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THEORY OF GENERALIZED CONNECTEDNESS ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-CONNECTEDNESS) IN GENERALIZED TOPOLOGICAL SPACES ($\mathscr{T}_{\mathfrak{g}}$-SPACES)

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Abstract. In this paper, the definitions of novel classes of generalized connected sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connected sets*) and generalized disconnected sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*disconnected sets*) in generalized topological spaces (briefly, $\mathscr{T}_{\mathfrak{g}}$-*spaces*) are defined in terms of generalized sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*sets*) and, their properties and characterizations with respect to set-theoretic relations are presented. The basic properties and characterizations of the notions of local, pathwise, local pathwise and simple $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness are also presented. The study shows that local pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness implies local $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a $\mathscr{T}_{\mathfrak{g}}$-property. Diagrams establish the various relationships amongst these types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness presented here and in the literature, and a nice application supports the overall theory.

1. Introduction

Among the most important topological properties (briefly, $\mathscr{T}$-properties relative to ordinary topology, and $\mathscr{T}_{\mathfrak{g}}$-properties relative to generalized topology), the $\mathscr{T}$-properties[1] called $\mathfrak{T}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}$-connectedness in $\mathscr{T}$-spaces (ordinary and generalized connectedness in ordinary topological spaces) and the $\mathscr{T}_{\mathfrak{g}}$-properties

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[1]Notes to the reader: $\mathfrak{T}=(\Omega,\mathscr{T})$, $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ are topological spaces (briefly, $\mathscr{T}$-space and $\mathscr{T}_{\mathfrak{g}}$-space) with ordinary and generalized topologies $\mathscr{T}$ and $\mathscr{T}_{\mathfrak{g}}$ (briefly, topology and $\mathfrak{g}$-topology). Subsets of $\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$-sets; subsets of $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$, respectively, are called $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-open sets, and their complements are called $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-closed sets. Generalizations of $\mathfrak{T}$-sets, $\mathscr{T}$-open and $\mathscr{T}$-closed sets, respectively, are called $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathscr{T}$-open and $\mathfrak{g}$-$\mathscr{T}$-closed sets; generalizations of $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets, respectively, are called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-closed sets. Connectedness in $\mathfrak{T}$ with $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-sets are called $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-connectedness, respectively; connectedness in $\mathfrak{T}_{\mathfrak{g}}$ with $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets are called $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, respectively.

called $\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces (ordinary and generalized connectedness in generalized topological spaces) are no doubt the most important invariant properties [1, 2, 3]. Indeed, $\mathfrak{T}$-connectedness is an absolute property of a $\mathfrak{T}$-set [1, 4, 5], and $\mathfrak{g}$-$\mathfrak{T}$-connectedness, $\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, respectively, are absolute properties of a $\mathfrak{g}$-$\mathfrak{T}$-set, a $\mathfrak{T}_{\mathfrak{g}}$-set, and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set [3, 6, 7, 8, 9, 10, 11]. Typical examples of $\mathfrak{g}$-$\mathfrak{T}$-connectedness in $\mathscr{T}$-spaces are α, β, γ-connectedness [12, 13, 14]; examples of $\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces are semi$^{*}\alpha$, s, gb-connectedness [2, 15, 16], whereas examples of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces are bT$^{\mu}$, μ-rgb, π p-connectedness [17, 18, 19], among others.

In the literature of $\mathscr{T}_{\mathfrak{g}}$-spaces, the study of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets by various authors has produced some new classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces, similar in descriptions to $\mathfrak{g}$-$\mathfrak{T}$-connectedness in $\mathscr{T}$-spaces [17, 20, 21]. By using the θ-modification generalized topology and $\gamma\theta$-operator introduced by [22], [23] have extended the notion of θ-connectedness [24] to the setting of $\mathscr{T}_{\mathfrak{g}}$-spaces and studied its $\mathscr{T}_{\mathfrak{g}}$-properties accordingly. Based on the work of [12], [20] have introduced a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces called hyperconnected and studied the $\mathscr{T}_{\mathfrak{g}}$-properties associated with it and its analogue in the generalized sense. In the same year, [25] have introduced, studied and exemplified the notion of extremally μ-disconnected $\mathscr{T}_{\mathfrak{g}}$-spaces, just to name a few.

In view of the above references, it would appear that, from every new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set introduced in a $\mathscr{T}_{\mathfrak{g}}$-space, there can be introduced a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in the $\mathscr{T}_{\mathfrak{g}}$-space. Having introduced a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets and studied from it some $\mathscr{T}_{\mathfrak{g}}$-properties in a $\mathscr{T}_{\mathfrak{g}}$-space [6, 7, 8, 9, 10], it seems, therefore, reasonable to introduce a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in the $\mathscr{T}_{\mathfrak{g}}$-space and discuss its $\mathscr{T}_{\mathfrak{g}}$-properties. In this paper, we attempt to make a contribution to such a development by introducing a new theory, called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Connectedness*, in which it is presented a new generalized version of $\mathfrak{T}_{\mathfrak{g}}$-connectedness in terms of the notion of $\mathfrak{g}$-$\mathfrak{T}$-set, discussing the fundamental properties and giving its characterizations on this ground.

The paper is organised as follows: In Sect. 2, preliminary notions are described in Sect. 2.1 and the main results of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in a $\mathscr{T}_{\mathfrak{g}}$-space are reported in Sect. 3. In Sect. 4, the establishment of the relationships among various types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness are discussed in Sect. 4.1. To support the work, a nice application of the concept of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in a $\mathscr{T}_{\mathfrak{g}}$-space is presented in Sect. 4.2. Finally, Sect. 5 provides concluding remarks and future directions of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in a $\mathscr{T}_{\mathfrak{g}}$-space.

2. Theory

2.1. **Preliminaries.** Notations and notions not presented below are found in the standard references [6, 7, 8, 9, 10]. Everywhere, $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces are designated by the topological structures $\mathfrak{T} \stackrel{\text{def}}{=} (\Omega, \mathscr{T})$ and $\mathfrak{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} (\Omega, \mathscr{T}_{\mathfrak{g}})$, respectively, on both of which no separation axioms are assumed unless otherwise mentioned [8, 10, 26, 27, 28]. The symbols I_n^0, $I_n^* \subset \mathbb{N}^0$ designate 0-included and 0-excluded finite index sets while I_∞^0, $I_\infty^* \subseteq \mathbb{N}^0$ the corresponding infinite index sets [10]. By $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}} \subseteq \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ are meant a pair of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets [10]. The operators $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda)$ carrying any $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ into its interior $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and closure $\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ are called $\mathfrak{g}$-interior and $\mathfrak{g}$-closure operators [9]. The totality of all possible compositions of these $\mathfrak{g}$-operators forms the class $\mathscr{L}_{\mathfrak{g}}[\Omega] \stackrel{\text{def}}{=} \{\mathbf{op}_{\mathfrak{g},\nu\mu}(\cdot) = (\text{op}_{\mathfrak{g},\nu}(\cdot), \neg\,\text{op}_{\mathfrak{g},\mu}(\cdot)) : (\nu,\mu) \in I_3^0 \times I_3^0\}$ [9]. Then, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set if and only if there exist $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ and $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathscr{L}_{\mathfrak{g}}[\Omega]$ such that the following statement holds:

$$(\exists\xi)\big[(\xi \in \mathscr{S}_{\mathfrak{g}}) \wedge \big((\mathscr{S}_{\mathfrak{g}} \subseteq \text{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})) \vee (\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\text{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}))\big)\big]. \tag{2.1}$$

