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RESEARCH ARTICLE

Connecting Poincaré Inequality with Sobolev Inequalities on Riemannian Manifolds

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

We connect the Poincaré inequality with the Sobolev inequality on Riemannian manifold in a family of integral inequalities (1.5)**. For these continuum of inequalities, we obtain topological structure theorems of manifolds generalizing previous unification theorems in both intrinsic and extrinsic settings ([33]). Manifolds which admit any of these integral inequalities are nonparabolic, affect topology, geometry, analysis, and admit nonconstant bounded harmonic functions of finite energy. As a consequence, we have proven a Conjecture of Schoen-Yau ([27, p.74]) to be true in dimension two with hypotheses weaker than that used in [1] and [33](which were weaker than the hypotheses set in the conjecture, (cf. Remark 1.5). In the same philosophy and spirit as in ([31]), we prove that if** M **is a complete** n**-manifold, satisfying** (i) **the volume growth condition** (1.1), (ii) **Liouville Theorem for harmonic functions, and either** (v) **a generalized Poincaré- Sobolev inequality** (1.5)**, or** (vi) **a general integral inequality** (1.6)**, and Liouville Theorem for harmonic map** $u : M \to K$ **with** $\mathrm{Sec}^K \leq 0$**, then** (1) M **has only one end and** (2) **there is no nontrivial homomorphism from fundamental group** $\pi_1(\partial D)$ **into** $\pi_1(K)$ **as stated in Theorem** 1.5**. Some applications in geometry** (§3)**, geometric analysis** (§4)**, nonlinear partial differential systems** (§5)**, integral inequalities on complete noncompact manifolds** (§6) **are made** (**cf. e.g., Theorems** 3.1, 4.1, 5.1**, and** 6.1)**. Whereas we made the first study in ([29, 32]) on how the existence of an essential positive supersolution of a second order partial differential systems** $Q(u) = 0$ **on a Riemannian manifold** M**, (by which we mean a** C^2 **function** $v \geq 0$ **on** M **that is positive almost everywhere on** M**, and that satisfies** $Q(v) = \mathrm{div}(A(x, v, \nabla v)\nabla v) + b(x, v, \nabla v)v \leq 0 \quad (5.1)$**) affects topology, geometry, analysis and variational problems on the manifold** M**. Whereas we generate the work in [35], under** p**-parabolic stable condition without assuming the** p**-th volume growth condition** $\lim_{r\to\infty} r^{-p} \mathrm{Vol}(B_r) = 0$**. The techniques, concepts, and results employed in this paper can be combined with those of essential positive supersolutions of degenerate nonlinear partial differential systems** (**cf. for example, Theorems 5.1 - 5.5, 6.1, etc.**) **generalizing previous work in [32, 4.11], which in term recaptures the work of Schoen-Simon-Yau ([25, Theorem 2]). The combined techniques, concepts and method of [32] and [35] can also be used in other new manifolds we found by an extrinsic average variational method ([34]).**

Keywords: *Poincáre inequality, topological end, Sobolev inequality, harmonic function, harmonic map, Ricci curvature.*
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1. Introduction

In [33], we generalize and unify topological structure theorems of complete noncompact Riemannian manifolds in both intrinsic and extrinsic settings. Striking a balance between polar opposites, i.e., two extremes of existence and nonexistence of nonconstant harmonic functions of finite energy or bounded value, we obtain

Theorem A ([33]) *Let* M *be a complete* n*-manifold satisfying*

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(i) *the volume growth condition (on large geodesic balls of radius s):*

$$\mathrm{Vol}(\mathrm{B(s)}) \geq c(n)h(s) \tag{1.1}$$

where $c(n) > 0$ is a constant depending only on n, and $h(s)$ is a continuous function of s with $\lim_{s\to\infty} h(s) = \infty$.

(ii) *Liouville Theorem for harmonic functions: Every harmonic function with finite energy, or every bounded harmonic function on M is constant.*

(iii) *The Poincaré inequality:*

There exists a constant $c(M) > 0$ depending on the geometry of M, such that

$$\int_M \phi^2\, dv \leq c(M) \int_M |\nabla\phi|^2\, dv \tag{1.2}$$

$\forall \phi \in C_0^\infty(M)$, where $n \geq 2$.
Or,

(iii′) *The Isoperimetric inequality:*

There exists a constant $c(n) > 0$ such that

$$(\mathrm{Vol}(\Omega))^{\frac{n-1}{n}} \leq c(n)\,\mathrm{Vol}(\partial\Omega), \tag{1.3}$$

for each open submanifold Ω of M, with compact closure in M, and smooth boundary $\partial\Omega$, where $n > 2$.
Then M has only one end (that is, the complement of any compact set has only one noncompact component).

Theorem A is sharp (cf [33, Remark 4]). By virtue of the Coarea formula, the Isoperimetric inequality (1.3) is equivalent to

(iv) **The Sobolev Inequality**:

There exists a constant $c(n) > 0$ such that $\forall \phi \in C_0^\infty(M)$,

$$\Big(\int_M |\phi|^{\frac{n}{n-1}}\, dv\Big)^{\frac{n-1}{n}} \leq c(n)\Big(\int_M |\nabla\phi|\, dv\Big). \tag{1.4}$$

As we assume that M satisfies **either** the Poincaré inequality (1.2) where $n \geq 2$, **or** the equivalent Sobolev inequality (1.4) via (1.3) where $n > 2$, *a natural question arises*: Whether Theorem A remains to be true for any other point on the straight line segment that connects the Poincaré with the Sobolev in some space of inequalities. More precisely:

Question 1.1. In Theorem A, can one replace (iii) the Poincaré inequality (1.2) and (iv) the Sobolev inequality (1.4) by a more general assumption, which we call

(v) **A generalized Poincaré-Sobolev inequality:**

There exists a constant $C > 0$, $p \in [1, 2]$ such that $\forall \phi \in C_0^\infty(M)$,

$$\Big(\int_M |\phi|^{\frac{pn-2p+2}{n-1}}\, dv\Big)^{\frac{n-1}{pn-2p+2}} \leq C\Big(\int_M |\nabla\phi|^p\, dv\Big)^{\frac{1}{p}}, \tag{1.5}$$

in which $p \in (1, 2)$? (Apparently, when $p = 1$ and $p = 2$, (1.5) becomes the equivalent Sobolev inequality (1.4) of the isoperimetric inequality (1.3), and the Poincaré inequality (1.2) respectively). Furthermore, if there exists such $p \in (1, 2)$, what is the difference between $n = 2$ and $n > 2$?

In the first part of this paper, we answer Question 1.1 in the affirmative for the entire continuum of inequalities (1.5), $p \in [0, 1]$ that connects the Poincaré with the Sobolev inequalities.

Theorem 1.1. *If M satisfies (i), (ii), and (v) the generalized Poincaré- Sobolev inequalities (1.5) in which $n > 2$, for $p \in [1, 2]$, or $n \geq 2$, for $p \in (1, 2]$, Then M has only one end.*

Remark 1.1. *Theorems* 1. 1 *is sharp. For example, catenoids M in $\mathbb{R}^3$ satisfies* (i), (ii), (1.5) *in which* $n = 2$, *and satisfies* (i), (ii), *and* (1.5) *in which* $n \geq 2$, *for* p = 1. *Yet* M *has two ends.*

In fact, Theorem 1.1 remains to be true for more general integral inequalities:

Theorem 1.2. *Let M be a complete n-manifold, satisfying* (i) *the volume growth condition* (1.1), (ii) *Liouville Theorem for harmonic functions, and*

(vi)

A general integral inequality:

There exists a constant $C > 0$ such that

$$\left(\int_M |\phi|^q\, dv\right)^{\frac{1}{q}} \leq C\left(\int_M |\nabla\phi|^p\, dv\right)^{\frac{1}{p}} \tag{1.6}$$

for every $\phi \in C_0^\infty(M)$, and $\frac{2q}{q+2} < p$, where $p \in [1, 2]$.
Then M has only one end.

