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# POWER SERIES SOLUTION OF DIFFERENTIAL ALGEBRAIC EQUATION

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**Özet**

Bu makalede, Lineer diferensiyel cebirsel denklemleri çözmek için kuvvet serisi yöntemi uygulandı. Bu yöntem diferensiyel cebirsel denklemlerin nümerik çözmek için bir keyfi mertebe verir.. Biz bu metodu test etmek için bir örnek verdik ve elde ettiğimiz sonuç ile analitik çözümü karşılaştırdık.
**Anahtar kelimeler:** Diferensiyel cebirsel denklem, keyfi mertebe, kuvvet serisi.

**Abstract**

In this paper, we apply the power series method to solve a linear differential algebraic equation. This method gives an arbitrary order for solving differential algebraic equation numerically. We have given an example to test the method and the result of the our method compared the exact solution of the given problem.

**Keywords**: Differential Algebraic Equation; Arbitrary Order; Power Series.

## 1. INTRODUCTION

A differential-algebraic equation has the form

$$F(y', y, x) = 0 \tag{1.1}$$

with initial values

$$y(x_0) = y_0, \quad y'(x_0) = y_1,$$

where $F$ and $y$ is a vector function for which we assumed sufficient differentiability[3,4,6], and the initial values to be consistent, i.e.

$$F(y_0, y_0', x_0) = 0. \tag{1.2}$$

The solutions of (1.1) can be assumed that

$$y = y_0 + y_1 x + ex^2, \tag{1.3}$$

where $e$ is a vector function which is the same size as $y_0$ and $y_0'$. Substitute (1.3) into (1.1) and neglect higher order term, we have the linear equation of $e$ in the form

$$Ae = B \tag{1.4}$$

where $A$ and $B$ are constant matrixes. Solving equation (1.4), the coefficients of $x^2$ in (1.3) can be determined. Repeating above procedure for higher order terms, we can get the arbitrary order power series of the solutions for (1.1) and we have numerical solution of differential algebraic equation in (1.1).

## 2. POWER SERIES FOR DIFFERENTIAL ALGEBRAIC EQUATIONS

We define another type power series in the form

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$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + (f_n + p_1 e_1 + \cdots + p_m e_m) x^n \tag{2.1}$$

where $p_1, p_2 \cdots p_m$ are constants $e_1, e_2 \cdots e_m$ are basis of vector $e$, $m$ is size of vector $e$. $y$ is a vector in (1.3) with $m$ elements. Every element can be represented by the Power series in (2.1). Therefore we can write

$$y_i = y_{i,0} + y_{i,1} x + y_{i,2} x^2 + \cdots + e_i x^n \tag{2.2}$$

from (1.3), where $y_i$ is $i$th element of $y$. Substitute (2.2) into (1.1), we can get

$$f_i = (f_{i,n} + p_{i,1} e_1 + \cdots + p_{i,m} e_m) x^{n-j} + Q(x^{n-j+1}), \tag{2.3}$$

where $f_i$ is $i$th element of $f(y, y', x)$ in (1.1) and if $f(y, y', x)$ have $y'$ then $j$ is 0, otherwise 1. From (2.3) and (1.4), we can determine the linear equation in (1.4) as follow

$$A_{i,j} = P_{i,j}. \tag{2.4}$$

$$B_i = -f_{i,n} \tag{2.5}$$

Solve this linear equation, we have $e_i$ $(i = 1, \cdots, m)$. Substitute $e_i$ into (2.2); we have $y_i$ $(i = 1, \cdots, m)$ which are polynomials of degree $n$. Repeating this procedure from (2.2) to (2.4), we can get the arbitrary order Power series solution of differential-algebraic equations in (1.1).
Let step size of $x$ to be $h$ and substitute it into the power series of $y$ and derivative of $y$, we have $y$ and $y'$ at $x = x_0 + h$. If we repeat above procedure, we can have numerical solution of differential-algebraic equations in (1.1)[1,2,7].