The derived class $\mathfrak{g}$-ν-$\text{S}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\text{E}\in\{\text{O,K}\}} \mathfrak{g}$-$\nu$-$\text{E}[\mathfrak{T}_{\mathfrak{g}}]$ is called the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of category $\nu \in I_3^0$ (briefly, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-*sets*) [9, 10]. Accordingly, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets [10] are

$$\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{(\nu,\text{E})\in I_3^0\times\{\text{O,K}\}} \mathfrak{g}\text{-}\nu\text{-E}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\text{E}\in\{\text{O,K}\}} \mathfrak{g}\text{-E}[\mathfrak{T}_{\mathfrak{g}}]. \tag{2.2}$$

Notations and notions utilized in the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in $\mathscr{T}_{\mathfrak{g}}$-spaces are now presented. By $\pi : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is meant a $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map between $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$. A map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is called a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map if and only if, for every $(\mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$, there exists $(\mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}) \in \mathscr{T}_{\mathfrak{g},\Sigma} \times \neg\mathscr{T}_{\mathfrak{g},\Sigma}$ such that:

$$\big[\pi_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\omega}) \subseteq \text{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\sigma})\big] \vee \big[\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \supseteq \neg\,\text{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\sigma})\big]. \tag{2.3}$$

It is said to be of category ν if and only if $\pi_{\mathfrak{g}} \in \mathfrak{g}$-$\nu$-$\text{M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ where,

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} :\ & (\forall \mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega})(\exists \mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)) \\ & \big[(\pi_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\omega}) \subseteq \text{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},\sigma})) \vee (\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \supseteq \neg\,\text{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},\sigma}))\big]\big\}. \end{aligned} \tag{2.4}$$

Let $\mathfrak{g}$-ν-$\text{M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] = \bigcup_{\text{E}\in\{\text{O,K}\}} \mathfrak{g}$-$\nu$-$\text{M}_{\text{E}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ where,

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-M}_{\text{O}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] &\stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} : (\forall \mathscr{O}_{\mathfrak{g},\omega})(\exists \mathscr{O}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)) \\ &\qquad \big[\pi_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\omega}) \subseteq \text{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},\sigma})\big]\big\}, \\ \mathfrak{g}\text{-}\nu\text{-M}_{\text{K}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] &\stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} : (\forall \mathscr{K}_{\omega})(\exists \mathscr{K}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)) \\ &\qquad \big[\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \supseteq \text{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},\sigma})\big]\big\}. \end{aligned} \tag{2.5}$$

Then, if $\pi_{\mathfrak{g}} \in \mathfrak{g}$-$\nu$-$\text{M}_{\text{O}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, it is called a $\mathfrak{g}$-ν-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-open map; if $\pi_{\mathfrak{g}} \in \mathfrak{g}$-$\nu$-$\text{M}_{\text{K}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, it is called a $\mathfrak{g}$-ν-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-closed map. Accordingly, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-maps [10] are

$$\begin{aligned} \mathfrak{g}\text{-M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] &= \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \\ &= \bigcup_{(\nu,\text{E})\in I_3^0\times\{\text{O,K}\}} \mathfrak{g}\text{-}\nu\text{-M}_{\text{E}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \\ &= \bigcup_{\text{E}\in\{\text{O,K}\}} \mathfrak{g}\text{-M}_{\text{E}}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]. \end{aligned} \tag{2.6}$$

A map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is said to be $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous if and only if, for every $(\mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}) \in \mathscr{T}_{\mathfrak{g},\Sigma} \times \neg\mathscr{T}_{\mathfrak{g},\Sigma}$, there exists $(\mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$ such that the following statement holds:

$$\big[\pi_{\mathfrak{g}}^{-1}(\mathscr{O}_{\mathfrak{g},\sigma}) \subseteq \text{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\omega})\big] \vee \big[\pi_{\mathfrak{g}}^{-1}(\mathscr{K}_{\mathfrak{g},\sigma}) \supseteq \neg\,\text{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega})\big]. \tag{2.7}$$

It is said to be of category ν if and only if $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ where,

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}}:\ &\left(\forall\mathscr{O}_{\mathfrak{g},\sigma},\mathscr{K}_{\mathfrak{g},\sigma}\right)\left(\exists\mathscr{O}_{\mathfrak{g},\omega},\mathscr{K}_{\mathfrak{g},\omega},\mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right)\\ &\left[\left(\pi_{\mathfrak{g}}^{-1}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right)\vee\left(\pi_{\mathfrak{g}}^{-1}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\supseteq\neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right)\right]\big\}.\end{aligned}\tag{2.8}$$

Obviously, $\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]=\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. A map $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\longrightarrow\mathfrak{T}_{\mathfrak{g},\Sigma}$ is said to be $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-irresolute if and only if, for every $\left(\mathscr{O}_{\mathfrak{g},\sigma},\mathscr{K}_{\mathfrak{g},\sigma}\right)\in\mathscr{T}_{\mathfrak{g},\Sigma}\times\neg\mathscr{T}_{\mathfrak{g},\Sigma}$, there exists $\left(\mathscr{O}_{\mathfrak{g},\omega},\mathscr{K}_{\mathfrak{g},\omega}\right)\in\mathscr{T}_{\mathfrak{g},\Omega}\times\neg\mathscr{T}_{\mathfrak{g},\Omega}$ such that the following statement holds:

$$\left[\pi_{\mathfrak{g}}^{-1}\left(\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\right)\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right]\vee\left[\pi_{\mathfrak{g}}^{-1}\left(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\right)\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right].\tag{2.9}$$

It is said to be a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-irresolute map of category ν if and only if $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-}\nu\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ where,

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}}:\ &\left(\forall\mathscr{O}_{\mathfrak{g},\sigma},\mathscr{K}_{\mathfrak{g},\sigma}\right)\left(\exists\mathscr{O}_{\mathfrak{g},\omega},\mathscr{K}_{\mathfrak{g},\omega},\mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right)\\ &\left[\left(\pi_{\mathfrak{g}}^{-1}\left(\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\right)\subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right)\vee\left(\pi_{\mathfrak{g}}^{-1}\left(\neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\right)\supseteq\right.\right.\\ &\left.\left.\neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right)\right]\big\}.\end{aligned}\tag{2.10}$$

Evidently, $\mathfrak{g}\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]=\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. The classes $\mathrm{M}_{\mathrm{O}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and $\mathrm{M}_{\mathrm{K}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ denote the families of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed maps, respectively, from $\mathfrak{T}_{\mathfrak{g},\Omega}$ into $\mathfrak{T}_{\mathfrak{g},\Sigma}$, with $\mathrm{M}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]=\bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}}\mathrm{M}_{\mathrm{E}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$.

Definition 2.1 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Separation, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Connected)**.** A $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation of category ν of two nonempty $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=\left(\Omega,\mathscr{T}_{\mathfrak{g}}\right)$ is realised if and only if there exists either a pair $\left(\mathscr{U}_{\mathfrak{g},\xi},\mathscr{U}_{\mathfrak{g},\zeta}\right)\in\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\times\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets or a pair $\left(\mathscr{V}_{\mathfrak{g},\xi},\mathscr{V}_{\mathfrak{g},\zeta}\right)\in\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\times\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that:

$$\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{U}_{\mathfrak{g},\lambda}=\mathscr{R}_{\mathfrak{g}}\sqcup\mathscr{S}_{\mathfrak{g}}\Big)\bigvee\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{V}_{\mathfrak{g},\lambda}=\mathscr{R}_{\mathfrak{g}}\sqcup\mathscr{S}_{\mathfrak{g}}\Big).\tag{2.11}$$

Two nonempty $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=\left(\Omega,\mathscr{T}_{\mathfrak{g}}\right)$ which are not $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated of category ν are said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected of category ν.

