$$

That means; the direction vector $\hat{A}$ of the dual line and its moment vector $\hat{A}^*$ are independent of choice of the points of the dual line. The two vectors $\hat{A}$ and $\hat{A}^*$ are not independent of one another; so they satisfy the equations

$$\left|\hat{A}\right|_D = 1 \text{ and } \left\langle \hat{A}, \hat{A}^* \right\rangle_D = 0. \tag{3.29}$$

The six dual components $A_i$, $A_i^*$ (for $i = 1, 2, 3$) of $\hat{A}$ and $\hat{A}^*$ are Plückerian homogeneous dual line coordinates. Hence the two dual vectors $\hat{A}$ and $\hat{A}^*$ determine the directed dual line. A point $\hat{Z}$ is on the dual line of dual vectors $\hat{A}$ and $\hat{A}^*$ if and only if

$$\hat{Z} \times_D \hat{A} = \hat{A}^*. \tag{3.30}$$

The set of directed dual lines is in one to one correspondence with pairs of dual vectors in $\mathbb{D}^3$ subject to the conditions (given by Equation (3.27)). Consequently; since $\hat{A}$ is a unit dual vector (i.e., $\left|\hat{A}\right|_D = 1$) and $\left\langle \hat{A}, \hat{A}^* \right\rangle_D = 0$, the unit hyper-dual vector $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ represents a dual line, see Figure 3. □

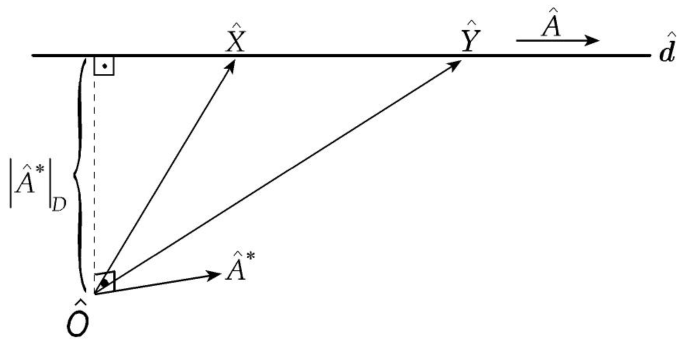


**Figure 3.** Geometric representation of E. Study mapping in $\mathbb{D}^3$

**Example 1.** (Application of E. Study mapping for unit hyper-dual vectors) Let us take the unit hyper-dual vector $\widetilde{\mathbb{A}} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} + \varepsilon, \frac{1}{\sqrt{3}} - \varepsilon) + \varepsilon^*(-2 + \varepsilon, 1, 1 - \varepsilon)$ that can be written in the form $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ for $\hat{A} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} + \varepsilon, \frac{1}{\sqrt{3}} - \varepsilon)$ and $\hat{A}^* = (-2 + \varepsilon, 1, 1 - \varepsilon)$. If $\hat{d}$ is the corresponding dual line in $\mathbb{D}^3$ to $\widetilde{\mathbb{A}}$, and $\hat{Z}$ is the nearest point from the origin $\hat{O}$ to the line $\hat{d}$, then the equalities

$$\hat{Z} \times_D \hat{A} = \hat{A}^* \quad \text{and} \quad \left\langle \hat{Z}, \hat{A} \right\rangle_D = \left\langle \hat{Z}, \hat{A}^* \right\rangle_D = 0 \tag{3.31}$$

can be given. From these equations, we get

$$\hat{Z} = \left(\varepsilon(2 - \tfrac{1}{\sqrt{3}}), -\sqrt{3} + \varepsilon(2 + \tfrac{2}{\sqrt{3}}), \sqrt{3} + \varepsilon(2 - \tfrac{1}{\sqrt{3}})\right). \tag{3.32}$$

Since the unit dual vector $\hat{A}$ is the direction vector of $\hat{d}$, and $\hat{Z}$ is a point on $\hat{d}$, we can give the corresponding dual line to unit hyper-dual vector $\widetilde{\mathbb{A}}$ as

$$\begin{aligned} \hat{d} = &\left(\varepsilon(2 - \tfrac{1}{\sqrt{3}}), -\sqrt{3} + \varepsilon(2 + \tfrac{2}{\sqrt{3}}), \sqrt{3} + \varepsilon(2 - \tfrac{1}{\sqrt{3}})\right) \\ &+ T\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} + \varepsilon, \tfrac{1}{\sqrt{3}} - \varepsilon\right), \end{aligned} \tag{3.33}$$

where the parameter $T$ is a dual variable.