Remark 1.2. *Theorems* 1. 2 *is sharp. For example, catenoids M in $\mathbb{R}^3$ satisfies* (i), (ii), *and* (1.6) *for every $\phi \in C_0^\infty(M)$, and $\frac{2q}{q+2} = p$, where $p = 2$. Yet M has two ends.*

Remark 1.3. *The general integral inequality* (1.6) *is a generalization of the generalized Poincaré- Sobolev inequality* (1.5), *where $q = \frac{pn-2p+2}{n-1}$, and $p \in [1, 2]$, we note $q = \frac{pn-2p+2}{n-1}$ increases in $\dim M = n\,(\geq 2\,)$. Thus, when* (1.5) *is viewed in terms of* (1.6), *in which $n > 2$, for $p \in [1, 2]$, $q > 2 = \frac{p(2-2p+2)}{2-1}$, When* (1.5) *is viewed in terms of* (1.6), *in which $n \geq 2$, for $p \in (1, 2]$, $q \geq 2 = \frac{p(2-2p+2)}{2-1}$,*

Example 1.1. (1) *Any orientable stable minimal hypersurface in $\mathbb{R}^{n+1}, n > 2$ (cf.* [3])

(2) *Some complete minimal submanifolds in complete manifolds of nonpositive curvature* (*cf.* [33, *Theorem* 1])

(3) *Some complete submanifolds with small mean curvature* (*in fact, bounded mean curvature in an L^n sense*) *in complete manifolds of nonpositive curvature. (cf. [33, Theorem 2])*

(4) *Some minimal hypersurfaces in $\mathbb{R}^{n+1}$ with Gauss image lying in an open hemisphere, and with p-th volume growth*

$$\mathrm{Vol}(B_r) = o(r^p) \quad \text{as} \quad r \to \infty, \quad \text{i. e.,} \quad \lim_{r\to\infty} \frac{\mathrm{Vol}(B_r)}{r^p} = 0, \quad p \in \left[4 + \sqrt{\frac{8}{n}}\right) \tag{0.1}$$

(cf. [32, Theorem 5.13]).

(5) *Some hypersurfaces of constant mean curvature in $\mathbb{R}^{n+1}$ with appropriate scalar curvature, Gauss image and the p-th volume growth* (0.1) (*cf.* [32, *Theorem 5.15*]).

We will introduce the notion of μ-integral inequality, derive and use the following crucial Transformation Lemma (vi) $\Longrightarrow$ (vii) that reduces the general integral inequality (1.6) to μ-integral inequality (1.7).

Lemma 1.1. *If* (vi) *the general integral inequality* (1.6) *holds for every $\phi \in C_0^\infty(M)$, $\frac{2q}{q+2} < p$, where $p \in [1, 2]$, then*

(vii)

μ-integral inequality:

$$\left(\int_M |\phi|^{\frac{2pq}{pq+2p-2q}}\, dv\right)^{\frac{pq+2p-2q}{pq}} \leq \mu^2 C^2 \int_M |\nabla\phi|^2\ \ dv \tag{1.7}$$

holds for every $\phi \in C_0^\infty(M)$, $\frac{2q}{q+2} < p$, where $\mu = \frac{2p}{2p+pq-2q}$ and $p \in [1, 2]$.

To see this, we observe $\frac{2p}{2p+pq-2q} > 0$, substitute ϕ for $\varphi = \phi^\mu$ into (1.6), and apply Hölder inequality. (cf. Proof of Lemma 1.1 in § 2 for details).

Example 1.2. *If* (vi) *the general integral inequality* (1.6) *holds for every* $\phi \in C_0^\infty(M)$, $\frac{2q}{q+2} = 1 < 2 = p = q$, *then*

(vii) **μ-integral inequality**

becomes $\mu = 1$ and Poincaře inequality $(1.2')$

$$\int_M |\phi|^2 \, dv \le C^2 \int_M |\nabla \phi|^2 \; dv \, . \tag{1.2'}$$

In the course of connecting the Poincaré inequality with the Sobolev inequality in the proof of Theorems 1.1 and 1.2, we have proved the following *existence theorem* of nonconstant harmonic functions under topological assumptions:

Theorem 1.3 (Existence Theorem)**.** *Let M be a complete n-manifold such that M admits general inequalities* (1.6) *for any $\frac{2q}{q+2} < p$, where $p \in [1,2]$, and has at least two ends each with infinite volume. Then M admits a nonconstant bounded harmonic function with finite energy.*

The process of connecting the Poincaré and the Sobolev inequalities in a space of general integral inequality (1.6) preserves nonparabolicity and other analytic and geometric properties:

Theorem 1.4. *Let M be a complete n-manifold such that M satisfies the general integral inequality* (1.6) *for any $\frac{2q}{q+2} < p$, where $p \in [1,2]$.*

Then

(a) *M is noncompact. and has infinite volume*

(b) *M is nonparabolic.*

(c)
$$\mathrm{Vol}(\{y \in M : G_x^\Omega(y) > t\}) \le (\mu C)^{\frac{2pq}{pq+2p-2q}}\, t^{-\frac{pq}{pq+2p-2q}}\, , \tag{1.8}$$
for any $x \in M$, any $t > 0$, where G_x is the positive minimal Green's function of pole x.

(d) *If $n = 2$, then M admits a nonconstant bounded harmonic function.*

Remark 1.4. *The case $p = q = 2$, $n = 2$, $\mu = 1$ in Theorem 1.4 is due to S.W. Wei* [33].

There are manifolds supporting nonconstant bounded harmonic functions, but for which every nonconstant harmonic functions has infinite energy [24]. Hence, combining Theorem 1.4.(d) and Theorem 1.3, we have

Corollary 1.1. (a) *Let M be a complete surface satisfying the general integral inequality* (1.6) *for any $\frac{2q}{q+2} < p$, where $p \in [1,2]$. Then M admits a nonconstant bounded harmonic function.*

(b) *If we assume further that M has at least two ends, and* (i) *the volume growth condition* (1.1)*, then M admits a nonconstant bounded harmonic function with finite energy.*

(c) *If we also assume further that* (ii) *Liouville Theorem for harmonic function, then M has only one end.*

This generalizes and extends [33, Corollary 1].

On the other hand, by considering the general integral inequalities (1.6) *in the large* or in a space of integral inequalities, we have proven the following Conjecture of Schoen-Yau to be true in dimension two with hypotheses weaker than that used in [1] and [33], (which were weaker than the hypotheses set in the conjecture, cf. Remark 1.5).

