

# A Robust Quintic Hermite Collocation Method for One-Dimensional Heat Conduction Equation

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## Abstract

In this work, a new robust numerical solution scheme constructed on Quintic Hermite Collocation Method (QHCM) utilizing the traditional Crank-Nicolson type approximation technique is developed for solving 1D heat conduction equation with certain initial and boundary conditions which is mostly handled as a prototype equation to support the reliability of many proposed new numerical methods. All temporal and spatial quantities in the equation are fully discretized using a usual Crank-Nicolson type finite difference approximation and a QHCM, respectively. In obtaining the present scheme, all the roots of the fourth degree Legendre and Chebyshev polynomials shifted to the unit interval are used as suitable inner collocation points. The obtained results from the developed scheme are found to be good enough and better than those from other schemes encountered in the literature. The scheme is also shown to be unconditionally stable by Fourier stability test.

## 1. Introduction

The considered problem in this study consists of one-dimensional heat conduction equation

$$\alpha^2 u_{xx} - u_t = 0, \quad t > 0, x \in [x_l, x_r] \quad (1.1)$$

given by the appropriate initial condition

$$u(x, t_{initial}) = f(x) \quad (1.2)$$

and homogenous Dirichlet boundary conditions

$$u(x_l, t) = u(x_r, t) = 0, \quad t > 0 \quad (1.3)$$

where  $t_{initial} = 0$ ,  $x_l = 0$ ,  $x_r = l$ ,  $\alpha$  and  $f(x)$  respectively stand for the thermal diffusivity coefficient and a smooth function to be prescribed in the process of numerical calculations. The initial and boundary value (IBV) problem governed by Eqs.(1.1)-(1.3) is generally considered as one of the most outstanding Partial Differential Equations (PDEs) appearing especially in physics and engineering mathematics. The current problem is among the widely-known second order linear partial differential equation. This problem illustrates the fact that heat equation defines irreversible process and also at the same time presents a distance between previous and the next steps. Those equations generally arise in various areas of engineering and science to describe the variation of the temperature on a predefined solution domain over a given time period. For more information about the characteristics of various heat equations, the reader may refer to Refs. [1, 2] and references therein.

Since the classical heat conduction equation (1.1) with various types of IBVs is handled by many researchers as a pioneer prototype test problem in the first applications of many new numerical techniques, there exist many proposed numerical techniques in the literature to compute its approximate solutions. For example, some of their frequently used are finite difference method [3–8], finite element

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method [9–14], wavelet method [15], spectral method [16]. To our knowledge, all studies constructed on finite elements method available in the literature have used one of the usual polynomial, trigonometric and exponential functions as B-spline bases. But recently, Kutluay *et al.* [17] have successfully applied collocation finite element method with cubic Hermite basis functions to generate more precise approximate numerical solutions of the heat conduction initial and boundary problem. There are also recently published and related articles with the presented method, problem and the equations such as given in Refs. [18–20] and therein. In this study, in order to obtain much more accurate and precise approximate solutions for the IBV problem, a new effective numerical scheme basically constructed on collocation finite element method using quintic Hermite spline basis functions, not the usual basis mentioned above, have been developed. The rest of the article is structured as follows: In Section 2, the solution domain of the problem is firstly divided into uniform partition in spatial and temporal directions. Then, the handled problem is fully discretized using Crank-Nicolson like difference approximation for the time quantities and QHCM utilizing the shifted roots of Chebyshev and Legendre polynomials as inner collocation points for the space quantities. Additionally, utilizing the initial and boundary conditions, the initial vector to be needed for starting the recursive scheme is also constructed. Thus, the heat problem is converted into a solvable system consisting of algebraic equations. In Section 3, it is shown that the scheme is unconditionally stable by using von-Neumann stability test for the modest values of spatial and temporal step sizes. In Section 4, the developed numerical scheme is used to obtain the approximate solutions of the experimental problem, and the computed results are compared with those of other researchers. In Section 5, this paper with a brief conclusion and future works is summarized.

## 2. Quintic Hermite Collocation Method

Throughout the manuscript, the 1D heat conduction equation which is widely given by Eq. (1.1) by the appropriate initial (1.2) and boundary conditions (1.3) is going to be handled. For generating the approximate numerical solutions of the problem, the finite element collocation method with quintic Hermite base functions is selected as a useful and powerful tool. To be able to apply this method, the solution domain  $[x_l, x_r] \times [0, t_f]$  of the problem should be discretized. For this purpose, the spatial interval  $[x_l, x_r]$  is divided into  $M$  equal width finite subintervals by means of the mesh points  $x_j, j = 1(1)M + 1$  such that  $x_l = x_1 < x_2 \cdots < x_M < x_{M+1} = x_r$  and  $\Delta x = x_{j+1} - x_j$ , and similarly the temporal interval  $[0, t_f]$  is divided into  $N$  equal width finite subintervals by means of the mesh points  $t_n, n = 0(1)N$  such that  $t_{initial} = t_0 < t_1 < \cdots < t_{N-1} < t_N = t_f$  and  $\Delta t = t_{n+1} - t_n$ . In fact, a non-uniform partition of the region could also be selected. But, since the non-uniform choice would increase the workload, computer storage capacity and running time, the uniform one has been preferred.