## 3. AN EXAMPLE

In this section, we consider the following differential algebraic equation as a test problem.

$$\begin{aligned} y_1'(x) &= e^x + y_2'(x) + x y_2'(x) \\ y_2(x) &= \cos x \end{aligned} \tag{3.1}$$

and initial values

$$y_1(0) = 1,\ y_2(0) = 1,\ y_1'(0) = 2,\ y_2'(0) = 0.$$

The exact solution is

$$\begin{aligned} y_1 &= e^x + x\cos x \\ y_2 &= \cos x \end{aligned}$$

From initial values, the solutions of (3.1) can be supposed as

$$\begin{aligned} y_1(x) &= y_{0,1} + y_{0,1}' x + e_1 x^2 = 1 + 2x + e_1 x^2 \\ y_2(x) &= y_{0,2} + y_{0,2}' x + e_2 x^2 = 1 + e_2 x^2 \end{aligned} \tag{3.2}$$

Substitute (3.2) into (3.1) and neglect higher order terms, we have

$$\begin{aligned} &(-1 + 2e_1) x + O(x^2) = 0 \\ &\left(\frac{1}{2} + e_2\right) x^2 + O(x^3) = 0, \end{aligned} \tag{3.3}$$

These formulas correspond to (2.3). The linear equation that corresponds to (2.4) can be given in the following:

$$Ae = B, \tag{3.4}$$

Where

$$A=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}, B=\begin{pmatrix}1\\ -0.5\end{pmatrix} \text{ and } e=\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

From Eq. (3.4), we have linear equation

$$\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}=\begin{pmatrix}1\\ -0.5\end{pmatrix}.$$

Solving this linear equation we have

$$e=\begin{pmatrix}0.5\\ -0.5\end{pmatrix}$$

and

$$\begin{aligned} y_1(x)&=1+2x+\frac{1}{2}x^2\\ y_2(x)&=1-\frac{1}{2}x^2 \end{aligned} \tag{3.5}$$

From (3.5) the solutions of (3.1) can be supposed as

$$\begin{aligned} y_1(x)&=1+2x+\frac{1}{2}x^2+e_1x^3\\ y_2(x)&=1-\frac{1}{2}x^2+e_2x^3 \end{aligned} \tag{3.6}$$

In like manner, substitute (3.6) into (3.1) and neglect higher order terms, then we have

$$\begin{aligned} &(3e_1+1)x^2+O(x^3)=0\\ &e_2x^3+O(x^4)=0 \end{aligned} \tag{3.7}$$

where

$$A=\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}, B=\begin{pmatrix}-1\\ 0\end{pmatrix} \text{ ve } e=\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

From (3.7) we have linear equation

$$\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}=\begin{pmatrix}-1\\ 0\end{pmatrix}.$$

Solving this linear equation, we have

$$e=\begin{pmatrix}-1/3\\ 0\end{pmatrix}$$

Therefore

$$\begin{aligned} y_1(x)&=1+2x+\frac{1}{2}x^2-\frac{1}{3}x^3\\ y_2(x)&=1-\frac{1}{2}x^2 \end{aligned} \tag{3.8}$$

Repeating above procedure we have

$$\begin{aligned} y_1^*(x)=\;&1+2.x+0.5000000000x^2-0.3333333333.x^3+0.04166666667x^4\\ &+0.05000000000x^5+0.001388888889x^6-0.001190476190x^7\\ &+0.00002480158730x^8+0.00002755731922x^9 \end{aligned}$$

$$y_2^*(x)=1-0.5000000000x^2+0.04166666667x^4-0.001388888889x^6$$

$+0.00002480158730x^8$.

We show tables 1 and 2 for the solution of (3.1) by above numerical method. The numerical values on Tables 1 and 2 are coinciding with the exact solutions of (3.1).

Table 1. Numerical solution of $y_1(x)$ in (3.1).

| $x$ | Exact $y_1(x)$ | $y_1^*(x)$ | $\lvert y_1(x)-y_1^*(x)\rvert$ |
|---|---|---|---|
| 0.1 | 1.204671335 | 1.204671335 | 0 |
| 0.2 | 1.417416074 | 1.417416074 | 0 |
| 0.3 | 1.636459755 | 1.636459755 | 0 |
| 0.4 | 1.860249096 | 1.860249096 | 0 |
| 0.5 | 2.087512552 | 2.087512551 | $0.1\ 10^{-8}$ |
| 0.6 | 2.317320169 | 2.317320169 | 0 |
| 0.7 | 2.549142238 | 2.549142236 | $0.2\ 10^{-8}$ |
| 0.8 | 2.782906295 | 2.782906288 | $0.7\ 10^{-8}$ |
| 0.9 | 3.019052082 | 3.019052064 | $0.18\ 10^{-7}$ |
| 1.0 | 3.258584134 | 3.258584106 | $0.28\ 10^{-7}$ |