Conjecture 1. ([27, *Schoen-Yau, p.74*]) *Let M^n be a simply-connected, complete Riemannian manifold with bounded sectional curvature $|K_M| \le 1$. Suppose that M^n has positive bottom of the spectrum $\lambda_1(M^n) > 0$ and positive injectivity radius Inj $(M^n) > 0$. Then M^n admits a nonconstant bounded harmonic function.*

Remark 1.5. *Theorem 1.4.*(d) *shows that Conjecture 1 is true in dimension $n = 2$, even without the original hypotheses that* (a) *M is simply-connected,* (b) *M has bounded sectional curvature $|K_M| \le 1$, and* (c) *M has positive injectivity radius, and* (d) *M is replaced with a more general assumption, the general integral inequality* (1.6)*, then the hypothesis $\lambda_1(M^2) > 0$. Furthermore, Conjecture 1 has been settled by I. Benjamini and J. G. Cao* [1] *in dimension $n > 2$ by their counterexamples.*

Just as striking a delicate balance between two polar opposites of existence and nonexistence of *harmonic functions* yields information on topology, so does averaging or striking a balance between two extremes of existence and nonexistence of *harmonic maps*. In the same philosophy and spirit as in [31], in particular, using harmonic functions $\big($for the conclusion $(1)\big)$ and harmonic maps $\big($for the conclusion $(2)\big)$ as catalysts, we have

Theorem 1.5. *Let M be a complete n-manifold, satisfying* (vi) *general integral inequality* (1.6), (i) *the volume growth condition* (1.1), (ii) *Liouville Theorem for harmonic functions, and M does not admit any nonconstant harmonic map $u : M \to K$ with finite energy, where K is compact with sectional curvature $\mathrm{Sec}^K \le 0$. Then*

(1) *M has only one end.*

(2) *For any compact domain D in M with continuous ∂D and $\pi_1(\partial D) = 0$ (∂D is not necessarily smooth and connected), there is no non-trivial homomorphism from the fundamental group $\pi_1(\partial D)$ into $\pi_1(K)$.*

Remark 1.6. *The case $1 \le p \le 2$, $q = \frac{n}{n-1}$, $n > 2$, and the case $p = q = 2$, $n \ge 2$ are treated in* [32, *Theorem 5.19*] *and* [33, *Theorem 5*].

As an application of Theorem 1.5, we have

Corollary 1.2. *Let M be a complete manifold, and K be a compact manifold with $\mathrm{Sec}^K \le 0$. Suppose there is no nonconstant harmonic map $u : M \to K$ with finite energy. Then for any compact domain D in M with continuous ∂D and $\pi_1(\partial D) = 0$ (∂D is not necessarily smooth and connected), $\pi_1(D)$ is finitely generated, and neither $\pi_1(D)$ nor any quotient group of $\pi_1(D)$ by any normal subgroup* H *of $\pi_1(D)$ can be isomorphic to the direct sum of any finite copies of Z. Furthermore, the index of the commutator subgroup $[\pi_1(D), \pi_1(D)]$ in $\pi_1(D)$ is finite.(cf.* [33, Corollary 3])

In the second part of this paper, we study some applications in geometry and geometric analysis. We prove, for example

Theorem 3.1 *Let M be a complete n-manifold, satisfying volume growth rate,* (i′), $\mathrm{Vol}\ (\mathrm{B(r)}) \ge c(n)\, r^n$, *where $c(n) > 0$ is a constant depending on n, and* (vi) *the general integral inequality* (1.6) *for any $\frac{2q}{q+2} < p$, where $p \in [1, 2]$. If G is uniform bounded in M, where G is as in* (3.1), *then M has only one end.*

Theorem 4.1 *Let M be a complete n-manifold with $\mathrm{Ric}^M \ge 0$ such that* (1.6) *holds for any $\frac{2q}{q+2} < p$, where $p \in [1, 2]$. Then*

(a) $\int_1^\infty \frac{s}{\mathrm{Vol}(B(s))} ds < \infty$.

(b) *M has only one end.*

(c) *For any compact domain D in M with continuous ∂D and $\pi_1(\partial D) = 0$ (∂D is not necessarily smooth and connected), there is no non-trivial homomorphism from the fundamental group $\pi_1(\partial D)$ into $\pi_1(K)$, where K is compact with $\mathrm{Sec}^K \le 0$.*

In the third part of paper, we discuss some application in nonlinear degenerate partial differential systems.

It is known in ([29, 32]), S.W. Wei made the first study on how the existence of **essential positive supersolutions** of nonlinear degenerate partial differential systems on a Riemannian manifold affects the topology, geometry, and analysis of the manifold. By an essential positive supersolution of a second order partial differential systems $Q(u) = 0$ on a Riemannian n-manifold M, we mean a C^2 nonnegative function v on M which is positive almost everywhere on M, and which satisfies

$$Q(v) = \mathrm{div}(A(x, v, \nabla v)\nabla v) + b(x, v, \nabla v)v \le 0 \tag{5.1}$$

on M, where A denotes a smooth cross section in the bundle whose fiber at each point x in M is a **nonnegative** linear transformation on the tangent space $T_x(M)$ into $T_x(M)$, b is a smooth real-valued function, and ∇v denotes the gradient of v.

Recall a complete noncompact Riemannian manifold M^n is said to be p**-parabolic** if it admits no nonconstant positive p-superharmonic function. In [35], S.W. Wei, J.F. Li and L.N. Wu prove that a manifold with p-moderate volume growth must be p-parabolic, (generalizing a result of S.Y. Cheng and S.T. Yau [9] for the case $p = 2$, $F(r) \equiv 1$), and construct an example of a p-parabolic manifold with exponential volume growth. Furthermore, we generate the work of Schoen-Simon-Yau under p-parabolic stable condition without assuming p-th volume growth condition $r^{-p}\,\mathrm{Vol}(B_r) \to 0$ as $r \to \infty$ (0.1). Our techniques, concepts, and results can be combined with those of essential positive supersolutions of degenerate nonlinear partial differential system, for example, (in Theorems 5.1 - 5.5, 6.1, etc.), and can be used in other new manifolds we found by an intrinsic average variational method ([34]).

Theorem 5.1 *Let M be a minimal hypersurface in a manifold N^{n+1} with constant sectional curvature $K_2 \geq 0$. Suppose M admits an essential positive supersolution of $Q(u) = 0$ with coefficients satisfying conditions*

(I) $||A|| \leq c_1$, *i.e.* $\langle A\nabla v, \nabla v\rangle \leq c_1|\nabla v|^2$, *and*

(VII) $b(x, u, \nabla u) \geq c_2|B|^2 + nK_2 \quad with \quad c_1 = c_2 = 1,$

where B is the second fundamental form of M in N.

If M is p-parabolic, where $p \in [4, 4 + \sqrt{8/n}\,)$, then M is totally geodesic.

Theorem 5.1 generalizes the work of Wei-Li-Wu [35, 4.11], which recaptures the work of Schoen-Simon-Yau for the case $A(x, u, \nabla u) =$ Identity map, $b(x, u, \nabla u) = |\nabla\nu|^2 + nK_2$, and $r^{-p}(B_r) \to 0$ as $r \to \infty \quad (0.1)$ in [25, Theorem 2].

In the fourth part of paper, we discuss integral inequalities on Riemannian manifolds and their links to curvature, essential positive supersolutions of partial differential systems, topology, variational problems, potential theory, etc. Based on the work of Blanc-Fiala-Huber [18], Greene-Wu [14], Theorem 1.4.(b), Varopoulos [28], Karp [19], Wei-Li-Wu [35, §2, §3], Milnor [21], Wei [32, 33, 31], Cheng-Yau [9], Do Carmo [10], Fischer-Colbrie - Schoen [11], Pogorelov, [23], Hoffman-Ossorman-Schoen [16], etc, we prove, for example,

Theorem 6.1 *Let M be a complete surface in $\mathbb{R}^3$ with a unit normal vector $\nu = (\nu_1, \nu_2, \nu_3)$. Suppose M satisfies the general integral inequality (1.6) for any $\frac{2q}{q+2} < p$, where $p \in [1, 2]$ Then M does not admit any essential positive supersolutions of*

$$Q(v) = \operatorname{div}(A(x, v, \nabla v)\nabla v) + b(x, v, \nabla v)v \leq 0 \tag{5.1}$$

with coefficients satisfying

(I) $||A|| \leq c_1$, *i.e.* $\langle A\nabla v, \nabla v\rangle \leq c_1|\nabla v|^2$, *and*

(IV) $b(x, u, \nabla u) \geq c_2|\nabla\nu|^2 \quad with \quad 0 < c_1 < 3c_2.$

Thus M can not be any of the following:

(a) *a stable minimal surface in $\mathbb{R}^3$,*

(b) *a surface of constant mean curvature in $\mathbb{R}^3$, such that the image under its Gauss map lies in an open hemisphere,*

(c) *the graph of a real-valued function on $\mathbb{R}^2$ with prescribed mean curvature function $\tilde{H} : \mathbb{R}^3 \to \mathbb{R}$ satisfying $\frac{\partial \tilde{H}}{\partial x_1} = \frac{\partial \tilde{H}}{\partial x_2} = 0$. and $\frac{\partial \tilde{H}}{\partial x_3} \leq 0$.*

2. Proofs of Lemma 1.1, and Theorems 1.1-1.5

We introduced the notion of μ-integral inequality (1.7) in §1. Now we begin with the Proof of the transformation Lemma 1.1 (vi) $\implies$ (vii), which reduces a general integral inequality (1.6) to μ-integral inequality (1.7).