### 2.1. Discretization of space variable

In this method, the approximate solution  $u_M(x, t)$  corresponding to analytical solution  $u(x, t)$  will be sought by means of quintic Hermite basis functions  $H_{ji}$  as [21]

$$u(x, t) \approx u_M(x, t) = \sum_{j=1}^{M+2} a_{j+4k-4}(t) H_{ji} \tag{2.1}$$

where  $a$ 's stand for the time dependent coefficients which are going to be determined later,  $k$  stands for the element number in each subinterval,  $H_{4j-R}(x)$  for  $R = -2(1)3$  are defined over the interval  $[x_{j-1}, x_{j+1}]$  as given in Ref. [22]

$$\begin{aligned} H_{4j-3}(x) &= 6 \frac{(x-x_{j-1})^5}{h^5} - 15 \frac{(x-x_{j-1})^4}{h^4} + 10 \frac{(x-x_{j-1})^3}{h^3}, & x_{j-1} \leq x \leq x_j \\ H_{4j+1}(x) &= 6 \frac{(x_{j-1}-x)^5}{h^5} - 15 \frac{(x_{j-1}-x)^4}{h^4} + 10 \frac{(x_{j-1}-x)^3}{h^3}, & x_j \leq x \leq x_{j+1} \\ H_{4j-2}(x) &= 3 \frac{(x-x_{j-1})^5}{h^4} - 7 \frac{(x-x_{j-1})^4}{h^3} + 4 \frac{(x-x_{j-1})^3}{h^2}, & x_{j-1} \leq x \leq x_j \\ H_{4j+2}(x) &= 3 \frac{(x_{j-1}-x)^5}{h^4} - 7 \frac{(x_{j-1}-x)^4}{h^3} + 4 \frac{(x_{j-1}-x)^3}{h^2}, & x_j \leq x \leq x_{j+1} \\ H_{4j-1}(x) &= \frac{1}{2} \frac{(x-x_{j-1})^5}{h^3} - \frac{(x-x_{j-1})^4}{h^2} + \frac{1}{2} \frac{(x-x_{j-1})^3}{h^1}, & x_{j-1} \leq x \leq x_j \\ H_{4j}(x) &= \frac{1}{2} \frac{(x_{j-1}-x)^5}{h^3} - \frac{(x_{j-1}-x)^4}{h^2} + \frac{1}{2} \frac{(x_{j-1}-x)^3}{h}, & x_j \leq x \leq x_{j+1} \end{aligned} \tag{2.2}$$

and finally  $i$  ( $i = 1(1)4$ ) stands for the inner-collocation points to be taken from the shifted roots of Legendre and Chebyshev polynomials. Throughout this study, the following shifted roots of the fourth-degree Legendre and Chebyshev polynomials [23] computed by the symbolic programming language Matlab are respectively used as inner-collocation points

$$\begin{aligned} \eta_1^L &= 0.069431844202974, \eta_2^L = 0.330009478207572 \\ \eta_3^L &= 0.669990521792428, \eta_4^L = 0.930568155797026. \\ \eta_1^C &= 0.038060233744357, \eta_2^C = 0.308658283817455 \\ \eta_3^C &= 0.691341716182545, \eta_4^C = 0.961939766255643 \end{aligned}$$

When the following local coordinate system has been utilized on the  $j^{th}$  element

$$\eta = \frac{x-x_j}{h}$$

the interval  $[x_j, x_{j+1}]$  is converted into an unit interval  $[0, 1]$  in which  $\eta$  represents both Chebyshev and Legendre shifted roots. Thus, from Eq. (2.2), quintic Hermite spline functions  $H_j(\eta)$  ( $j = 1(1)6$ ) in terms of the local coordinate  $\eta$  are written as

$$\begin{aligned}
H_1(\eta) &= 1 - 10\eta^3 + 15\eta^4 - 6\eta^5, & H_2(\eta) &= (\eta - 6\eta^3 + 8\eta^4 - 3\eta^5)\Delta x, \\
H_3(\eta) &= (0.5\eta^2 - 1.5\eta^3 + 1.5\eta^4 - 0.5\eta^5)(\Delta x)^2, & H_4(\eta) &= (0.5\eta^3 - \eta^4 + 0.5\eta^5)(\Delta x)^2, \\
H_5(\eta) &= 10\eta^3 - 15\eta^4 + 6\eta^5, & H_6(\eta) &= (-4\eta^3 + 7\eta^4 - 3\eta^5)\Delta x.
\end{aligned} \tag{2.3}$$