**Theorem 4.** Let us take a subset of unit hyper-dual sphere $\tilde{\mathbb{S}}$ as

$$\tilde{\mathbb{S}}_1 = \left\{ \widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* : \left|\hat{A}^*\right|_D = 1, \widetilde{\mathbb{A}} \in \tilde{\mathbb{S}} \right\}. \tag{3.34}$$

Then, there exists a one to one correspondence between the points of $\tilde{\mathbb{S}}_1$ and any two intersecting perpendicular directed lines in $\mathbb{R}^3$.

*Proof.* Since $\widetilde{\mathbb{A}} \in \tilde{\mathbb{S}}_1$; $\hat{A}$ and $\hat{A}^*$ are unit dual vectors and $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ is a unit hyper-dual vector satisfying $\left|\hat{A}\right|_D = 1$ and $\left\langle \hat{A}, \hat{A}^* \right\rangle_D = 0$. According to Theorem 1, let $\hat{A}$ and $\hat{A}^*$ represent the directed lines $d_1$ and $d_2$ in $\mathbb{R}^3$, respectively. Thus, from Equation (2.18), the property $\left\langle \hat{A}, \hat{A}^* \right\rangle_D = 0$ shows that $d_1$ and $d_2$ are perpendicular intersecting directed lines. □

**Example 2.** (Application of the subset $\tilde{\mathbb{S}}_1$) Let us take the unit hyper-dual vector $\widetilde{\mathbb{A}} = (\varepsilon, 1, 0) + \varepsilon^*(-\varepsilon, 0, 1)$ that can be written in the form $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ for

$$\hat{A} = (\varepsilon, 1, 0) = (0, 1, 0) + \varepsilon(1, 0, 0), \tag{3.35}$$

$$\hat{A}^* = (-\varepsilon, 0, 1) = (0, 0, 1) + \varepsilon(-1, 0, 0). \tag{3.36}$$

Since $\left|\hat{A}\right|_D = \left|\hat{A}^*\right|_D = 1$; $\hat{A}$ and $\hat{A}^*$ are unit dual vectors, and thus $\widetilde{\mathbb{A}} \in \tilde{\mathbb{S}}_1$. According to Theorem 4, unit hyper-dual vector $\widetilde{\mathbb{A}}$ represents two perpendicular intersecting directed lines in $\mathbb{R}^3$. And according to E. Study mapping, each of these lines correspond to a unit dual vector (one of them corresponds to $\hat{A}$ and the other to $\hat{A}^*$), [11]. These lines will be obtained, respectively, as

$$d_1 = (0, 0, -1) + t_1\,(0, 1, 0)\,, \tag{3.37}$$

$$d_2 = (0, -1, 0) + t_2\,(0, 0, 1)\,, \tag{3.38}$$

where the parameters $t_1$ and $t_2$ are real variables. Direction vectors of $d_1$ and $d_2$ are $\boldsymbol{v}_1 = (0, 1, 0)$ and $\boldsymbol{v}_2 = (0, 0, 1)$, respectively. Since $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$; $d_1$ and $d_2$ are perpendicular. And for $t_1 = -1$ and $t_2 = -1$; $d_1$ and $d_2$ intersect at the point $(0, -1, -1)$.

**Definition 4.** (**Hyper-dual angle**)

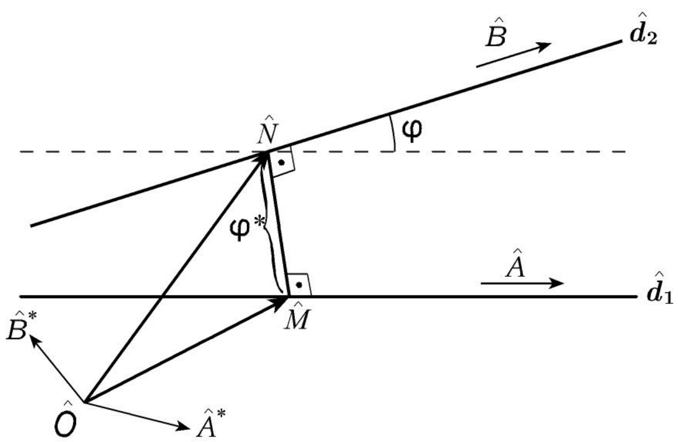


**Figure 4.** Geometric representation of hyper-dual angle between the directed dual lines $\hat{d}_1$ and $\hat{d}_2$ in $\mathbb{D}^3$