Proof of Lemma 1.1. Since $p \in [1, 2]$,

$$\begin{aligned} \frac{2q}{q+2} < p &\Longrightarrow 2p + qp - 2q > 0 \\ &\Longrightarrow \frac{2p}{2p + pq - 2q} > 0, \end{aligned}$$

one can select $\mu = \frac{2p}{2p+pq-2q} > 0$ and substitute ϕ for $\varphi = \phi^\mu$ into (1.6). We obtain

$$\Big(\int_M |\varphi|^q \;dv\Big)^{\frac{1}{q}} \leq \mu C\Big(\int_M |\phi|^{(\mu-1)p}|\nabla\phi|^p \;dv\Big)^{\frac{1}{p}}. \tag{2.1}$$

Applying Hölder inequality to the rigt-hand side of (2.1), we have

$$\begin{aligned}\left(\int_M |\phi|^{\mu q}\, dv\right)^{\frac{1}{q}} &\leq \mu C\left(\int_M |\phi|^{(\mu-1)p\cdot\frac{\mu q}{(\mu-1)p}}\, dv\right)^{\frac{(\mu-1)}{\mu q}} \\ &\quad \cdot \left(\int_M |\nabla\phi|^{p\cdot\frac{\mu q}{\mu(q-p)+p}}\, dv\right)^{\frac{\mu(q-p)+p}{\mu p q}}.\end{aligned} \tag{2.2}$$

Observing the exponent $p\cdot\frac{\mu q}{\mu(q-p)+p}=2$ in (2.2), we acquire

$$\left(\int_M |\phi|^{\mu q}\, dv\right)^{\frac{1}{q}} \leq \mu C\left(\int_M |\phi|^{\mu q}\, dv\right)^{\frac{(\mu-1)}{\mu q}} \cdot \left(\int_M |\nabla\phi|^2\, dv\right)^{\frac{1}{2}}. \tag{2.3}$$

Dividing and squaring, we have the desired

$$\left(\int_M |\phi|^{\frac{2pq}{pq+2p-2q}}\, dv\right)^{\frac{pq+2p-2q}{pq}} \leq \mu^2 C^2 \int_M |\nabla\phi|^2\ dv. \tag{1.7}$$

□

Proof of Theorems 1.1 and 1.2 We first prove Theorem 1.2: Suppose M has more than one end, i.e. M is covered by an exhaustion $\{D_i\}$ of M, $M\backslash D_i=\cup_{j=1}^{s}F_j^{(i)}$ is the disjoint union of connected components $F_j^{(i)}, j=1,\ldots s$ with $s\geq 2$. Let $F_1^{(i_0)}$ and $F_2^{(i_0)}$ have infinite volume. For each $i\geq i_0$, let $u_i: D_i\to\mathbb{R}$ be the minimizer of the energy functional $E(u_i)=\frac{1}{2}\int_{D_i}|du_i|^2\, dv$ among all functions u such that $u|_{\partial F_1^{(i)}}=1$ and $u|_{\partial F_k^{(i)}}=0$ for all $k\geq 2$. Then by the maximum principle of harmonic functions, $0\leq u_i\leq 1$. For any $j<i$, we extend u_j to $\overline{u}_j: D_i\to\mathbb{R}$ continuously such that $\overline{u}_j=1$ or 0 on the complement D_i-D_j. Then $\overline{u}_j$ has the same boundary condition as u_i on ∂D_i. Hence by the minimality of the energy $E(u_i)$ of u_i over D_i, one has the following monotonicity:

$$\begin{aligned}\int_{D_i}|\nabla u_i|^2\, dv &\leq \int_{D_i}|\nabla\overline{u}_j|^2\, dv \\ &= \int_{D_j}|\nabla u_j|^2\, dv \quad \text{for} \quad i>j.\end{aligned}$$

Thus there exists a constant $c_1>0$ such that

$$\int_{D_i}|\nabla u_i|^2\, dv\leq c_1 \quad \text{for} \quad i>i_0$$

Therefore, we can find a harmonic function u on M such that

$$\lim_{i\to\infty}u_i(x)=u(x), \quad \forall x\in M,$$

$0\leq u\leq 1$ and $\int_M|\nabla u|^2\, dv\leq c_1$.

By virtue of the Transformation Lemma 1.1, we may assume μ-integral inequality (1.7) holds for $\frac{2q}{q+2}<p$, where $p\in[1,2]$.

Substituting $\phi=u_i(1-u_i)$ into (1.7), we have

$$\begin{aligned}\left(\int_{D_i}(u_i(1-u_i))^{\frac{2pq}{pq+2p-2q}}\, dv\right)^{\frac{pq+2p-2q}{pq}} &\leq \mu^2C^2\int_{D_i}2|\nabla u_i|^2(1-u_i)^2 \\ &\quad + 2u_i^2|\nabla u_i|^2\, dv \\ &\leq \mu^2C^2c_2,\end{aligned} \tag{2.4}$$

where $c_2=4c_1$. By assumption (i), $\mathrm{Vol}(D_i)\to\infty$, as $i\to\infty$, hence we find that if u is a constant, then $u\equiv 0$ or $u\equiv 1$. If $u\equiv 1$, we choose $\phi=u_i\psi$ where

$$\psi=\begin{cases}1 & \text{on } F_2^{(i_0)}\\ 0 & \text{on } \quad F_k^{(i_0)},\ k\neq 2\end{cases}$$

$|\nabla\psi| \le c_3, \quad 0 \le \psi \le 1$, and $|\nabla\psi|$ vanishes outside D_{i_0}, then inequality (1.7) implies that

$$\begin{aligned}\Big(\int_{D_i} (u_i\psi)^{\frac{2pq}{pq+2p-2q}}\, dv\Big)^{\frac{pq+2p-2q}{pq}} &\le \mu^2 C^2 \int_{D_i} 2\psi^2|\nabla u_i|^2 + 2u_i^2|\nabla\psi|^2\, dv \\ &\le c_4\,,\end{aligned} \tag{2.5}$$

where the constant $c_4 = 2\mu^2C^2c_1 + 2\mu^2C^2c_3^2\cdot \mathrm{Vol}(D_{i_0})$. It follows that

$$\Big(\int_{F_2^{(i_0)}\cap D_i} u_i^{\frac{2pq}{pq+2p-2q}}\, dv\Big)^{\frac{pq+2p-2q}{pq}} \le c_4\,.$$

As $i \to \infty$, we find that

$$\begin{aligned}&\mathrm{Vol}(F_2^{(i_0)})^{\frac{pq+2p-2q}{pq}} \\ &= \Bigg(\int_{F_2^{(i_0)}} u^{\frac{2pq}{pq+2p-2q}}\, dv\Bigg)^{\frac{pq+2p-2q}{pq}} \le c_4\,,\end{aligned}$$

which contradicts a consequence of (i) (cf. [33]). Similarly, $u \equiv 0$ can not happen by replacing u and u_i by $1-u$ and $1-u_i$, respectively in the same argument. Consequently u is not a constant. This contradicts (ii) and *proves* the assertion in the general case (1.6), or Theorem 1.2.