and therefore the approximate solution  $u_M(x, t)$  given by Eq. (2.1) becomes

$$u_M(\eta, t) = \sum_{j=1}^6 a_{j+4k-4}(t) H_j(\eta). \tag{2.4}$$

From (2.3), the second order derivatives of  $H_j(\eta)$  are found as follows

$$\begin{aligned}
H_1''(\eta) &= -60\eta + 180\eta^2 - 120\eta^3, & H_2''(\eta) &= (-36\eta + 96\eta^2 - 60\eta^3)\Delta x, \\
H_3''(\eta) &= (1 - 9\eta + 18\eta^2 - 10\eta^3)(\Delta x)^2, & H_4''(\eta) &= (3\eta - 12\eta^2 + 10\eta^3)(\Delta x)^2, \\
H_5''(\eta) &= 60\eta - 180\eta^2 + 120\eta^3, & H_6''(\eta) &= (-24\eta + 84\eta^2 - 60\eta^3)\Delta x.
\end{aligned}$$

Thus, the point wise values of the approximation (2.4) with its second order derivative at inner collocation points  $\eta_i$  for  $i = 1(1)4$  are found as follows

$$\begin{aligned}
u_i &= u(\eta_i) = a_{4k-3}H_1(\eta_i) + a_{4k-2}H_2(\eta_i) + a_{4k-1}H_3(\eta_i) + a_{4k}H_4(\eta_i) + a_{4k+1}H_5(\eta_i) + a_{4k+2}H_6(\eta_i), \\
(\Delta x)^2 u_i'' &= (\Delta x)^2 u''(\eta_i) = a_{4k-3}H_1''(\eta_i) + a_{4k-2}H_2''(\eta_i) + a_{4k-1}H_3''(\eta_i) + a_{4k}H_4''(\eta_i) + a_{4k+1}H_5''(\eta_i) + a_{4k+2}H_6''(\eta_i)
\end{aligned} \tag{2.5}$$

where  $H_1(\eta) = H_5(1 - \eta)$ ,  $H_2(\eta) = -H_6(1 - \eta)$ ,  $H_3(\eta) = -H_4(1 - \eta)$  and  $H_{ji} = H_j(\eta_i)$ .

In the building of the numerical scheme, the finite element method constructed on quintic Hermite spline basis functions in the discretization of spatial quantities and the finite difference method constructed on Crank-Nicolson like approximation in the discretization of temporal quantities will be used. This choice has the advantages of several vital characteristics such as easy to handle algorithms produced by quintic Hermite spline basis functions and low level of storage requirement. Besides those advantages, both of the non-linear and linear systems resulted from the usage of splines are usually not ill-conditioned and thus permit the required coefficients to be found out in an easy manner. Furthermore, the newly obtained approximate solutions generally don't result in numerical instability.

## 2.2. Discretization of time variable

Now, we are ready to discretize the aforementioned 1D heat conduction equation given by (1.1). To do this, one can use Crank-Nicolson type formula. First of all, one discretizes Eq. (1.1) as

$$\frac{u^{n+1} - u^n}{\Delta t} - \alpha^2 \left[ \frac{(u_{xx})^n + (u_{xx})^{n+1}}{2} \right] = 0.$$

Before proceeding more, let us separate the above equation such that the unknown values at  $(n + 1)$ . time level are on the left-hand side and the known values  $n$ . time level are on the right-hand side as

$$\frac{u^{n+1}}{\Delta t} - \alpha^2 \frac{(u_{xx})^{n+1}}{2} = \frac{u^n}{\Delta t} + \alpha^2 \frac{(u_{xx})^n}{2}. \tag{2.6}$$

If one puts Eq. (2.5) in Eq. (2.6), the following fully discretized difference equation system with  $4M$  difference equations and  $4M + 2$  coefficients in both time and space variables is obtained for the coefficients  $\mathbf{a}$  to be calculated

$$\begin{aligned}
& \frac{1}{\Delta t} \left[ a_{4k-3}^{n+1} H_{1i} + a_{4k-2}^{n+1} H_{2i} + a_{4k-1}^{n+1} H_{3i} + a_{4k}^{n+1} H_{4i} + a_{4k+1}^{n+1} H_{5i} + a_{4k+2}^{n+1} H_{6i} \right] \\
& - \frac{\alpha^2}{2(\Delta x)^2} \left[ a_{4k-3}^{n+1} B_{1i} + a_{4k-2}^{n+1} B_{2i} + a_{4k-1}^{n+1} B_{3i} + a_{4k}^{n+1} B_{4i} + a_{4k+1}^{n+1} B_{5i} + a_{4k+2}^{n+1} B_{6i} \right] \\
& = \frac{1}{\Delta t} \left[ a_{4k-3}^n H_{1i} + a_{4k-2}^n H_{2i} + a_{4k-1}^n H_{3i} + a_{4k}^n H_{4i} + a_{4k+1}^n H_{5i} + a_{4k+2}^n H_{6i} \right] \\
& + \frac{\alpha^2}{2(\Delta x)^2} \left[ a_{4k-3}^n B_{1i} + a_{4k-2}^n B_{2i} + a_{4k-1}^n B_{3i} + a_{4k}^n B_{4i} + a_{4k+1}^n B_{5i} + a_{4k+2}^n B_{6i} \right].
\end{aligned} \tag{2.7}$$