The scalar product of any unit hyper-dual vectors $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ and $\widetilde{\mathbb{B}} = \hat{B} + \varepsilon^* \hat{B}^*$ is

$$\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = \left\langle \hat{A}, \hat{B} \right\rangle_D + \varepsilon^* \left( \left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D \right). \tag{3.39}$$

If $\hat{d}_1$ and $\hat{d}_2$ are the directed dual lines in $\mathbb{D}^3$ corresponding, respectively, to the unit hyper-dual vectors $\widetilde{\mathbb{A}}$ and $\widetilde{\mathbb{B}}$, then then, $\hat{A}$ and $\hat{A}^*$represent, respectively, the direction vector and the position of $\hat{d}_1$; and $\hat{B}$, $\hat{B}^*$represent, respectively, the direction vector and the position of $\hat{d}_2$ in $\mathbb{D}^3$. In Equation (3.39), $\left\langle \hat{A}, \hat{B} \right\rangle_D$ is equal to

$$\left\langle \hat{A}, \hat{B} \right\rangle_D = \cos\varphi, \tag{3.40}$$

where $\varphi$ is the dual angle between the unit dual vectors $\hat{A}$ and $\hat{B}$. Let $\hat{M} \in \hat{d}_1$ and $\hat{N} \in \hat{d}_2$ be the two closest points on $\hat{d}_1$ and $\hat{d}_2$. Then, using $\hat{M} \times_D \hat{A} = \hat{A}^*$ and $\hat{N} \times_D \hat{B} = \hat{B}^*$ in Equation (3.39), we get

$$\begin{aligned} \left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D &= \left\langle \hat{A}, \left(\hat{N} \times_D \hat{B}\right) \right\rangle_D + \left\langle \left(\hat{M} \times_D \hat{A}\right), \hat{B} \right\rangle_D \\ &= -\left\langle \hat{N}, \left(\hat{A} \times_D \hat{B}\right) \right\rangle_D + \left\langle \hat{M}, \left(\hat{A} \times_D \hat{B}\right) \right\rangle_D \\ &= \left\langle \left(\hat{M} - \hat{N}\right), \left(\hat{A} \times_D \hat{B}\right) \right\rangle_D . \end{aligned} \tag{3.41}$$

Let $\varphi^*$ be the distance between the points $\hat{M}$ and $\hat{N}$, then we can write

$$\left|\hat{M} - \hat{N}\right|_D = \varphi^*. \tag{3.42}$$

It is obvious that $\varphi^*$ is a dual number, because the modulus of $\hat{M}$ and $\hat{N}$ are dual numbers (see Equation (2.12)). Using $\left|\hat{M} - \hat{N}\right|_D = \varphi^*$ in Equation (3.41), we get

$$\begin{aligned} \left\langle \left(\hat{M} - \hat{N}\right), \left(\hat{A} \times_D \hat{B}\right) \right\rangle_D &= \left\langle \left(\frac{\hat{M} - \hat{N}}{\left|\hat{M} - \hat{N}\right|_D} \varphi^*\right), \left(\hat{A} \times_D \hat{B}\right) \right\rangle_D \\ &= \left\langle \left(\frac{\hat{A} \times_D \hat{B}}{\left|\hat{A} \times_D \hat{B}\right|_D} \varphi^*\right), \left(\hat{A} \times_D \hat{B}\right) \right\rangle_D \\ &= \frac{\varphi^*}{\left|\hat{A} \times_D \hat{B}\right|_D} \left\langle \left(\hat{A} \times_D \hat{B}\right), \left(\hat{A} \times_D \hat{B}\right) \right\rangle_D \\ &= \pm\varphi^* \left|\hat{A} \times_D \hat{B}\right|_D \\ &= \pm\varphi^* \sin\varphi. \end{aligned} \tag{3.43}$$

Inserting Equations (3.40) and (3.43) in Equation (3.39), we also get

$$\begin{aligned} \left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} &= \left\langle \hat{A}, \hat{B} \right\rangle_D + \varepsilon^* \left( \left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D \right) \\ &= \cos\varphi - \varepsilon^* \varphi^* \sin\varphi. \end{aligned} \tag{3.44}$$