Proof of Theorem 1.1. This follows from Remark 1.3. Indeed, for the case $q = \frac{pn-2p+2}{n-1}$ in (1.5) and $p = 1$, $q = \frac{1\cdot n - 2\cdot 1 + 2}{n-1} = \frac{n}{n-1}$. Hence,

$$\frac{2q}{q+2} = \frac{2\cdot\frac{n}{n-1}}{\frac{n}{n-1}+2} = \frac{2n}{3n-2} < 1 = p \le 2,$$

since $n > 2$.
Similarly, for $p \in (1,2]$, $n > \frac{2p}{3p-2}$ and $q = \frac{pn-2p+2}{n-1} > \frac{1\cdot n-2\cdot 1+2}{n-1} = \frac{n}{n-1}$. Hence,

$$\begin{aligned}&\frac{2p}{2-p} > q > \frac{n}{n-1} \quad \text{for} \quad p > 1 \\ &\Longrightarrow pq + 2p > 2q \quad \text{and} \quad n > \frac{2p}{3p-2} \quad \text{for} \quad p > 1 \\ &\Longrightarrow \frac{2q}{q+2} < p \quad \text{and} \quad n \ge \sup_{p>1} \frac{2p}{3p-2} = 2\end{aligned}$$

Now we are ready to apply Theorem 1.2 and *complete the proof of* Theorem 1.1.

Just as the Poincaré and the Sobolev inequalities preserve the nonparabolicity and other properties, so does the general inequality (1.6) (cf. Theorem 1.4). Now we give a

Proof of Theorem 1.4. (a). If M were compact, then substituting $\phi \equiv 1$ into (1.6) would lead to $\mathrm{Vol}(M) = 0$, a contradiction. Similarly, if M had finite volume, then substituting

$$\phi = \begin{cases} 1 & \text{on} \quad B(r) \\ 0 & \text{off} \quad B(2r) \end{cases} \tag{2.6}$$

and $|\nabla\phi| < \frac{1}{r}$ into (1.6), would lead to

$$\mathrm{Vol}(B(r)) \le \int_M \phi^2\, dv \le \frac{C}{r^2}\mathrm{Vol}(M) \tag{2.7}$$

which in turn would lead to a contradiction $\mathrm{Vol}(M) = 0$, by letting $r \to \infty$ (cf. [32, p.157]). This proves (a).

Now use the idea in [33, p.686], let Ω be a regular open bounded subset of M such that $x \in \Omega$, and let G be the solution of $\triangle_M G = \delta_x$ in Ω, and $G = 0$ on Ω. Set $G_x^\Omega(y) = G(y)$ if $y \in \Omega$, and $G_x^\Omega(y) = 0$ otherwise. Since $\frac{2q}{q+2} < p$, where $p \in [1,2]$, we apply Lemma 1.1, substitute ϕ for $\phi_t(y) = \min\{G_x^\Omega(y), t\}$, where $t > 0$ into (1.7), and obtain

$$\begin{aligned}&\mathrm{Vol}(\{y\in M : G_x^\Omega(y) > t\})^{\frac{pq+2p-2q}{pq}}\, t^2 \\ &= \Big(\int_{\{y\in M: G_x^\Omega(y)>t\}} t^{\frac{2pq}{pq+2p-2q}}\, dv\Big)^{\frac{pq+2p-2q}{pq}} \\ &\le \mu^2C^2\int_M |\nabla\phi_t|^2\, dv.\end{aligned} \tag{2.8}$$

On the other hand, in view of the Divergence Theorem and the fact that $\triangle_M G_x^\Omega(y) = 0$ in $\Omega\backslash\{x\}$,

$$\begin{aligned}\int_M |\nabla\phi_t|^2\, dv &= \int_{\{y\in M: G_x^\Omega(y)<t\}} |\nabla G_x^\Omega(y)|^2\, dv \\ &= -t\int_{\{y\in M: G_x^\Omega(y)=t\}} \frac{\partial G_x^\Omega}{\partial n}\, ds \\ &= t\end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$\mathrm{Vol}(\{y\in M : G_x^\Omega(y) > t\}) \le (\mu C)^{\frac{2pq}{pq+2p-2q}} t^{-\frac{pq}{pq+2p-2q}} \tag{2.10}$$

for any $x \in M$, any $t > 0$, and any bounded domain Ω in M with $x \in \Omega$. It follows that M admits the positive minimal Green's function G_x of pole x, where $G_x = \sup_{x\in\Omega} G_x^\Omega(y) < \infty, \forall y \in \Omega\backslash\{x\}$. That is, M is nonparabolic. This proves (b) and (c). Now we use the assumption $n = 2$, in applying the Uniformization theorem to the universal cover $\tilde{M}$ of M. Consequently, $\tilde{M}$ is conformally equivalent to the unit disk, and hence $\tilde{M}$ admits nonconstant bounded harmonic functions, and so does M. This completes the proof of (d). □

Proof of Theorem 1.5

(1) This follows immediately from Theorems 1.1, 1.2, 1.3, or [33, Theorem 5].

(2) Since K is compact and $\mathrm{Sec}^K \le 0$, K is $K(\pi,1)$ and for any homomorphism $f : \pi_1(D) \to \pi_1(K)$, there exists a continuous map $u : D \to K$ such that $u_* = f$. Then as $\pi_1(\partial D) = 0$ and the universal cover $\widetilde{K}$ of K is contractible, for each component C_i of ∂D, $1 \le i \le k$ there exists a continuous lifting $\widetilde{u}_i$ of $u|_{C_i}$ such that $\widetilde{u}_i : C_i \to \widetilde{K}$ is homotopic to a constant map $\widetilde{c}_i \equiv \widetilde{y}_0$ for some fixed point $\widetilde{y}_0 \in \widetilde{K}$. Extend $\widetilde{u}_i$ (resp. $\widetilde{c}_i$) to $\widetilde{u}$ (resp. $\widetilde{c}$) defined on $\partial D = C_1 \cup \cdots \cup C_k$. Then $\widetilde{u}$ is homotopic to $\widetilde{c} \equiv \widetilde{y}_0$. Hence there exists a continuous homotopy $\mathsf{F} : \partial D \times [0,1] \to K$ such that $\mathsf{F}(x,0) = u(x)$ and $\mathsf{F}(x,1) \equiv y_0$ via the covering map. Let $(D \subset)D'$ be a compact domain in M such that $D' - int\ \ D$ is a collar of ∂D, and $\mathsf{g} : D' - int\ \ D \to \partial D \times [0,1]$ be the corresponding homomorphism with $\mathsf{g}(\partial D) = \partial D \times \{0\}$ (cf. [22, 2.26, p. 26] for the existence of the collar). Then one can extend u to a continuous map $\overline{u} : M \to K$ by defining $\overline{u} = \mathsf{F}\circ\mathsf{g}$ on $D' - int\ \ D$ and $\overline{u} \equiv y_0$ off D'. It follows that there exists a smooth map $u_0 : M \to K$ such that u_0 is homotopic to $\overline{u}$ and $u_0(x) = \overline{u}(x) \equiv y_0$ for $x \notin D''$, where $(D' \subset\subset)D''$ is a compact domain in M. Thus $E(u_0) < \infty$ and $(u_{0|_D})_* = f$. By an existence theorem (cf. [26, 2]), in the homotopy class of u_0, there is a smooth harmonic map h with $E(h) \le E(u_0) < \infty$. By the nonexistence theorem of harmonic map $u : M \to K$ with finite energy in the assumption, we find that h is a constant and hence f is trivial. This completes the proof of (2) (cf. [32, Corollary 5.10]).