These newly obtained equations are clearly recursive in nature. Thus, the unknown vector  $\mathbf{a}^n = (a_1^n, \dots, a_{4M+1}^n, a_{4M+2}^n)$  can be recursively determined up to the requested final time  $t_f$ . If one utilizes the conditions given at the boundary of the solution domain by Eq. (1.3) and eliminates the coefficients  $a_1^n, a_{4M+1}^n$  in Eq. (2.7), the following statements are easily obtained. Using the boundary condition given at the left of the solution domain, the value of  $u(x_l, t)$  is written as

$$u(x_l, t) = a_1^n H_{11} + a_2^n H_{21} + a_3^n H_{31} + a_4^n H_{41} + a_5^n H_{51} + a_6^n H_{61} = 0.$$



where  $\varphi = \beta h$ ,  $\beta$  is the mode number,  $h$  is the spatial step size,  $i = \sqrt{-1}$  and

$$\begin{aligned}\alpha_1 &= H_{1i} - \frac{\alpha^2}{2h^2} kB_{1i} & \beta_1 &= H_{1i} + \frac{\alpha^2}{2h^2} kB_{1i} \\ \alpha_2 &= H_{2i} - \frac{\alpha^2}{2h^2} kB_{2i} & \beta_2 &= H_{2i} + \frac{\alpha^2}{2h^2} kB_{2i} \\ \alpha_3 &= H_{3i} - \frac{\alpha^2}{2h^2} kB_{3i} & \beta_3 &= H_{3i} + \frac{\alpha^2}{2h^2} kB_{3i} \\ \alpha_4 &= H_{4i} - \frac{\alpha^2}{2h^2} kB_{4i} & \beta_4 &= H_{4i} + \frac{\alpha^2}{2h^2} kB_{4i} \\ \alpha_5 &= H_{5i} - \frac{\alpha^2}{2h^2} kB_{5i} & \beta_5 &= H_{5i} + \frac{\alpha^2}{2h^2} kB_{5i} \\ \alpha_6 &= H_{6i} - \frac{\alpha^2}{2h^2} kB_{6i} & \beta_6 &= H_{6i} + \frac{\alpha^2}{2h^2} kB_{6i}\end{aligned}$$

Again, if one makes the required simplification and mathematical operations, one obtains

$$\xi = \frac{M_1 - iM_2}{M_3 - iM_4} \quad (3.1)$$

where

$$\begin{aligned}M_1 &= (\beta_3 + \beta_5) \cos \varphi + (\beta_2 + \beta_6) \cos 2\varphi + \beta_1 \cos 3\varphi + \beta_4 \\ M_2 &= (\beta_3 - \beta_5) \sin \varphi + (\beta_2 - \beta_6) \sin 2\varphi + \beta_1 \sin 3\varphi + \beta_4 \\ M_3 &= (\alpha_3 + \alpha_5) \cos \varphi + (\alpha_2 + \alpha_6) \cos 2\varphi + \alpha_1 \cos 3\varphi + \alpha_4 \\ M_4 &= (\alpha_3 - \alpha_5) \sin \varphi + (\alpha_2 - \alpha_6) \sin 2\varphi + \alpha_1 \sin 3\varphi + \alpha_4.\end{aligned}$$

When the modulus of Eq. (3.1) is taken, it is seen that the condition  $|\xi| \leq 1$  is satisfied. Thus, it is concluded that the numerical scheme is unconditionally stable.

#### 4. Numerical Example and Results

In this section, the obtained scheme (2.8) will be applied to the 1D heat conduction problem given by Eqs. (1.1)-(1.3) for  $x \in [0, 1]$ ,  $t \in [0, t_f]$ ,  $f(x) = \sin(\pi x)$  and  $\alpha = 1$ , as a test problem. The exact solution of the test problem is [7, 8]

$$u(x, t) = \sin(\pi x) e^{-\alpha^2 \pi^2 t}.$$

Since the test problem has an exact solution, the error norms  $L_2$  and  $L_\infty$  given as follows, respectively, are going to be used to test the accuracy and validity of the scheme

$$L_2 = \sqrt{\left( h \sum_{i=1}^M |u_i - (u_M)_i|^2 \right)}, \quad L_\infty = \max_{1 \leq i \leq M} |u_i - (u_M)_i|.$$

This manuscript carries out all of the numerical computations using both Quintic Hermite Collocation Method with Legendre roots (QHCM-L) and Quintic Hermite Collocation Method with Chebyshev roots (QHCM-C). All calculations have been made with MATLAB R2021a on 13th Gen Intel(R) Core(TM) i9-13900HX 2.20 GHz computer having 32.0 GB of RAM.