The definitions of classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets of category ν follow.

Definition 2.2. Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=\left(\Omega,\mathscr{T}_{\mathfrak{g}}\right)$. Then:

– I. The $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ is said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-connected sets:

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right] \stackrel{\text{def}}{=} \Big\{&\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}:\left(\forall\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi,\zeta}\in\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\times\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right)\\ &\Big[\neg\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{U}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big)\bigwedge\neg\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{V}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big)\Big]\Big\}.\end{aligned}\tag{2.12}$$

– II. The $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-separated sets:

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right] \overset{\mathrm{def}}{=} \Big\{ \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \left(\exists \left(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right)$$
$$\left[\left(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\right) \bigvee \left(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\right)\right]\Big\}. \tag{2.13}$$

The dependence of $\mathfrak{g}$-ν-$\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-ν-$\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ on both $\mathfrak{g}$-ν-$\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-ν-$\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is immediate. Thus, to define the pairs $\left(\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right], \nu\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right)$, $\left(\mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}\right], \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}\left[\mathfrak{T}\right]\right)$, and $\left(\nu\text{-}\mathrm{Q}\left[\mathfrak{T}\right], \nu\text{-}\mathrm{D}\left[\mathfrak{T}\right]\right)$, respectively, it suffices to let them be dependent on the pairs $\left(\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right], \nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right)$, $\left(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}\right], \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}\right]\right)$, and $\left(\nu\text{-}\mathrm{O}\left[\mathfrak{T}\right], \nu\text{-}\mathrm{K}\left[\mathfrak{T}\right]\right)$; the characters of these classes are found in our previous works [9, 10]. The notations $\mathfrak{g}$-$\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-$\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ stand for $\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, respectively.

Remark 2.3. In defining the classes $\mathfrak{g}$-ν-$\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}$-ν-$\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it is clear that by the statement $\left(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is meant a pair of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets. Furthermore, by $\Omega \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ or $\Omega \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is meant a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connection of category ν or a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation of category ν of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is realised.

Definition 2.4. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then:

– I. $\mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} \overset{\mathrm{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\right)$ if and only if it is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected of category ν.

– II. $\mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} \overset{\mathrm{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\right)$ if and only if it is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated of category ν.

In the sequel, by $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle} \overset{\mathrm{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle}\right)$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle} \overset{\mathrm{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle}\right)$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LPC}\rangle} \overset{\mathrm{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LPC}\rangle}\right)$, and $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{SC}\rangle} \overset{\mathrm{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{SC}\rangle}\right)$, respectively, are meant *locally*, *pathwise*, *locally pathwise*, and *simply* $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces. Finally, by a $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{A}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{A}\rangle} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{A}\rangle}\right)$ is meant $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{A}\rangle} = \bigvee_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{A}\rangle} = \left(\Omega, \bigvee_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{A}\rangle}\right) = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{A}\rangle}\right)$, where $\mathrm{A} \in \{\mathrm{C}, \mathrm{LC}, \mathrm{PC}, \mathrm{LPC}, \mathrm{SC}, \mathrm{D}\}$.

By omitting the subscript $\mathfrak{g}$ in almost all symbols of the above descriptions, very similar descriptions are obtained but in a $\mathscr{T}$-space [10]. In the following sections, the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness are presented.

3. Main Results

If for all $\left(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ neither $\mathscr{U}_{\mathfrak{g},\xi} \sqcup \mathscr{V}_{\mathfrak{g},\zeta} = \Omega$ nor $\mathscr{U}_{\mathfrak{g},\zeta} \sqcup \mathscr{V}_{\mathfrak{g},\xi} = \Omega$ is satisfied, then a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. Hence, the following theorem:

Theorem 3.1. *If* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\right)$ *be a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$*-space, then there exists a pair* $\left(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}\right) \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *or a pair* $\left(\mathscr{U}_{\mathfrak{g},\zeta}, \mathscr{V}_{\mathfrak{g},\xi}\right) \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *such that*

$$\left[\mathscr{U}_{\mathfrak{g},\xi} \sqcup \mathscr{V}_{\mathfrak{g},\zeta} = \Omega\right] \vee \left[\mathscr{U}_{\mathfrak{g},\zeta} \sqcup \mathscr{V}_{\mathfrak{g},\xi} = \Omega\right]. \tag{3.1}$$

Proof. Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle})$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space. Then, there exists a pair $(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ such that

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big).$$

If $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ then $\mathscr{U}_{\mathfrak{g},\zeta} = \complement(\mathscr{U}_{\mathfrak{g},\xi}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, and if $\mathscr{V}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ then $\mathscr{V}_{\mathfrak{g},\zeta} = \complement(\mathscr{V}_{\mathfrak{g},\xi}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$. Thus, if $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$, it suffices to set $\mathscr{V}_{\mathfrak{g},\zeta} = \complement(\mathscr{U}_{\mathfrak{g},\xi})$, and if $\mathscr{V}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, it suffices to set $\mathscr{U}_{\mathfrak{g},\zeta} = \complement(\mathscr{V}_{\mathfrak{g},\xi})$. By substitutions, it follows, then, that

$$[\mathscr{U}_{\mathfrak{g},\xi} \sqcup \mathscr{V}_{\mathfrak{g},\zeta} = \Omega] \vee [\mathscr{U}_{\mathfrak{g},\zeta} \sqcup \mathscr{V}_{\mathfrak{g},\xi} = \Omega],$$

which was to be proved. □

Remark 3.2. Given $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{T}_{\mathfrak{g}}$ and $\neg\,\mathrm{op}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, the statement $(\mathscr{R}_{\mathfrak{g}} \cap \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})) \cup (\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{R}_{\mathfrak{g}}) \cap \mathscr{S}_{\mathfrak{g}}) = \emptyset$, when $\nu = 0$, may be called the *Hausdorff-Lennes Separation Condition* in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$.