Proof of Corollary 1.2

Since K is compact with $\mathrm{Sec}^K \le 0$, $\pi_1(K)$ contains no element of finite order other than the identity (see e.g. [30, p.667] for the proof by p-harmonic maps). If there were a compact domain $D \subset M$ such that $\pi_1(D) \simeq Z\bigoplus\cdots\bigoplus Z$, then there would exist a nontrivial homomorphism from $\pi_1(D)$ into $\pi_1(K)$ by assigning only one generator $1 \in Z$ to any nonidentity element in $\pi_1(K)$, all other $1 \in Z$ the identity element, and extending it linearly. This contradicts Theorem 1.5.(2). Similarly, via the projection map from $\pi_1(D)$ onto $\pi_1(D)/\mathsf{H}$, one can show $\pi_1(D)/\mathsf{H} \simeq Z\bigoplus\cdots\bigoplus Z$. Furthermore, since D is a compact domain in M, $\pi_1(D)$ is finitely generated. In view of the fundamental theorem of finitely generated abelian group, the commutator quotient group $\pi_1(D)/[\pi_1(D),\pi_1(D)]$ is torsion and finite. This completes the proof. (cf. [33, Corollary 3])

3. Applications in Geometry

In [33], we prove

Theorem 3.1. *Let M be a complete minimal n-submanifold in a complete simply-connected manifold of nonpositive sectional curvature, where $n > 2$. If G is uniformly bounded in M, then M has only one end. In particular, if $\mathrm{Ric}^M\left(\sigma'(t)\right)t^{2+\epsilon}$ is uniformly bounded for some $\epsilon > 0$, then M has only one end.*

We recall G is a real-valued function defined on a compact manifold M with boundary ∂M , such that

$$G = \sup_\sigma\ \exp\left(\int_0^\ell \left(-\frac{1}{t^2}\int_0^t \mathrm{Ric}^M(\sigma'(\tau))\ \tau^2\, d\tau\right) dt\right) \tag{3.1}$$

where σ ranges over all minimal geodesic segments in M with length $\ell \leq \frac{2}{3}r$, r is the distance $d(x_0, \partial M)$, and $d(x_0, \sigma(0)) \leq \frac{1}{3}r$. (cf. [37, p. 505])

In this section, we extend the result on minimal submanifolds to manifolds admitting the general integral inequality (1.6).

Theorem 3.2. *Let M be a complete n-manifold, admitting* (vi) *the general integral inequality* (1.6). *If G is uniform bounded in M with volume growth rate,*

$$(\mathrm{i}')\ \mathrm{Vol}((B(r)) \geq c(n)\, r^n$$

for some constant $c(n) > 0$ depending on n. Then M has only one end.

Proof. Suppose contrary, M had at least two ends, then proceed as in the proof of Theorems 1.1 and 1.2, M would admit a nonconstant bounded harmonic function. On the other hand, for every Lipschitz function f,

$$\inf_{\beta\in\mathbb{R}} \int_{B_{x_0}(\frac{1}{3}r)} |f-\beta| \leq \frac{4r^{n+1}}{3} n\omega(n)\,\mathrm{Vol}\left(B_{x_0}(\frac{1}{3}r)\right)^{-1} G \int_{B_{x_0}(\frac{2}{3}r)} |\nabla f|. \tag{3.2}$$

where $B_{x_0}(s)$ be a geodesic ball of M of radius s centered at x_0, and $\omega(n)$ is the volume of the unit ball in $\mathbb{R}^n$. (cf. [37, 6.24, p. 505], [33, (2), p. 676]).

Substituting the volume growth rate (i') $\mathrm{Vol}((B(r)) \geq c(n)\, r^n$ into (3.2), one has

$$\inf_{\beta\in\mathbb{R}} \int_{B_{x_0}(\frac{1}{3}r)} |f-\beta| \leq c_n r G \int_{B_{x_0}(\frac{2}{3}r)} |\nabla f|. \tag{3.3}$$

where $c_n > 0$ is a constant depending only on n.

In view of Lemma 1.1, we have μ-integral inequality (1.7). One can apply standard De Giorge-Nash-Moser result ([13]) to prove a *Harnack inequality* on M. Since M is complete and G is uniformly bounded on M, such a Harnack inequality will imply *Liouville theorem* on M. This contradicts the existence theorem and completes the proof.

□

4. Applications in Geometric Analysis

Gromoll and Meyer [15] prove that every manifold with positive Ricci curvature has only one end. It follows from the splitting theorem of Cheeger-Gromoll [6] that every manifold with nonnegative Ricci curvature has at most two ends. In this section we first prove that if manifolds with nonnegative Ricci curvature support the integral inequality (1.6), then M has only one end. We then weaken the condition of nonnegative Ricci curvature to the condition that along each geodesic ray the integral of the Ricci curvature is nonnegative.

Theorem 4.1. *Let M be a complete n-manifold with $\mathrm{Ric}^M \geq 0$ such that (1.6) holds for any $\frac{2q}{q+2} < p$, where $p \in [1,2]$. Then*

(a) $\int_1^\infty \frac{s}{\mathrm{Vol}(B(s))} ds < \infty$.

(b) M has only one end.

(c) For any compact domain D in M with continuous ∂D and $\pi_1(\partial D) = 0$ (∂D is not necessarily smooth and connected), there is no non-trivial homomorphism from the fundamental group $\pi_1(\partial D)$ into $\pi_1(K)$, where $\mathrm{Sec}^K \leq 0$.

First Proof. Conclusion (a) follows from Bishop Volume Comparison Theorem. To prove Conclusion (b), suppose there were more than two ends. Since $\mathrm{Ric}^M \geq 0$, each end of M has infinite volume, it follows from Theorem 1.4 that there would exist a bounded nonconstant harmonic function with finite energy. This contradicts both a Theorem of Yau [36] on the boundedness, and a Theorem of Greene and Wu [14] on the finiteness. Conclusion (c) follows from Theorem 1.5.(2) due to nonexistence of harmonic maps $u : M \to K$ of finite energy with $\mathrm{Sec}^K \leq 0$. □

Second Proof. Since $\mathrm{Ric}^M \geq 0$, by the Cheeger-Gromoll Splitting Theorem ([5]), M has at most two ends. If M had two ends, then M would be isometric to $\mathbb{R} \times N$, where N is a compact set. This implies that M would have linear volume growth and would be parabolic, contradicting Theorem 1.4.(b). Conclusion (c) follows from Theorem 1.5.(2) due to nonexistence of harmonic maps $u : M \to K$ of finite energy with $\mathrm{Sec}^K \leq 0$. □

Theorem 4.2. *Let M be a complete n-manifold with $\mathrm{Ric}^M \geq 0$ such that $\int_1^\infty \frac{s}{\mathrm{Vol}(B(s))} ds < \infty$ holds. Then M has only one end.*

Proof. If M had two ends, then M would be isometric to $R \times N$, where N is a compact set. This in turn would imply $\int_1^\infty \frac{s}{\mathrm{Vol}(B(s))} ds = \infty$ which contradicts the hypothesis. □

Apparently, the converse of Theorem 4.2 does not hold, as $\mathbb{R}^2$ has only one end, but $\int_1^\infty \frac{s}{\mathrm{Vol}(B(s))} ds = \infty$.