Some numerical results are presented for different spatial and temporal step sizes with final desired values of time to check the efficiency and accuracy of the scheme. The newly obtained results are also compared with some of the existing ones using the same parameter values.

$\Delta t$	$L_2$			
	QHCM-L	QHCM-C	[7]	[17]CHCM-L
0.01	$5.8454 \times 10^{-7}$	$5.8454 \times 10^{-7}$	$4.2273 \times 10^{-4}$	$4.1333 \times 10^{-7}$
0.005	$1.4641 \times 10^{-7}$	$1.4641 \times 10^{-7}$	$1.0560 \times 10^{-4}$	$1.0353 \times 10^{-7}$
0.0025	$3.6621 \times 10^{-8}$	$3.6621 \times 10^{-8}$	$2.6395 \times 10^{-5}$	$2.5895 \times 10^{-8}$

**Table 4.1:** A comparison of the computed error norms  $L_2$  of the present scheme with those in Refs. [7, 17], for  $M = 1000$  and  $\Delta t = 0.01, 0.005, 0.0025$  ( $t_f = 1$ ).

The values of the error norm  $L_2$  are calculated from the scheme (2.8) for the parameters  $\Delta t = 1/100, 5/1000, 25/10000$  and  $\Delta x = 1/1000$  at  $t_f = 1$ , and are listed in Table 4.1 with a comparison of those given in Refs. [7, 17]. The newly obtained error norm  $L_2$  is remarkably small enough for using both QHCM-C and QHCM-L schemes. They are also in very good agreement with those in Ref. [17] and much better than those in Ref. [7].

In Table 4.2, a clear comparison of the newly computed error norm values  $L_2$  of the scheme with those in Refs. [8, 17] for several values of  $\Delta x = 1/5, 1/10, 1/20$  and  $\Delta t = 1/10^6$  at  $t_f = 1$  is displayed. It can be easily seen from the table that the obtained results are extremely small

$\Delta x$	$L_2$				
	QHCM-L	QHCM-C	[8](CN-I)	[8](CN-II)	[17]CHCM-L
1/5	$3.1385 \times 10^{-12}$	$2.2952 \times 10^{-9}$	$4.7696 \times 10^{-6}$	$8.5859 \times 10^{-3}$	$3.5716 \times 10^{-8}$
1/10	$4.7257 \times 10^{-14}$	$1.5029 \times 10^{-10}$	$7.7143 \times 10^{-9}$	$2.1412 \times 10^{-3}$	$2.2848 \times 10^{-9}$
1/20	$1.1057 \times 10^{-14}$	$9.4983 \times 10^{-12}$	$1.8820 \times 10^{-11}$	$5.3498 \times 10^{-4}$	$1.4361 \times 10^{-10}$
1/40	$1.3947 \times 10^{-15}$	$5.9155 \times 10^{-13}$			
1/80	$5.1467 \times 10^{-16}$	$3.3364 \times 10^{-14}$			

**Table 4.2:** A comparison of the computed error norms  $L_2$  of the present scheme with those in Refs. [8, 17], for various values of  $M = 5, 10, 20, 40, 80$  and  $\Delta t = 1/10^6$  at  $t_f = 1$ .

$\Delta x$	$L_2$			
	QHCM-L	[7]		[17]CHCM-L
		$\theta = 0.1$	$\theta = 0.2$	
1/10	$3.5183 \times 10^{-10}$	$1.6534 \times 10^{-4}$	$2.4968 \times 10^{-4}$	$1.6957 \times 10^{-6}$
1/20	$8.5012 \times 10^{-12}$	$4.3906 \times 10^{-7}$	$7.8152 \times 10^{-7}$	$1.0426 \times 10^{-7}$
1/40	$1.0432 \times 10^{-12}$	$8.6154 \times 10^{-10}$	$8.0111 \times 10^{-10}$	$6.4916 \times 10^{-9}$
		$\theta = 0.3$	$\theta = 0.4$	
1/10	$3.5183 \times 10^{-10}$	$3.2357 \times 10^{-4}$	$3.6276 \times 10^{-4}$	$1.6957 \times 10^{-6}$
1/20	$8.5012 \times 10^{-12}$	$1.0791 \times 10^{-6}$	$1.2556 \times 10^{-6}$	$1.0426 \times 10^{-7}$
1/40	$1.0432 \times 10^{-12}$	$9.2171 \times 10^{-10}$	$1.1166 \times 10^{-9}$	$6.4916 \times 10^{-9}$
		$\theta = 0.5$	$\theta = 0.6$	
1/10	$3.5183 \times 10^{-10}$	$3.5225 \times 10^{-4}$	$2.8316 \times 10^{-4}$	$1.6957 \times 10^{-6}$
1/20	$8.5012 \times 10^{-12}$	$1.2494 \times 10^{-6}$	$1.0109 \times 10^{-6}$	$1.0426 \times 10^{-7}$
1/40	$1.0432 \times 10^{-12}$	$1.2167 \times 10^{-9}$	$1.1107 \times 10^{-9}$	$6.4916 \times 10^{-9}$
		$\theta = 0.7$	$\theta = 0.8$	
1/10	$3.5183 \times 10^{-10}$	$1.5954 \times 10^{-4}$	$1.9640 \times 10^{-4}$	$1.6957 \times 10^{-6}$
1/20	$8.5012 \times 10^{-12}$	$5.4419 \times 10^{-7}$	$9.1503 \times 10^{-7}$	$1.0426 \times 10^{-7}$
1/40	$1.0432 \times 10^{-12}$	$7.4804 \times 10^{-10}$	$8.6751 \times 10^{-10}$	$6.4916 \times 10^{-9}$
		$\theta = 0.9$	$\theta = 0.95$	
1/10	$3.5183 \times 10^{-10}$	$4.5197 \times 10^{-4}$	$6.1645 \times 10^{-4}$	$1.6957 \times 10^{-6}$
1/20	$8.5012 \times 10^{-12}$	$2.0820 \times 10^{-6}$	$2.8818 \times 10^{-6}$	$1.0426 \times 10^{-7}$
1/40	$1.0432 \times 10^{-12}$	$2.4668 \times 10^{-9}$	$3.6858 \times 10^{-9}$	$6.4916 \times 10^{-9}$