If a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected, then either $\complement(\mathscr{U}_{\mathfrak{g},\lambda}) = \mathscr{U}_{\mathfrak{g},\eta}$, so that $\mathscr{U}_{\mathfrak{g},\lambda} \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ or, $\complement(\mathscr{V}_{\mathfrak{g},\lambda}) = \mathscr{V}_{\mathfrak{g},\eta}$, so that $\mathscr{V}_{\mathfrak{g},\lambda} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$, where $(\lambda, \eta) \in \{(\xi,\zeta), (\zeta,\xi)\}$. Therefore, $\mathfrak{T}_{\mathfrak{g}}$ is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected if it has no nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$. Hence, these theorems follow:

Theorem 3.3. *If a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ has a nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, then it is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle})$:*

$$\exists \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}). \tag{3.2}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ be a nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed set in $\mathfrak{T}_{\mathfrak{g}}$. Then, there exists $(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ such that

$$[\mathscr{U}_{\mathfrak{g},\xi} \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \complement(\mathscr{U}_{\mathfrak{g},\zeta})] \vee [\complement(\mathscr{V}_{\mathfrak{g},\xi}) \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g},\zeta}].$$

Consequently, the following relation holds:

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} \supseteq \mathscr{S}_{\mathfrak{g}} \cup \mathscr{U}_{\mathfrak{g},\zeta} \supseteq \complement(\mathscr{U}_{\mathfrak{g},\zeta}) \cup \mathscr{U}_{\mathfrak{g},\zeta}\Big)$$
$$\bigvee \Big(\complement(\mathscr{V}_{\mathfrak{g},\xi}) \cup \mathscr{V}_{\mathfrak{g},\xi} \supseteq \mathscr{S}_{\mathfrak{g}} \cup \mathscr{V}_{\mathfrak{g},\xi} \supseteq \bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda}\Big).$$

Since $\complement(\mathscr{U}_{\mathfrak{g},\zeta}) \cup \mathscr{U}_{\mathfrak{g},\zeta} = \Omega$, $\mathscr{S}_{\mathfrak{g}} \cup \mathscr{U}_{\mathfrak{g},\zeta} = \Omega$ and, consequently, $\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega$; observe that, $\mathscr{U}_{\mathfrak{g},\xi} = \mathscr{S}_{\mathfrak{g}} = \complement(\mathscr{U}_{\mathfrak{g},\zeta})$ because $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ is a nonempty proper $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed set in $\mathfrak{T}_{\mathfrak{g}}$. Since $\complement(\mathscr{V}_{\mathfrak{g},\xi}) \sqcup \mathscr{V}_{\mathfrak{g},\xi} = \Omega$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, $\complement(\mathscr{V}_{\mathfrak{g},\xi}) = \mathscr{S}_{\mathfrak{g}} = \mathscr{V}_{\mathfrak{g},\zeta}$. Therefore, $\complement(\mathscr{V}_{\mathfrak{g},\xi}) \cup \mathscr{V}_{\mathfrak{g},\xi} = \mathscr{S}_{\mathfrak{g}} \cup \mathscr{V}_{\mathfrak{g},\xi} = \bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda}$. By substitutions, it consequently follows that

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big),$$

which was to be proved. □

The converse of the above theorem also holds as demonstrated below.

Theorem 3.4. *If* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}\big)$ *be a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}$*-space, then it has a nonempty proper* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open-closed set* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$*:*

$$\exists \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \iff \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}\big). \tag{3.3}$$

Proof. Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}\big)$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}$-space. Then, there exists a pair $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\bigg) \bigvee \bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\bigg).$$

But $\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega$ implies either $\mathscr{U}_{\mathfrak{g},\xi} = \complement\left(\mathscr{U}_{\mathfrak{g},\zeta}\right)$ or $\mathscr{U}_{\mathfrak{g},\zeta} = \complement\left(\mathscr{U}_{\mathfrak{g},\xi}\right)$, and on the other hand, $\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega$ implies either $\mathscr{V}_{\mathfrak{g},\xi} = \complement\left(\mathscr{V}_{\mathfrak{g},\zeta}\right)$ or $\mathscr{V}_{\mathfrak{g},\zeta} = \complement\left(\mathscr{V}_{\mathfrak{g},\xi}\right)$. Consequently, there exists a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ such that

$$\big[\mathscr{U}_{\mathfrak{g},\xi} \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \complement\left(\mathscr{U}_{\mathfrak{g},\zeta}\right)\big] \vee \big[\complement\left(\mathscr{V}_{\mathfrak{g},\xi}\right) \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g},\zeta}\big].$$

Hence, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}$ has a nonempty proper $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$; this completes the proof of the theorem. □

Combined together, the above theorems establish the necessary and sufficient conditions for a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, and hence the following corollary.

Corollary 3.5. *Let* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *be a* $\mathscr{T}_{\mathfrak{g}}$*-space. Then it is a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}$*-space* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}\big)$ *if and only if* $\mathfrak{T}_{\mathfrak{g}}$ *has a nonempty proper* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open-closed set* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$*:*

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}\big) \iff \exists \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]. \tag{3.4}$$

A $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation is realised if the only $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathfrak{T}_{\mathfrak{g}}$ which are both $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed sets are the improper $\mathfrak{T}_{\mathfrak{g}}$-sets $\emptyset, \Omega \subseteq \mathfrak{T}_{\mathfrak{g}}$. The theorem follows.

Theorem 3.6. *A* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is said to be a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}$*-space* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{D}\rangle}\big)$ *if the only* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-sets in* $\mathfrak{T}_{\mathfrak{g}}$ *which are both* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open-closed sets are the improper* $\mathfrak{T}_{\mathfrak{g}}$*-sets* $\emptyset, \Omega \subseteq \mathfrak{T}_{\mathfrak{g}}$.

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set in $\mathfrak{T}_{\mathfrak{g}}$. Then, there exists $\big(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that $\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g}}$. Consequently, $\complement\left(\mathscr{U}_{\mathfrak{g}}\right) \subseteq \complement\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \complement\left(\mathscr{V}_{\mathfrak{g}}\right)$. Since $\big(\complement\left(\mathscr{V}_{\mathfrak{g}}\right), \complement\left(\mathscr{U}_{\mathfrak{g}}\right)\big) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it follows that, $\complement\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Since $\mathscr{S}_{\mathfrak{g}} \cap \complement\left(\mathscr{S}_{\mathfrak{g}}\right) = \emptyset$, implying $\mathscr{S}_{\mathfrak{g}} \cup \complement\left(\mathscr{S}_{\mathfrak{g}}\right) = \Omega$, it results, obviously, that,

$$\big[\big(\mathscr{S}_{\mathfrak{g}}, \complement\left(\mathscr{S}_{\mathfrak{g}}\right)\big) \in (\emptyset, \Omega)\big] \vee \big[\big(\mathscr{S}_{\mathfrak{g}}, \complement\left(\mathscr{S}_{\mathfrak{g}}\right)\big) \in (\Omega, \emptyset)\big].$$

This completes the proof of the theorem. □

The logical relationship between $\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is contained in the following theorem.

Theorem 3.7. *If* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *be a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$*-space* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}\big)$*, then it is also a* $\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$*-space* $\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}\big)$*:*

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}\big) \implies \mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}\big). \tag{3.5}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$-space $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big)$. Then it has a nonempty proper $\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$. Since $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$. This proves that $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$ is also a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big)$. In other words, if $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big)$, then it is also a $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big)$, and the proof is complete. □

By virtue of the above theorem, the following corollary follows.

Corollary 3.8. *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$-space $\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big)$, then it is also a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big)$:*

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big) \impliedby \mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big). \tag{3.6}$$

A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected if and only if it is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected as a $\mathscr{T}_{\mathfrak{g}}$-subspace. The theorem follows.