By using the Taylor series expansion given by Equation (3.6) in Equation (3.44), we obtain

$$\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = \cos\varphi - \varepsilon^* \varphi^* \sin\varphi = \cos\tilde{\varphi} \tag{3.45}$$

where $\tilde{\varphi} = \varphi + \varepsilon^* \varphi^*$ is a hyper dual angle, see Figure 4. Similarly, the modulus of the vector product of any unit hyper-dual vectors $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ and $\widetilde{\mathbb{B}} = \hat{B} + \varepsilon^* \hat{B}^*$ can be given as

$$\left|\widetilde{\mathbb{A}} \times_{HD} \widetilde{\mathbb{B}}\right|_{HD} = \sin\varphi + \varepsilon^* \varphi^* \cos\varphi = \sin\tilde{\varphi}. \tag{3.46}$$

**Proposition 1.** If $\widetilde{\mathbb{A}}$ and $\widetilde{\mathbb{B}}$ are hyper-dual vectors, then

$$\widetilde{\mathbb{V}} = \frac{\widetilde{\mathbb{A}}}{\left|\widetilde{\mathbb{A}}\right|_{HD}} \text{ and } \widetilde{\mathbb{U}} = \frac{\widetilde{\mathbb{B}}}{\left|\widetilde{\mathbb{B}}\right|_{HD}} \tag{3.47}$$

are unit hyper-dual vectors. From the equations

$$\left\langle \widetilde{\mathbb{V}}, \widetilde{\mathbb{U}} \right\rangle_{HD} = \cos\tilde{\varphi}, \tag{3.48}$$

$$\left|\widetilde{\mathbb{V}} \times_{HD} \widetilde{\mathbb{U}}\right|_{HD} = \sin\tilde{\varphi} \tag{3.49}$$

we can give

$$\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = \left| \widetilde{\mathbb{A}} \right|_{HD} \left| \widetilde{\mathbb{B}} \right|_{HD} \cos \tilde{\varphi}, \tag{3.50}$$

$$\left| \widetilde{\mathbb{A}} \times_{HD} \widetilde{\mathbb{B}} \right|_{HD} = \left| \widetilde{\mathbb{A}} \right|_{HD} \left| \widetilde{\mathbb{B}} \right|_{HD} \sin \tilde{\varphi}, \tag{3.51}$$

where $\tilde{\varphi}$ is a hyper-dual angle.

Let $\tilde{\varphi}$, $\varphi$ and $\theta$ be, respectively, a hyper-dual angle, a dual angle and a real angle. Then, the following four cases can be given related to the hyper-dual angle $\tilde{\varphi}$ by using $\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = \cos \tilde{\varphi} = \cos \varphi - \varepsilon^* \varphi^* \sin \varphi$, where $\cos \varphi = \cos \theta - \varepsilon \theta^* \sin \theta$ and $\sin \varphi = \sin \theta + \varepsilon \theta^* \cos \theta$:

1. If
$$\cos \varphi = 0 \text{ and } \varphi^* \neq 0, \tag{3.52}$$
then $\theta = \frac{\pi}{2}$ and $\theta^* = 0$. Hence, $\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = -\varepsilon^* \varphi^*$. Thus, dual lines $\hat{d}_1$ and $\hat{d}_2$ are perpendicular but not intersecting.
2. If
$$\varphi^* = 0, \tag{3.53}$$
then $\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = \cos \varphi$. Thus, dual lines $\hat{d}_1$ and $\hat{d}_2$ are intersecting.
3. If
$$\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = 0, \tag{3.54}$$
then $\cos \varphi = 0$ and $\varphi^* = 0$. Hence, $\theta = \frac{\pi}{2}$ and $\theta^* = 0$. Thus, $\hat{d}_1$ and $\hat{d}_2$ are perpendicular and intersecting lines.
4. If
$$\left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} = 1, \tag{3.55}$$
then $\theta = 0$ and $\varepsilon^* \varepsilon \varphi^* \theta^* = 0$. Thus, the following two cases can be given:
$(i)$ If $\theta^* = 0$, then $\varphi = 0$. Hence, $\hat{d}_1$ and $\hat{d}_2$ are parallel lines.
$(ii)$ If $\varphi^* = 0$, then $\tilde{\varphi} = \varphi = \varepsilon \theta^*$.