**Table 4.3:** A comparison of the calculated error norms  $L_2$  of the present scheme with those in Refs. [7, 17] for  $M = 10, 20, 40$  and  $\Delta t = 1/10^6$  at  $t_f = 0.1$ .

and also when the element number is increased, the error norm  $L_2$  decreases. Again, from the table it is obvious that the newly obtained results are much more better than those in Refs. [8, 17].

Table 4.3 shows a clear comparison of the  $L_2$  error norms computed from the new scheme using QHCM-L with those in Refs. [7, 17] for values of  $\Delta x = 1/10, 1/20, 1/40$  and  $\Delta t = 1/10^6$  at  $t_f = 0.1$ . One can obviously see that the scheme produces good enough results and clearly, the obtained  $L_2$  error norms are relatively smaller than those in Refs. [7, 17].

$\Delta x = \Delta t$	$L_2$				$L_\infty$	
	QHCM-L	QHCM-C	[17]CHCM-L	[11]	[6](CN)	[6](CBVM)
0.2	$5.1455 \times 10^{-5}$	$5.1439 \times 10^{-5}$	$5.8498 \times 10^{-5}$	$1.4145 \times 10^{-1}$	$1.1 \times 10^{-1}$	$2.8 \times 10^{-2}$
0.1	$3.1581 \times 10^{-5}$	$3.1589 \times 10^{-5}$	$3.1584 \times 10^{-5}$	$3.7195 \times 10^{-2}$	$3.0 \times 10^{-2}$	$3.8 \times 10^{-3}$
0.05	$9.7102 \times 10^{-6}$	$9.7106 \times 10^{-6}$	$9.7065 \times 10^{-6}$	$8.4588 \times 10^{-3}$	$6.9 \times 10^{-3}$	$2.7 \times 10^{-4}$
0.025	$2.5488 \times 10^{-6}$	$2.5489 \times 10^{-6}$	$2.5485 \times 10^{-6}$	$2.0698 \times 10^{-3}$	$1.7 \times 10^{-3}$	$1.3 \times 10^{-5}$
0.0125	$6.4490 \times 10^{-7}$	$6.4490 \times 10^{-7}$	$6.4488 \times 10^{-7}$	$5.1473 \times 10^{-4}$	$4.2 \times 10^{-4}$	$5.1 \times 10^{-7}$
0.00625	$1.6171 \times 10^{-7}$	$1.6171 \times 10^{-7}$	$1.6171 \times 10^{-7}$		$1.1 \times 10^{-4}$	$3.6 \times 10^{-8}$
0.01	$4.1332 \times 10^{-7}$	$4.1333 \times 10^{-7}$	$4.1332 \times 10^{-7}$			
0.005	$1.0353 \times 10^{-7}$	$1.0353 \times 10^{-7}$	$1.0353 \times 10^{-7}$			
0.0025	$2.5895 \times 10^{-8}$	$2.5895 \times 10^{-8}$	$2.5895 \times 10^{-8}$			
0.002	$1.6574 \times 10^{-8}$	$1.6574 \times 10^{-8}$	$1.6574 \times 10^{-8}$			
0.001	$4.1437 \times 10^{-9}$	$4.1437 \times 10^{-9}$	$4.1437 \times 10^{-9}$			

**Table 4.4:** A comparison of the calculated error norms  $L_\infty$  of the present scheme with those in Refs. [6, 11, 17], for various values of  $\Delta x = \Delta t$  at  $t_f = 1$ .