Theorem 3.9. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Gamma}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$, then $\mathscr{S}_{\mathfrak{g}}$ is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Gamma}$-connected if and only if it is $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}$-connected:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \iff \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]. \tag{3.7}$$

Proof. – *Necessity.* Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Gamma}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and suppose that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$. Then, $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$ and, hence, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$,

$$\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big),$$

But, $\mathscr{T}_{\mathfrak{g},\Gamma} \times \neg\mathscr{T}_{\mathfrak{g},\Gamma} \subseteq \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$, and on the other hand, $\mathbf{op}_{\mathfrak{g},\Gamma}(\cdot) \in \mathscr{L}_{\mathfrak{g}}\big[\Gamma\big]$ implies $\mathrm{op}_{\mathfrak{g},\Gamma}(\mathscr{O}_{\mathfrak{g},\lambda}) = \Gamma \cap \mathrm{op}_{\mathfrak{g},\Omega}(\mathscr{O}_{\mathfrak{g},\lambda}) = \mathrm{op}_{\mathfrak{g},\Omega}(\mathscr{O}_{\mathfrak{g},\lambda})$ and $\neg\,\mathrm{op}_{\mathfrak{g},\Gamma}(\mathscr{K}_{\mathfrak{g},\lambda}) = \Gamma \cap \neg\,\mathrm{op}_{\mathfrak{g},\Omega}(\mathscr{K}_{\mathfrak{g},\lambda}) = \neg\,\mathrm{op}_{\mathfrak{g},\Omega}(\mathscr{K}_{\mathfrak{g},\lambda})$ for any $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g},\Gamma} \times \neg\mathscr{T}_{\mathfrak{g},\Gamma}$, where $\mathbf{op}_{\mathfrak{g},\Omega}(\cdot) \in \mathscr{L}_{\mathfrak{g}}\big[\Omega\big]$. Thus, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$,

$$\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big).$$

Consequently, $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ and, hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$.

– *Sufficiency.* Conversely, suppose that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$. This implies that $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$. Therefore, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$,

$$\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big).$$

But the statement $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \subseteq \big(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\big) \subseteq \mathfrak{T}_{\mathfrak{g},\Gamma} \times \mathfrak{T}_{\mathfrak{g},\Gamma}$ implies, evidently, $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \times \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$, since $\mathbf{op}_{\mathfrak{g},\Gamma}(\cdot) \in \mathscr{L}_{\mathfrak{g}}\big[\Gamma\big]$ is equivalent to $\mathbf{op}_{\mathfrak{g},\Gamma}(\cdot) = \Gamma \cap \mathbf{op}_{\mathfrak{g},\Omega}(\cdot) \in \mathscr{L}_{\mathfrak{g}}\big[\Omega\big]$. Consequently, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in$

$\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$,

$$\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big),$$

Therefore, $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$ and, thus, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$. □

There are some very fundamental $\mathscr{T}_{\mathfrak{g}}$-properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets which follow from the next theorem.

Theorem 3.10. *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set of a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$, then there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that*

$$\Big(\bigvee_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g},\lambda}\big)\Big) \bigvee \Big(\bigvee_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g},\lambda}\big)\Big). \tag{3.8}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$. Then, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\sigma,\zeta_\sigma} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$,

$$\begin{aligned} &\neg\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{S}_{\mathfrak{g}}\Big) \\ \Rightarrow\ &\neg\Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \emptyset\Big) \bigwedge \neg\Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset\Big) \\ \Rightarrow\ &\Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} \neq \emptyset\Big) \bigwedge \Big(\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} \neq \emptyset\Big). \end{aligned}$$

Since $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$, there exists, therefore, pairs $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\begin{aligned} &\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big) \\ \Rightarrow\quad &\Big(\bigsqcup_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\lambda}\big) = \mathscr{S}_{\mathfrak{g}}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g},\lambda}\big) = \mathscr{S}_{\mathfrak{g}}\Big) \\ \Rightarrow\quad &\Big(\bigvee_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g},\lambda}\big)\Big) \bigvee \Big(\bigvee_{\lambda=\xi,\zeta} \big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g},\lambda}\big)\Big). \end{aligned}$$

Since $\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda}, \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} \neq \emptyset$ hold, and, moreover, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}\big)$, the proof at once follows. □

Equivalently stated, the following proposition states that, any $\mathfrak{T}_{\mathfrak{g}}$-set which is contained in a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.

Proposition 1. Let $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and $\mathscr{R}_{\mathfrak{g}}$ satisfies

$$\big[\mathscr{R}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big] \vee \big[\mathscr{R}_{\mathfrak{g}} \subseteq \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big], \tag{3.9}$$

then $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

Proof. Let $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$, and, by hypothesis, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$. Since $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$, there exists, then, $(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_\rho,\zeta_\rho} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ such that,

$$\Big(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}}\Big).$$

Since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$, it must be contained in either of $\mathscr{U}_{\mathfrak{g},\xi_\rho}$, $\mathscr{U}_{\mathfrak{g},\zeta_\rho}$, or in either of $\mathscr{V}_{\mathfrak{g},\xi_\rho}$, $\mathscr{V}_{\mathfrak{g},\zeta_\rho}$. Consequently,

$$\begin{aligned} &\Big(\bigvee_{\lambda=\xi_\rho,\zeta_\rho} (\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g},\lambda})\Big) \bigvee \Big(\bigvee_{\eta=\xi_\rho,\zeta_\rho} (\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g},\eta})\Big) \\ \Rightarrow &\Big(\bigvee_{\lambda=\xi_\rho,\zeta_\rho} (\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{U}_{\mathfrak{g},(\rho,\lambda)})\Big) \bigvee \Big(\bigvee_{\eta=\xi_\rho,\zeta_\rho} (\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g},(\rho,\eta)})\Big), \end{aligned}$$

where $\mathscr{U}_{\mathfrak{g},(\rho,\lambda)} = \mathrm{op}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g},\lambda})$ and $\mathscr{V}_{\mathfrak{g},(\rho,\eta)} = \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g},\eta})$ for every pair $(\lambda, \eta) \in \{(\xi_\rho, \zeta_\rho), (\zeta_\rho, \xi_\rho)\}$. With no loss of generality, let it be supposed that

$$[\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{U}_{\mathfrak{g},(\rho,\lambda)}] \vee [\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g},(\rho,\eta)}]$$

holds for a $(\lambda, \eta) \in \{(\xi_\rho, \zeta_\rho), (\zeta_\rho, \xi_\rho)\}$. Then, since the relations $\mathscr{R}_{\mathfrak{g}} = \bigsqcup_{\sigma=\lambda,\eta} \mathscr{U}_{\mathfrak{g},\sigma} \subseteq \bigsqcup_{\sigma=\lambda,\eta} \mathscr{U}_{\mathfrak{g},(\rho,\sigma)}$ and $\emptyset = \bigcap_{\sigma=\lambda,\eta} \mathscr{V}_{\mathfrak{g},\sigma} \supseteq \bigcap_{\sigma=\lambda,\eta} \mathscr{V}_{\mathfrak{g},(\rho,\sigma)}$ hold, it follows that,

$$\begin{aligned} \mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\eta} &\subseteq \mathscr{U}_{\mathfrak{g},(\rho,\lambda)} \cap \mathscr{U}_{\mathfrak{g},\eta} \subseteq \bigcap_{\sigma=\lambda,\eta} \mathscr{U}_{\mathfrak{g},(\rho,\sigma)} = \emptyset; \\ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{V}_{\mathfrak{g},\lambda} &\subseteq \mathscr{V}_{\mathfrak{g},(\rho,\eta)} \cap \mathscr{V}_{\mathfrak{g},\lambda} \subseteq \bigcap_{\sigma=\lambda,\eta} \mathscr{V}_{\mathfrak{g},\sigma} = \emptyset. \end{aligned}$$

Therefore, $\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\eta}, \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset$. On the other hand, since $\mathscr{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ satisfies $[\mathscr{R}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})] \vee [\mathscr{R}_{\mathfrak{g}} \subseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]$, it results that,

$$\begin{aligned} \mathscr{U}_{\mathfrak{g},\eta} &= \mathscr{R}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\eta} = \mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\eta}, \\ \mathscr{V}_{\mathfrak{g},\lambda} &= \mathscr{R}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g},\lambda} = \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\lambda}. \end{aligned}$$

From these and $\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{U}_{\mathfrak{g},\eta}$, $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset$, it follows that, $\mathscr{U}_{\mathfrak{g},\eta}$, $\mathscr{V}_{\mathfrak{g},\lambda} = \emptyset$, which contradict the hypothesis that $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$. □

The following proposition states that, if it be given a collection of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets with non-void intersection, then $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is preserved under the operation of union.