Analogously, one has

Theorem 4.3. *Let M be a complete n-manifold such that (1.6) holds for any $\frac{2q}{q+2} < p$, where $p \in [1, 2]$. Suppose along each ray $\sigma : [0, \infty) \to M$, $\liminf_{t\to\infty} \int_0^\infty \mathrm{Ric}^M(\sigma'(s))ds \geq 0$. Then M has only one end. In particular, every complete manifold of nonnegative Ricci curvature with positive Cheeger constant has only one end.*

5. Applications in nonlinear partial differential systems

In ([18, 24]), S.W. Wei made the first study on how the existence of **essential positive supersolutions** of nonlinear degenerate partial differential systems on a manifold affects the topology, geometry, analysis, and variational probems on the manifold. By an essential positive supersolution of a second order differential differential systems $Q(u) = 0$ on a Riemannian n-manifold M, we mean a C^2 nonnegative function v on M which is positive almost everywhere on M, and which satisfies

$$Q(v) = \mathrm{div}\left(A(x, v, \nabla v)\nabla v\right) + b(x, v, \nabla v)v \leq 0 \tag{5.1}$$

on a Riemannian n-manifold M, where A denotes a smooth cross section in the bundle whose fiber at each point x in M is a **nonnegative** linear transformation on the tangent space $T_x(M)$ into $T_x(M)$, b is a smooth real-valued function, and ∇v denotes the gradient of v.

In [35], we generate the work of Schoen-Simon-Yau under p-parabolic stable condition without assuming p-th volume growth condition $r^{-p}\,\mathrm{Vol}(B_r) \to 0$ as $r \to \infty \quad (0.1)$. Our techniques, concepts, and results can be combined with those of essential supersolutions of degenerate nonlinear partial differential systems and used in other type of manifolds. The combined techniques, concepts and method of [32] and [35] can also be used in other new manifolds we found by an extrinsic average variational method ([34]).

Theorem 5.1. *Let M be a minimal hypersurface in a manifold N^{n+1} with constant sectional curvature $K_2 \geq 0$. Suppose M admits an essential positive supersolution of $Q(u) = 0$ of with coefficients satisfying conditions*

(I) $||A|| \leq c_1$, *i.e.* $\langle A\nabla v, \nabla v\rangle \leq c_1|\nabla v|^2$, *and*

(VII) $b(x, u, \nabla u) \geq c_2|B|^2 + nK_2 \quad$ with $\quad c_1 = c_2 = 1$,

where B is the second fundamental form of M in N. If M is p-parabolic, where $p \in [4, 4 + \sqrt{8/n}\,)$, then M is totally geodesic.

Theorem 5.1 generalizes our previous work [35, 4.11], which in term recaptures the work of Schoen-Simon-Yau for the case $A(x, u, \nabla u) = \text{Identity map}$, $b(x, u, \nabla u) = |\nabla \nu|^2 + nK_2$, and $r^{-p}\,\mathrm{Vol}(B_r) \to 0$ as $r \to \infty \quad (0.1)$ in [25, Theorem 2].

Proof. We recall, A generalized Ricatti identity ([32, 2.1, p.150])

$$\mathrm{div}(A\nabla \log v_\epsilon) = \frac{1}{v_\epsilon}\,\mathrm{div}(A\nabla v) - \frac{1}{{v_\epsilon}^2}\langle A\nabla v, \nabla v\rangle \tag{5.2}$$

where $v \geq 0$ is C^2 and $v_\epsilon = v + \epsilon > 0$.

Multiplying $\frac{1}{v_\epsilon}$ in (5.1) and moving terms in (5.2), we have

$$\frac{v}{v_\epsilon}b \leq -\frac{1}{v_\epsilon}\,\mathrm{div}(A\nabla v) = -\,\mathrm{div}(A\nabla \log v_\epsilon) - \frac{1}{{v_\epsilon}^2}\langle A\nabla v, \nabla v\rangle. \tag{5.3}$$

Multiplying (5.3) by ϕ^2, integrating by parts, and applying Cauchy-Schwarz inequality, we have for every $\phi \in C_0^\infty(M)$,

$$\begin{aligned}\int_M \frac{v}{v_\epsilon} b\phi^2\,dv &\leq 2\int_M \frac{\phi}{v_\epsilon}\langle \nabla\phi, A\nabla v\rangle\,dv - \int_M \frac{\phi^2}{v_\epsilon^2}\langle A\nabla v, \nabla v\rangle\,dv \\ &\leq \int_M \frac{\phi^2}{{v_\epsilon}^2}\langle A\nabla v, \nabla v\rangle\,dv + \int_M \langle A\nabla\phi, \nabla\phi\rangle\,dv \\ &\qquad - \int_M \frac{\phi^2}{v_\epsilon^2}\langle A\nabla v, \nabla v\rangle\,dv \\ &= \int_M \langle A\nabla\phi, \nabla\phi\rangle\,dv.\end{aligned} \tag{5.4}$$

Letting $\epsilon \to 0$ in (5.4), and using the assumptions (I) and (VII), we have

$$\int_M (nK_2 + |B|^2)\phi^2\,dv \leq \int_M |\nabla\phi|^2\,dv, \tag{5.5}$$

for every $\phi \in C_0^\infty(M)$. It follows from the curvature estimates in [25, p. 281], in which $K_1 = K_2 \geq 0$ and $c = 0$ that for each $p \in [4, 4 + \sqrt{8/n}\,)$ and for each nonnegative function $\phi \in C_0^\infty(M)$,

$$\int_M \phi^p |B|^p\,dv \leq c_7 \int_M |\nabla\phi|^p\,dv \tag{5.6}$$

where c_7 is constant depending only on n and p. Applying

Proposition 5.1. $[35, (5.4)]$ *Let $b \geq 0$ be a continuous function on a p-parabolic manifold M. Suppose there exists a constant $C > 0$, such that*

$$\int_M b|\varphi|^p\,dv \leq C\int_M |\nabla\varphi|^p\,dv, \forall\varphi \in C_0^\infty(M)\,. \tag{5.7}$$

Then $b \equiv 0$,

we conclude $B \equiv 0$, i.e., M is totally geodesic. □

Similarly, we prove

Theorem 5.2. *Let M be a p-parabolic stable minimal hypersurface in a manifold N^{n+1} with nonnegative constant sectional curvature, where $p \in [4, 4 + \sqrt{8/n}\,)$. Then M is totally geodesic.*

Theorem 5.2 generalizes our previous work [32, Theorem 4.12].

Analogously, by virtue of Proposition 5.1, we have

Theorem 5.3. *Let M be a hypersurface of constant mean curvature in $\mathbb{R}^{n+1}$ with Ric$^M \geq 0$ and either $3 \leq n \leq 4$ or M is p-parabolic for some $p \in [4, 4 + \frac{1}{n-1}]$. If M admits an essential positive supersolution of $Q(u) = 0$ of which the coefficients satisfy* (I) *and*
(IV) *$b(x, u, \nabla u) \geq c_2|\nabla\nu|^2$ with $c_1 = c_2 = 1$, where ν is the Gauss map of M.*
Then M is a hyperplane.

Theorem 5.3 generalizes the previous work [32, Theorem 4.14]. Likewise, use the method in Wei-Li-Wu [35], we have

Theorem 5.4. *Every stable harmonic map $u : M \to N$ is constant provided $\mathrm{Ric}^M \geq 0$ and M is p-parabolic for $p \in [4, 2 + 2w\frac{n}{n-1}]$, where N is an* SSU *manifold and w is an* SSU*-index (as in [30, 3.6, p.644 and 3.25, p.648] respectively).*

Theorem 5.4 generalizes the previous work [32, Theorem 4.15]. Using Theorem 5.4 to consider the case $A(x, u, \nabla u) =$ Identity map and $b(x, u, \nabla u) = |\nabla\nu|^2$ in Theorem 5.3, we obtain

Theorem 5.5. *Let M be a hypersurface of constant mean curvature in $\mathbb{R}^{n+1}$ with $\mathrm{Ric}^M \geq 0$ and either $3 \leq n \leq 4$ or M is p-parabolic for some $p \in [4, 4 + \frac{1}{n-1}]$. If the image of M under the Gauss map lies in some open hemisphere then M is a hyperplane. If the image of M under the Gauss map lies in some closed hemisphere, then M is a hyperplane, or a cylinder $\times M'$, where M' is a hypersurface of constant mean curvature H' in $\mathbb{R}^n$.*

Theorem 5.5 generalizes previous work [32, Theorem 4.16].