**Example 3.** (Application of hyper-dual angle) Let us take the unit hyper-dual vectors $\widetilde{\mathbb{A}} = (1, \varepsilon, \varepsilon) + \varepsilon^*(\varepsilon, \varepsilon, -1)$ and $\widetilde{\mathbb{B}} = (0, 1, \varepsilon) + \varepsilon^*(2\varepsilon, \varepsilon, -1)$ that can be written in the form $\widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^*$ and $\widetilde{\mathbb{B}} = \hat{B} + \varepsilon^* \hat{B}^*$ for $\hat{A} = (1, \varepsilon, \varepsilon)$, $\hat{A}^* = (\varepsilon, \varepsilon, -1)$, $\hat{B} = (0, 1, \varepsilon)$ and $\hat{B}^* = (2\varepsilon, \varepsilon, -1)$. Hyper-dual angle $\tilde{\varphi}$ between $\widetilde{\mathbb{A}}$ and $\widetilde{\mathbb{B}}$ will be obtained as

$$\begin{aligned} \left\langle \widetilde{\mathbb{A}}, \widetilde{\mathbb{B}} \right\rangle_{HD} &= \left\langle \hat{A}, \hat{B} \right\rangle_D + \varepsilon^* \left( \left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D \right) \\ &= \cos \varphi - \varepsilon^* \varphi^* \sin \varphi \\ &= \cos (\varphi + \varepsilon^* \varphi^*) \\ &= \cos \tilde{\varphi}. \end{aligned} \tag{3.56}$$

Here; $\left\langle \hat{A}, \hat{B} \right\rangle_D$ is equal to

$$\begin{aligned} \left\langle \hat{A}, \hat{B} \right\rangle_D &= \langle (1, \varepsilon, \varepsilon), (0, 1, \varepsilon) \rangle_D \\ &= \varepsilon \end{aligned} \tag{3.57}$$

and from the Equation (2.15), the equality

$$\begin{aligned} \left\langle \hat{A}, \hat{B} \right\rangle_D &= \cos \varphi \\ &= \cos (\theta + \varepsilon \theta^*) \\ &= \cos \theta - \varepsilon \theta^* \sin \theta \end{aligned} \tag{3.58}$$

can be given. Since Equations (3.57) and (3.58) are equal, we get

$$\theta = \frac{\pi}{2} \text{ and } \theta^* = -1 \text{ so } \varphi = \frac{\pi}{2} - \varepsilon. \tag{3.59}$$

We can obtain $\left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D$ as

$$\begin{aligned} \left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D &= \langle (1,\varepsilon,\varepsilon), (2\varepsilon,\varepsilon,-1) \rangle_D + \langle (\varepsilon,\varepsilon,-1), (0,1,\varepsilon) \rangle_D \\ &= \varepsilon. \end{aligned} \tag{3.60}$$

Using the Taylor series expansion given by Equation (2.7), we get

$$\begin{aligned} \sin\varphi &= \sin(\theta + \varepsilon\theta^*) \\ &= \sin\theta + \varepsilon\theta^* \cos\theta. \end{aligned} \tag{3.61}$$

And using Equation (3.59) in Equation (3.61), we obtain

$$\sin\varphi = 1. \tag{3.62}$$

From the equality of the Equations (3.41) and (3.43), we can write

$$\left\langle \hat{A}, \hat{B}^* \right\rangle_D + \left\langle \hat{A}^*, \hat{B} \right\rangle_D = -\varphi^* \sin\varphi. \tag{3.63}$$

Inserting Equations (3.60) and (3.62) in Equation (3.63), we obtain

$$\varepsilon = -\varphi^*. \tag{3.64}$$

Finally; from Equations (3.59) and (3.64), hyper-dual angle $\tilde{\varphi}$ is obtained as

$$\tilde{\varphi} = \left(\frac{\pi}{2} - \varepsilon\right) - \varepsilon^*\varepsilon. \tag{3.65}$$

## 4. Conclusions

In this paper, the basic and kinematic concepts of hyper-dual numbers are given by using properties of dual numbers. The concept "dual line" is defined to represent the geometric interpretation of unit hyper-dual vectors in $\mathbb{D}^3$. Using these concepts, the geometric interpretations of E. Study mapping and hyper-dual angle are given. Furthermore; by taken $\left|\hat{A}^*\right|_D = 1$ in the unit hyper-dual vectors set $\tilde{\mathbb{S}} = \left\{ \widetilde{\mathbb{A}} = \hat{A} + \varepsilon^* \hat{A}^* : \left|\widetilde{\mathbb{A}}\right|_{HD} = 1;\ \hat{A},\ \hat{A}^* \in \mathbb{D}^3 \right\}$, we have defined the subset $\tilde{\mathbb{S}}_1$. And it is shown that there exists a one to one correspondence between the points of the subset $\tilde{\mathbb{S}}_1$ and any two intersecting perpendicular directed lines in $\mathbb{R}^3$.

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