Table 2. Numerical solution of $y_2(x)$ in (3.1).

| $x$ | $y_2(x)$ | $y_2^*(x)$ | $\lvert y_2(x)-y_2^*(x)\rvert$ |
|---|---|---|---|
| 0.1 | 0.9950041653 | 0.9950041653 | 0 |
| 0.2 | 0.9800665778 | 0.9800665779 | $-0.1\ 10^{-9}$ |
| 0.3 | 0.9553364891 | 0.9553364891 | 0 |
| 0.4 | 0.9210609940 | 0.9210609941 | $-0.1\ 10^{-9}$ |
| 0.5 | 0.8775825619 | 0.8775825622 | $-0.3\ 10^{-9}$ |
| 0.6 | 0.8253356149 | 0.8253356166 | $-0.17\ 10^{-8}$ |
| 0.7 | 0.7648421873 | 0.7648421951 | $-0.78\ 10^{-8}$ |
| 0.8 | 0.6967067093 | 0.6967067388 | $-0.295\ 10^{-7}$ |
| 0.9 | 0.6216099683 | 0.6216100638 | $-0.955\ 10^{-7}$ |
| 1.0 | 0.5403023059 | 0.5403025794 | $-0.2735\ 10^{-6}$ |

The graph of $y_1(x)$, $y_2(x)$ and their power series approximant are simultaneously shown in Fig. 1 and Fig. 2. As can be seen from the graphics, the accuracy of the approximation by using power series which agree with given exact solution of the equation systems.

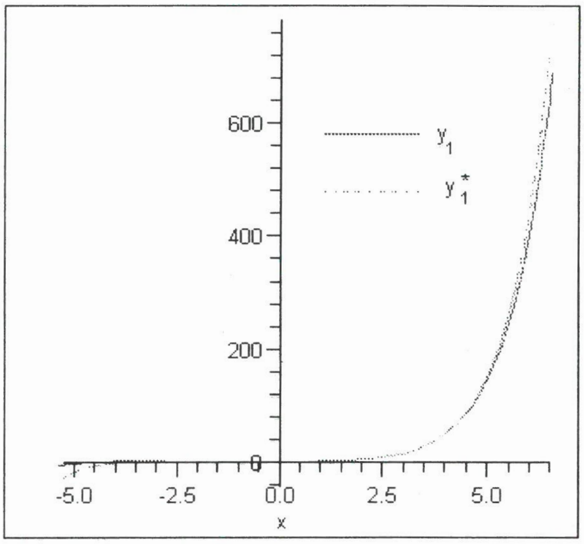


**Fig. 1.** Graph of $y_1(x)$ and its power approximant in the interval [-6,6]

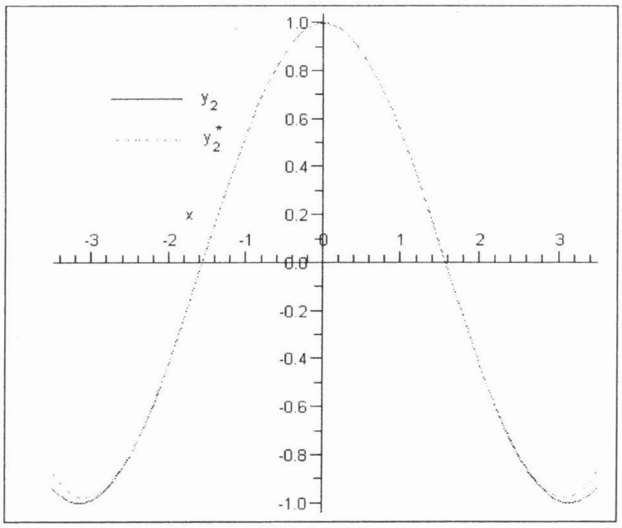


**Fig. 2.** Graph of $y_2(x)$ and its power approximant in the interval $[-4,4]$.

## 4. CONCLUSION

A Power approximation method has proposed for solving Differential algebraic equations in this study. This method is very simple an effective for most of differential algebraic equations.

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