Table 4.4 displays a clear comparison of the  $L_\infty$  error norms obtained from the presented scheme with those given in Refs. [6, 11, 17] for some decreasing values of  $\Delta x = \Delta t$  at  $t_f = 1$ . The table indicates that the computed  $L_\infty$  error norms are sufficiently small and in good harmony with those in Refs. [6, 17], but much more better than those in Ref. [11].

Finally, Table 4.5 compares the discrete values of both  $L_2$  and  $L_\infty$  error norms for  $M = 16$  and  $\Delta t = 0.01$  at various  $t_f$  with those in Refs. [15, 17]. The table shows that the newly computed error norms are again small enough and better than both of those in Refs. [15, 17].

$t_f$	$L_2$			$L_\infty$		
	QHCM-L	[17]CHCM-L	[15]	QHCM-L	[17]CHCM-L	[15]
0.1	$4.2273 \times 10^{-4}$	$2.9917 \times 10^{-4}$	$4.86 \times 10^{-3}$	$2.9889 \times 10^{-4}$	$2.9891 \times 10^{-4}$	$6.79 \times 10^{-3}$
0.3	$1.7602 \times 10^{-4}$	$1.2457 \times 10^{-4}$	$8.87 \times 10^{-5}$	$1.2446 \times 10^{-4}$	$1.2447 \times 10^{-4}$	$3.76 \times 10^{-4}$
0.5	$4.0720 \times 10^{-5}$	$2.8818 \times 10^{-5}$	$1.73 \times 10^{-3}$	$2.8791 \times 10^{-5}$	$2.8793 \times 10^{-5}$	$2.44 \times 10^{-4}$
0.7	$7.9127 \times 10^{-6}$	$5.5999 \times 10^{-6}$	$2.04 \times 10^{-4}$	$5.5446 \times 10^{-6}$	$5.5953 \times 10^{-6}$	$3.17 \times 10^{-4}$
0.9	$1.4121 \times 10^{-6}$	$9.9934 \times 10^{-7}$	$2.14 \times 10^{-3}$	$9.9840 \times 10^{-7}$	$9.9856 \times 10^{-7}$	$3.14 \times 10^{-3}$
1.0	$5.8454 \times 10^{-7}$	$4.1368 \times 10^{-7}$	$2.15 \times 10^{-3}$	$4.1329 \times 10^{-7}$	$4.1338 \times 10^{-7}$	$3.32 \times 10^{-3}$

**Table 4.5:** A comparison of the calculated error norms  $L_2$  and  $L_\infty$  of the present scheme with those in Ref. [15, 17], for  $M = 16$ ,  $\Delta t = 0.01$  at various  $t_f$ .

From the presented tables one can clearly see that the newly calculated numerical results are in good harmony with both the analytical and other previously published numerical results. In fact, the present scheme in general produces better results with a smaller number of elements compared to other studies based on classical and trigonometric B-splines.

From the results given in the above tables, it is clearly seen that the approximate numerical solutions are getting closer and closer to exact ones as the mesh step sizes are refined. It should not be forgotten that the mesh step sizes must be chosen modestly small enough to avoid increase in the CPU simulation time and storage requirement capacity.

## 5. Conclusion

In this work, the developed fully discretized numerical scheme constructed using the conventional Crank-Nicolson-like finite difference technique for the integration of temporal quantities and a quintic Hermite B-spline finite element collocation technique for the integration of spatial quantities has been successfully applied to obtain approximate numerical solutions of the 1-D heat conduction equation for certain initial and boundary conditions. In order to confirm the accuracy and reliability of the suggested scheme, the error norms  $L_2$  and  $L_\infty$  calculated from the exact and numerical solutions are compared with the results given by the previous ones existing in the literature. It is seen that the results obtained by applying the developed scheme to the example problem are mostly better than the previous results and at least in some cases in good agreement with the previous ones. Generally speaking, it is found a good quality harmony between the existing and the above aforesaid results. For the stability test of the current scheme, von-Neumann (Fourier) technique is performed which demonstrates that the scheme is unconditionally stable. The main contribution of this paper is the ability of the newly presented scheme to produce much better results by using less computer storage capacity and requiring less CPU time. In conclusion, the suggested scheme is producing stable, efficient and accurate results, and can be successfully used to find approximate numerical solutions of initial and boundary value problems consisting of many linear and especially nonlinear PDEs which have an important role in describing natural phenomena encountered in most branches of applied and engineering sciences.