Proposition 2. Let $\{\mathscr{S}_{\mathfrak{g},\nu} : \nu \in I_n^*\} \subseteq \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ be a collection of $n \geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\bigcap_{\nu\in I_n^*} \mathscr{S}_{\mathfrak{g},\nu} \neq \emptyset$, then $\bigcup_{\nu\in I_n^*} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$:

$$\bigcap_{\nu\in I_n^*} (\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]) \neq \emptyset \implies \bigcup_{\nu\in I_n^*} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]. \tag{3.10}$$

Proof. Let $\{\mathscr{S}_{\mathfrak{g},\nu} : \nu \in I_n^*\} \subseteq \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ be a collection of $n \geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, and suppose, by hypothesis, that $\bigcup_{\nu\in I_n^*} \mathscr{S}_{\mathfrak{g},\nu} \in$

$\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, where $\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\neq\emptyset$. Since $\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, there exists, then, $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi,\zeta}\in\mathfrak{g}$-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}$-$\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{U}_{\mathfrak{g},\lambda}=\textstyle\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\Big)\bigvee\Big(\displaystyle\bigsqcup_{\lambda=\xi,\zeta}\mathscr{V}_{\mathfrak{g},\lambda}=\textstyle\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\Big).$$

Since $\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\neq\emptyset$, there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\eta\}\subset\mathfrak{T}_{\mathfrak{g}}$ satisfying $\{\eta\}\subseteq\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\neq\emptyset$. But, by hypothesis, $\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Consequently,

$$\begin{aligned}&\Big(\bigvee_{\lambda=\xi,\zeta}\Big(\{\eta\}\subseteq\mathscr{U}_{\mathfrak{g},\lambda}\cap\Big(\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\Big)\Big)\Big)\\ \bigvee&\Big(\bigvee_{\lambda=\xi,\zeta}\Big(\{\eta\}\subseteq\mathscr{V}_{\mathfrak{g},\lambda}\cap\Big(\bigcap_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\Big)\Big)\Big).\end{aligned}$$

Clearly, for every $\nu\in I_n^*$,

$$\begin{aligned}&\Big(\bigvee_{\lambda=\xi,\zeta}\big(\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{U}_{\mathfrak{g},\lambda}\neq\emptyset\big)\Big)\bigvee\Big(\bigvee_{\lambda=\xi,\zeta}\big(\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{V}_{\mathfrak{g},\lambda}\neq\emptyset\big)\Big)\\ \Rightarrow&\Big(\bigvee_{\lambda=\xi,\zeta}\big(\mathscr{S}_{\mathfrak{g},\nu}\subseteq\mathscr{U}_{\mathfrak{g},\lambda}\big)\Big)\bigvee\Big(\bigvee_{\lambda=\xi,\zeta}\big(\mathscr{S}_{\mathfrak{g},\nu}\subseteq\mathscr{V}_{\mathfrak{g},\lambda}\big)\Big).\end{aligned}$$

Therefore,

$$\Big(\bigvee_{\lambda=\xi,\zeta}\Big(\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\subseteq\mathscr{U}_{\mathfrak{g},\lambda}\Big)\Big)\bigvee\Big(\bigvee_{\lambda=\xi,\zeta}\Big(\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\subseteq\mathscr{V}_{\mathfrak{g},\lambda}\Big)\Big),$$

which contradicts the hypothesis that $\bigcup_{\nu\in I_n^*}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. □

Stated differently, the following proposition states that, if every two-point $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathfrak{T}_{\mathfrak{g}}$-set is a $\mathfrak{T}_{\mathfrak{g}}$-subset of some $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected subset of the $\mathfrak{T}_{\mathfrak{g}}$-set, then the $\mathfrak{T}_{\mathfrak{g}}$-set is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set.

Proposition 3. Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. If every two-point $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}$ satisfies the relation $\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{R}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}$, where $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}$-$\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}$-$\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$:

$$\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\supseteq\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\implies\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big].\tag{3.11}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and suppose that every two-point $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}$ satisfies the relation $\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{R}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}$, where $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}$-$\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and by hypothesis, $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}$-$\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Then, there exists a pair $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\sigma,\zeta_\sigma}\in\mathfrak{g}$-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}$-$\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{U}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big)\bigvee\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{V}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Big).$$

Since $\mathscr{U}_{\mathfrak{g},\lambda}$, $\mathscr{V}_{\mathfrak{g},\lambda}\neq\emptyset$ for every $\lambda\in\{\xi_\sigma,\zeta_\sigma\}$, assume that

$$\begin{aligned}\{\xi\}&=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{U}_{\mathfrak{g},\xi_\sigma}=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{V}_{\mathfrak{g},\xi_\sigma},\\ \{\zeta\}&=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{U}_{\mathfrak{g},\zeta_\sigma}=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{V}_{\mathfrak{g},\zeta_\sigma},\ \mathscr{Q}_{\mathfrak{g}}=\{\xi\}\cup\{\zeta\}.\end{aligned}$$

In other words, $\mathscr{Q}_{\mathfrak{g}} \subset \bigtimes_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathscr{U}_{\mathfrak{g},\lambda}$ or $\mathscr{Q}_{\mathfrak{g}} \subset \bigtimes_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathscr{V}_{\mathfrak{g},\lambda}$. Since $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, for all $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_{\rho},\zeta_{\rho}} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$,

$$\begin{aligned} &\neg\Big(\bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}} \mathscr{U}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}} \mathscr{V}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g}}\Big) \\ &\Rightarrow \neg\Big(\bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}} \big(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\lambda}\big) = \mathscr{Q}_{\mathfrak{g}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}} \big(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g},\lambda}\big) = \mathscr{Q}_{\mathfrak{g}}\Big) \\ &\Rightarrow \neg\big(\mathscr{Q}_{\mathfrak{g}} = \mathscr{Q}_{\mathfrak{g}}\big) \wedge \neg\big(\mathscr{Q}_{\mathfrak{g}} = \mathscr{Q}_{\mathfrak{g}}\big), \end{aligned}$$

which contradicts the hypothesis that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. □

A $\mathscr{T}_{\mathfrak{g}}$-space is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected if any two-point $\mathfrak{T}_{\mathfrak{g}}$-set can be enclosed in some $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set, and hence the following proposition.

Proposition 4. If any two-point $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{Q}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ can be enclosed in some $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, then the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big)$:

$$\big(\forall \mathscr{Q}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}\big)\big(\exists \mathscr{S}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}} \cup \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big) \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big). \tag{3.12}$$

Proof. Let $\xi \in \mathfrak{T}_{\mathfrak{g}}$ be fixed and, for every $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, let $\mathscr{Q}_{\mathfrak{g}(\xi,\zeta)} \subset \mathfrak{T}_{\mathfrak{g}}$ be a two-point $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ containing ξ, $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Then, $\bigcup_{\zeta \in \mathfrak{T}_{\mathfrak{g}}} \mathscr{Q}_{\mathfrak{g}(\xi,\zeta)} \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and, by hypothesis, it is the entire $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Hence $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big)$. □

The theorem given below states that, any pair of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets which is contained in some pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.