6. Integral inequalities on complete noncompact manifolds

In this section, we discuss integral inequalities on Riemannian manifolds and their links to curvature, essential positive supersolutions of partial differential systems, topology, variational problems, potential theory, etc.

Applying the work of Blanc-Fiala-Huber [18], Greene-Wu [14], Theorem 1.4.(b), Varopoulos [28], Karp [19], Wei-Li-Wu [35, §2, §3], Milnor [21], Wei [32, 33, 31], Cheng-Yau [9], Do Carmo [10], Fischer-Colbrie - Schoen [11], Pogorelov [23], Hoffman-Ossorman-Schoen [16], etc, we have for example,

Theorem 6.1. *Let M be a complete surface in $\mathbb{R}^3$ with a unit normal vector ν. Suppose M satisfies the general integral inequality (1.6) for any $\frac{2q}{q+2} < p$, where $p \in [1, 2]$. Then M does not admit any essential positive supersolutions of*

$$Q(v) = \mathrm{div}(A(x, v, \nabla v)\nabla v) + b(x, v, \nabla v)v \leq 0 \tag{5.1}$$

with coefficients satisfying

(I) $||A|| \leq c_1$, *i.e.* $\langle A\nabla v, \nabla v\rangle \leq c_1|\nabla v|^2$, *and*

(IV) $b(x, u, \nabla u) \geq c_2|\nabla \nu|^2 \quad$ *with* $\quad 0 < c_1 < 3c_2$.

Thus M can not be any of the following:

(a) a stable minimal surface in $\mathbb{R}^3$,

(b) a surface of constant mean curvature in $\mathbb{R}^3$, such that the image under its Gauss map lies in an open hemisphere,

(c) the graph of a real-valued function on $\mathbb{R}^2$ with prescribed mean curvature function $\tilde{H} : \mathbb{R}^3 \to \mathbb{R}$ satisfying $\frac{\partial \tilde{H}}{\partial x_1} = \frac{\partial \tilde{H}}{\partial x_2} = 0$. and $\frac{\partial \tilde{H}}{\partial x_3} \leq 0$.

Proof. Suppose contrary, M admitted an essential positive supersolutions of $Q(v) \leq 0 \quad (5.1)$ with coefficients satisfying (I), (IV) and $0 < c_1 < 3c_2$. It follows from [32, Theorem 3.12] that M is a plane. On the other hand, according to Theorem 1.4.(d), M satisfies the general integral inequality (1.6) for any $\frac{2q}{q+2} < p$, where $p \in [1, 2]$ implies that M is conformal to a disk. This is a contradiction.

Thus, suppose contrary,

(a) M were a stable minimal surface in $\mathbb{R}^3$. It follows from a Theorem of Do Carmo-Peng [10], a Theorem of Fischer-Colbrie - Schoen [11], or a Theorem of Pogorelov [23] that M were a plane. This contradicts M being conformal to a disk.

(b) A surface of constant mean curvature in $\mathbb{R}^3$ such that the image under its Gauss map would lie in an open hemisphere. It follows from a Theorem of Hoffman-Ossorman-Schoen [16] that M were a plane, contradicting M being conformal to a disk.

(c) The graph of a real-valued function on $\mathbb{R}^2$ with prescribed mean curvature function $\tilde{H} : \mathbb{R}^3 \to \mathbb{R}$ would satisfy $\frac{\partial \tilde{H}}{\partial x_1} = \frac{\partial \tilde{H}}{\partial x_2} = 0$. and $\frac{\partial \tilde{H}}{\partial x_3} \leq 0$. It follows from a Theorem of Cheng-Yau [9] that M would be a plane, contradicting M being conformal to a disk.

□

Combining a theorem of Blanc-Fiala-Huber [18], a theorem of Greene-Wu [14], and Theorem 1.4.(b), we obtain

Corollary 6.1. *Let M be a complete surface which satisfies the general integral inequality* (1.6) *for any* $\frac{2q}{q+2} < p$, *where* $p \in [1,2]$. *Then*

(a) *M has negative Gaussian curvature $K(x)$ at some point $x \in M$.*

(b) *For every $r_0 > 0, x_0 \in M$, there exists $x_1 \in M$ such that $K(x_1) < \frac{-1}{r^2 \log r}$ where $r = \operatorname{dist}(x_0, x_1) \geq r_0$.*

Proof. Suppose contrary,

(a) M had nonnegative Gaussian curvature $K(x)$ at every point $x \in M$. It follows from a Theorem of Blanc-Fiala-Huber [18] that M were parabolic, contradicting M being conformal to a disk..

(b) For every $r_0 > 0, x_0 \in M$, every $x_1 \in M$ $K(x_1) \geq \frac{-1}{r^2 \log r}$ where $r = \operatorname{dist}(x_0, x_1) \geq r_0$. It follows from a Theorem of Greene-Wu [14], (cf. also Karp [19, Corollary 3.5.1′]) that M would be parabolic, contradicting M being conformal to a disk.

, □

We define $F(r)$ to be any positive nondecreasing function satisfying

$$\int_a^\infty \frac{dr}{rF(r)} = \infty \tag{6.1}$$

for some $a > 0$. Then we have, by a result of Karp [19],

Corollary 6.2. *Let M be the graph of a smooth real valued function u over $\mathbb{R}^2$ with mean curvature $H(x)$ at $(x, u(x))$. Suppose M satisfies the general integral inequality* (1.6) *for any* $\frac{2q}{q+2} < p$, *where* $p \in [1,2]$. *Then either* $|u| \neq O(rF(r))$, *or* $|uH| \neq O(F(r))$. *In particular, if M is the graph of $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying* (1.6) *for any* $\frac{2q}{q+2} < p$ *where* $p \in [1,2]$, *then either u is unbounded, or* $\frac{\sum_{i,j=1}^2 |\frac{\partial^2 u}{\partial x_i \partial x_j}|}{(1+|\nabla u|^2)^{\frac{1}{2}}}$ *is unbounded.*

For general dimensions, applying Varopoulos [28], Karp [19], Wei-Li-Wu [35, §2, §3], or Theorem $1.4.\,(b)$, we have

Theorem 6.2. *Let M be a complete n-manifold such that*

$$\int_1^\infty \frac{s}{\operatorname{Vol}(B(s))} ds = \infty \tag{6.2}$$

where $B(s)$ is a geodesic ball of radius s in M. Then M does not satisfies the general integral inequality (1.6) *for any* $\frac{2q}{q+2} < p$, *where* $p \in [1,2]$.

Combining a Theorem of Milnor [21], and Theorem 1.4, we have

Theorem 6.3. *If M isometrically covers a compact Riemannian manifold N and the growth function $\gamma(s)$ of $\pi_1(N)$ satisfies* $\limsup \frac{\gamma(s)}{s^2 F(s)} < \infty$ *for $F(s)$ as in* (6.1), *then M does not satisfies the integral inequality* (1.6) *for any* $\frac{2q}{q+2} < p$, *where* $p \in [1,2]$.

Proof. Suppose contrary, in view of Theorem 1.4, M would be nonparabolic. On the other hand, the assumption on the growth function $\gamma(s)$ of $\pi_1(N)$ imply, via a Theorem of Milnor [21], M would have moderate growth, and hence M would be parabolic, a contradiction. □

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