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## References

- [1] A. D. Amin, E. Amin, S. Hasan, A. Meysam, L. Yueming, W. Lian-Ping, J. Dengwei, X. Gongnan, *A comprehensive review on multi-dimensional heat conduction of multi-layer and composite structures: analytical solutions*, J. Therm. Sci., **30**(6) (2021), 1875-1907.
- [2] H. J. Xu, Z. B. Xing, F. Q. Wangb, Z. M. Cheng, *Review on heat conduction, heat convection, thermal radiation and phase change heat transfer of nanofluids in porous media: fundamentals and applications*, Chem. Eng. Sci., **195** (2019), 462-483.
- [3] F. Suarez-Carreno, L. Rosales-Romero, *Convergence and stability of explicit and implicit schemes in the simulation of the heat equation*, Appl.Sci. **11** (2021).
- [4] N. F. Kaskar, *Modified implicit method for solving one dimensional heat equation*, Int. J. Eng. Res. Comp. Sci. Eng., **8**(9) (2021), 1-6.
- [5] G. Lozande-Cruz, C.E. Rubio-Mercedes, J. Rodrigues-Riberio, *Numerical solution of heat equation with singular robin boundary condition*, Tendencias em Matematica Aplicada e Computacional, **19**(2) (2018), 209-220.
- [6] H. Sun, J. Zhang, *A high-order compact boundary value method for solving one-dimensional heat equations*, Numer. Methods Partial Differential Equations, **19**(6) (2003), 846-857.
- [7] A. Yosaf, S. U. Rehman, F. Ahmad, M. Z. Ullah, A. S. Alshomrani, *Eight-order compact finite difference scheme for 1D heat conduction equation*, Adv. Numer. Anal., (2016), Article ID 8376061, 12 pages.
- [8] F. Han, W. Dai, *New higher-order compact finite difference schemes for 1D heat conduction equations*, Appl. Math. Modell., **37** (2013), 7940-7952.
- [9] S. Dhawan, S. Kumar, *A numerical solution of one dimensional heat equation using cubic b-spline basis functions*, Int. J. Res. Rev. App. Sci., **1** (2009), 71-77.



- [10] H. Çağlar, M. Özer, N. Çağlar, *The numerical solution of the one-dimensional heat equation by using third degree b-spline functions*, Chaos Solitons Fractals, **38** (2008), 1197–1201.
- [11] J. Goh, A. A. Majid, A. I. Ismail, *Cubic b-spline collocation method for one-dimensional heat and advection-diffusion equations*, J. Appl. Math., (2012), Article ID 458701, 8 pages.
- [12] M. H. Khabir, R. A. Farah, *Cubic b-spline collocation method for one-dimensional heat equation*, Pure Appl. Math. J., **6**(1) (2017), 51-58.
- [13] B. Mebrate, *Numerical solution of a one dimensional heat equation with Dirichlet boundary conditions*, Amer. J. Appl. Math., **3**(6) (2015), 305-311.
- [14] N. Patel, J. U. Pandya, *One-dimensional heat equation subject to both Neumann and Dirichlet initial boundary conditions and used a spline collocation method*, Kalpa Publ. Comput., **2** (2017), 107-112.
- [15] M. R. Hooshmandasl, M. H. Heydari, F. M. Maalek Ghaini, *Numerical solution of the one-dimensional heat equation by using Chebyshev wavelets method*, J. Appl. Comp. Math., **1**(6) (2012), 1-7.
- [16] T. Tarmizi, E. Safitri, S. Munzir, M. Ramli, *On the numerical solutions of a one-dimensional heat equation: spectral and crank Nicolson method*, AIP Conference Proceedings, (2020).
- [17] S. Kutluay, N. M. Yağmurlu, A. S. Karakaş, *An effective numerical approach based on cubic hermite b-spline collocation method for Solving the 1D heat conduction equation*, New Trends in Math. Sci., **10**(4) (2022), 20-31.
- [18] H. Zeybek, S. B. G. Karakoç, *A collocation algorithm based on quintic b-splines for the solitary wave simulation of the GRLW equation*, Sci. Iran., **26**(6) (2019), 3356-3368.
- [19] W. Wu, J. Manafian, K. A. Khalid, S. B. G. Karakoç, A. H. Taqi, M. A. Mahmoud, *Numerical and analytical results of the 1D BBM equation and 2D coupled BBM-system by finite element method*, Int. J. Mod. Phys. B, **36**(28) (2022).
- [20] S. B. G. Karakoç, Y. Uçar, N. M. Yağmurlu, *Different linearization techniques for the numerical solution of the MEW equation*, Selcuk J. Appl. Math., **13**(2) (2012), 43-62.
- [21] S. Arora, I. Kaur, W. Tilahun, *An exploration of quintic Hermite splines to solve burgers' equation*, Arab. J. Math., (2020), 19–36.
- [22] S. Arora, I. Kaur, *Applications of quintic Hermite collocation with time discretization to singularly perturbed problems*, Appl. Math. Comput., **316** (2018) 409–421.
- [23] S. P. Kaur, A. K. Mittal, V. K. Kukreja, A. Kaundal, N. Parumasur, P. Singh, *Analysis of a linear and non-linear model for diffusion–dispersion phenomena of pulp washing by using quintic hermite interpolation polynomials*, Afr. Mat., **32** (2021), 997–1019.