Theorem 3.11. *Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. If $(\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets and $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a pair of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets satisfying the statement $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \subseteq (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta})$, then $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$:*

$$\left(\mathscr{S}_{\mathfrak{g},\lambda} = \mathscr{R}_{\mathfrak{g},\lambda} \cup \mathscr{S}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g}}\right] \implies \left(\mathscr{R}_{\mathfrak{g},\lambda}\right)_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g}}\right]. \tag{3.13}$$

Proof. Let $(\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets and let $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a pair of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets satisfying $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \subseteq (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta})$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\begin{aligned} &\Big(\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathscr{U}_{\mathfrak{g},\lambda} = \bigsqcup_{\eta=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\eta}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathscr{V}_{\mathfrak{g},\lambda} = \bigsqcup_{\eta=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\eta}\Big) \\ &\Leftrightarrow \Big(\bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathscr{U}_{\mathfrak{g},\lambda} = \emptyset\Big) \bigvee \Big(\bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset\Big). \end{aligned}$$

Since $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \subseteq (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta})$, $\bigcap_{\eta=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\eta} \subseteq \bigcap_{\eta=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\eta}$. If the relation $\bigsqcup_{\eta=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\eta} = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathscr{U}_{\mathfrak{g},\lambda}$ is satisfied, then $\bigcap_{\eta=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\eta} \subseteq \bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathscr{U}_{\mathfrak{g},\lambda} = \emptyset$; if

$\bigsqcup_{\eta=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\eta}=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{V}_{\mathfrak{g},\lambda}$, then $\bigcap_{\eta=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\eta}\subseteq\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{V}_{\mathfrak{g},\lambda}=\emptyset$ holds. Hence, there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\rho,\zeta_\rho}\in\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Bigg(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho}\mathscr{U}_{\mathfrak{g},\lambda}=\bigsqcup_{\eta=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\eta}\Bigg)\bigvee\Bigg(\bigsqcup_{\lambda=\xi_\rho,\zeta_\rho}\mathscr{V}_{\mathfrak{g},\lambda}=\bigsqcup_{\eta=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\eta}\Bigg).$$

This proves the theorem. □

The basic relation between $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separateness follows:

Theorem 3.12. *In order that a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected it is necessary and sufficient that there exists no $(\mathscr{R}_{\mathfrak{g},\alpha},\mathscr{R}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ or $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that it be expressible as*

$$\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Bigg)\bigvee\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Bigg).\tag{3.14}$$

Proof. – *Necessity.* Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and let there exists $(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Bigg)\bigvee\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Bigg).$$

Since $(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that $\big(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}=\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}$. Consequently,

$$\Bigg(\bigsqcup_{\lambda=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Bigg)\bigvee\Bigg(\bigsqcup_{\lambda=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Bigg).$$

This shows that $(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Hence, $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. The condition of the theorem is, therefore, necessary.

– *Sufficiency.* Conversely, suppose that $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, there exists, then, a pair $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\Bigg(\bigsqcup_{\lambda=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Bigg)\bigvee\Bigg(\bigsqcup_{\lambda=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\lambda}=\mathscr{S}_{\mathfrak{g}}\Bigg).$$

But, $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Therefore, it follows that there exists $\big(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that the relations expressible by $\big(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}=\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}$ hold. Hence, the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ is expressible as

$$\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{R}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Bigg)\bigvee\Bigg(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Bigg).$$

The condition of the theorem is, therefore, sufficient. □

If $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets in $\mathscr{S}_{\mathfrak{g}}$ are clearly also in $\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and conversely. Likewise, if $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in $\mathscr{S}_{\mathfrak{g}}$ are clearly also in $\in\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and conversely. Hence, an immediate consequence of the above theorem is the following corollary:

Corollary 3.13. *Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:*

– I. *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then in order that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it is necessary and sufficient that there exists no $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that it be expressible as $\mathscr{S}_{\mathfrak{g}} = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\lambda}$.*

– II. *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, then in order that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it is necessary and sufficient that there exists no $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that it be expressible as $\mathscr{S}_{\mathfrak{g}} = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\lambda}$.*

The following remark contains classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness with respect to openness and closedness.

Remark 3.14. Suppose $\bigsqcup_{\sigma=\alpha,\beta} \mathscr{R}_{\mathfrak{g},\sigma} = \mathscr{S}_{\mathfrak{g}}$ hold, then it is no error to call $\mathscr{S}_{\mathfrak{g}}$ a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connected open set* if $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*connected closed set* if $(\mathscr{R}_{\mathfrak{g},\alpha}, \mathscr{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

From the above corollary, it would appear that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness depends on the existence of certain $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets or, equivalently, on the existence of certain disjoint $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open, closed sets. As another simple ways of characterizing $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, the proposition follows.

Proposition 5. A $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big)$ if and only if any one of the following statements holds:

– I. $\exists\, (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 : \quad \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda} = \Omega$;

– II. $\exists\, (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 : \quad \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda} = \Omega$;

– III. $\exists\, (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 : \quad \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda} = \Omega$.

Proof. – *Necessity.* Let $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}\big)$ be a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D} \rangle}$-space. Then, there exists $\mathscr{S}_{\mathfrak{g}} \in \big(\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \setminus \{\emptyset, \Omega\}$. Consequently, there exists $\big(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that

$$\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g}} \;\Rightarrow\; \complement\,(\mathscr{U}_{\mathfrak{g}}) \subseteq \complement\,(\mathscr{S}_{\mathfrak{g}}) \subseteq \complement\,(\mathscr{V}_{\mathfrak{g}})\,.$$

Therefore, $\complement\,(\mathscr{S}_{\mathfrak{g}}) \in \big(\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \setminus \{\emptyset, \Omega\}$. Hence,

$$\big(\mathscr{S}_{\mathfrak{g}}, \complement\,(\mathscr{S}_{\mathfrak{g}})\big) \in \big(\mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 \cup \big(\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 \cup \big(\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2.$$

– *Sufficiency.* Conversely, suppose that

$$\exists\, (\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 \cup \big(\mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2 \cup \big(\mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2,$$

such that $\Omega = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda}$. Then if $(\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \in \big(\mathfrak{g}\text{-D}\big[\mathfrak{T}_{\mathfrak{g}}\big] \setminus \{\emptyset\}\big)^2$, there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ such that,

$$\begin{aligned} &\Big(\bigsqcup_{\lambda=\alpha,\beta} \mathscr{U}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda}\Big) \bigvee \Big(\bigsqcup_{\lambda=\alpha,\beta} \mathscr{V}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda}\Big) \\ \Rightarrow \quad &\Big(\bigsqcup_{\lambda=\alpha,\beta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\alpha,\beta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big). \end{aligned}$$

If $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\big(\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\setminus\{\emptyset\}\big)^2$, there exists $\big(\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that the statement $\big(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta}\big)\subseteq\big(\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta}\big)$ holds. Consequently, it follows that $\Omega=\bigsqcup_{\lambda=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\lambda}\subseteq\bigsqcup_{\lambda=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\lambda}$. Hence, $\bigsqcup_{\lambda=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\lambda}=\Omega$.

If $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\big(\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\setminus\{\emptyset\}\big)^2$, then there exists $\big(\mathscr{V}_{\mathfrak{g},\alpha},\mathscr{V}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ such that the statement $\big(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta}\big)\supseteq\big(\mathscr{V}_{\mathfrak{g},\alpha},\mathscr{V}_{\mathfrak{g},\beta}\big)$ holds. Thus, it results that $\emptyset=\bigcap_{\lambda=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\lambda}\supseteq\bigcap_{\lambda=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\lambda}$. Hence, $\bigsqcup_{\lambda=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\lambda}=\Omega$. These complete the proof of the proposition. □

The following lemma is a useful tool for the proof of the theorem following it.

Lemma 3.15. *Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, and let $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in $\mathfrak{T}_{\mathfrak{g}}$. If there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\{\xi\}$, then $\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\exists\{\xi\}=\bigcap_{\sigma=\alpha,\beta}\big(\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]\big)\;\Rightarrow\;\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}].\tag{3.15}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, let $\mathscr{S}_{\mathfrak{g}}=\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}$, and suppose that there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\{\xi\}=\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}$, where $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$, and assume that

$$\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Big)\bigvee\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}\Big),$$

for some $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\alpha,\beta}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Since $\{\xi\}\subseteq\mathscr{S}_{\mathfrak{g}}$,

$$\Big(\bigvee_{\sigma=\alpha,\beta}\big(\{\xi\}\subseteq\mathscr{U}_{\mathfrak{g},\sigma}\big)\Big)\bigvee\Big(\bigvee_{\sigma=\alpha,\beta}\big(\{\xi\}\subseteq\mathscr{V}_{\mathfrak{g},\sigma}\big)\Big),$$

meaning that, with respect to $\big(\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, either $\xi\in\mathscr{U}_{\mathfrak{g},\alpha}$ or $\xi\in\mathscr{U}_{\mathfrak{g},\beta}$; with respect to $\big(\mathscr{V}_{\mathfrak{g},\alpha},\mathscr{V}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, either $\xi\in\mathscr{V}_{\mathfrak{g},\alpha}$ or $\xi\in\mathscr{V}_{\mathfrak{g},\beta}$. Therefore, set

$$\big(\{\xi\}\subseteq\mathscr{U}_{\mathfrak{g},\alpha}\big)\vee\big(\{\xi\}\subseteq\mathscr{V}_{\mathfrak{g},\alpha}\big).$$

Clearly, $\mathscr{U}_{\mathfrak{g},\beta}$, $\mathscr{V}_{\mathfrak{g},\beta}\neq\emptyset$; $\mathscr{U}_{\mathfrak{g},\beta}\subseteq\bigsqcup_{\sigma=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}$ and, $\mathscr{V}_{\mathfrak{g},\beta}\subseteq\bigsqcup_{\sigma=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\sigma}=\mathscr{S}_{\mathfrak{g}}$. Therefore, for at least a $\sigma\in\{\alpha,\beta\}$,

$$\big(\mathscr{U}_{\mathfrak{g},\beta}\cap\mathscr{S}_{\mathfrak{g},\sigma}\neq\emptyset\big)\vee\big(\mathscr{V}_{\mathfrak{g},\beta}\cap\mathscr{S}_{\mathfrak{g},\sigma}\neq\emptyset\big).$$

Choose a $\eta\in\{\alpha,\beta\}$. Then, for every $\sigma\in\{\alpha,\beta\}$,

$$\big(\mathscr{U}_{\mathfrak{g},\sigma}\cap\mathscr{S}_{\mathfrak{g},\eta}\subseteq\mathscr{U}_{\mathfrak{g},\sigma}\big)\vee\big(\mathscr{V}_{\mathfrak{g},\sigma}\cap\mathscr{S}_{\mathfrak{g},\eta}\subseteq\mathscr{V}_{\mathfrak{g},\sigma}\big).$$

Therefore, with respect to $\big(\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathscr{U}_{\mathfrak{g},\alpha}\cap\mathscr{S}_{\mathfrak{g},\eta}$ and $\mathscr{U}_{\mathfrak{g},\beta}\cap\mathscr{S}_{\mathfrak{g},\eta}$ are $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets; with respect to $\big(\mathscr{V}_{\mathfrak{g},\alpha},\mathscr{V}_{\mathfrak{g},\beta}\big)\in\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathscr{V}_{\mathfrak{g},\alpha}\cap\mathscr{S}_{\mathfrak{g},\eta}$ and $\mathscr{V}_{\mathfrak{g},\beta}\cap\mathscr{S}_{\mathfrak{g},\eta}$ are also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated sets. Consequently,

$$\Big(\mathscr{S}_{\mathfrak{g},\eta}\cap\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{U}_{\mathfrak{g},\sigma}\Big)=\mathscr{S}_{\mathfrak{g},\eta}\Big)\bigvee\Big(\mathscr{S}_{\mathfrak{g},\eta}\cap\Big(\bigsqcup_{\sigma=\alpha,\beta}\mathscr{V}_{\mathfrak{g},\sigma}\Big)=\mathscr{S}_{\mathfrak{g},\eta}\Big)$$

$$\Rightarrow\Big(\bigsqcup_{\sigma=\alpha,\beta}\big(\mathscr{U}_{\mathfrak{g},\sigma}\cap\mathscr{S}_{\mathfrak{g},\eta}\big)=\mathscr{S}_{\mathfrak{g},\eta}\Big)\bigvee\Big(\bigsqcup_{\sigma=\alpha,\beta}\big(\mathscr{V}_{\mathfrak{g},\sigma}\cap\mathscr{S}_{\mathfrak{g},\eta}\big)=\mathscr{S}_{\mathfrak{g},\eta}\Big).$$

Therefore, $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\notin\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, contrary to hypothesis. Hence, it follows that $\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}$ must be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g}}$, that is $\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. □

For the case of $n\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets, the theorem follows.

Theorem 3.16. *Let $\mathscr{S}_{\mathfrak{g},1},\ \mathscr{S}_{\mathfrak{g},2},\ \ldots,\ \mathscr{S}_{\mathfrak{g},n}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be $n\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. If, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\{\xi\}$, then $\bigcup_{\sigma\in I_n^*}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\exists\,\{\xi\}=\bigcap_{\sigma\in\{\alpha,\beta\}\subseteq I_n^*\times I_n^*}\big(\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\implies\bigcup_{\sigma\in I_n^*}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big].$$

(3.16)

Proof. Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, let $\mathscr{S}_{\mathfrak{g}}=\bigcup_{\sigma\in I_n^*}\mathscr{S}_{\mathfrak{g},\sigma}$, and suppose that, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\{\xi\}$, where $\mathscr{S}_{\mathfrak{g},1},\ \mathscr{S}_{\mathfrak{g},2},\ \ldots,\ \mathscr{S}_{\mathfrak{g},n}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ are $n\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets in $\mathfrak{T}_{\mathfrak{g}}$. If $(\xi_\alpha,\xi_\beta)\in\mathscr{S}_{\mathfrak{g}}\times\mathscr{S}_{\mathfrak{g}}$ be any pair of elements of $\mathscr{S}_{\mathfrak{g}}$, then there is a pair $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets such that $(\xi_\alpha,\xi_\beta)\in(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})$. Set $\mathscr{Q}_{\mathfrak{g},(\alpha,\beta)}=\{\xi_\alpha,\xi_\beta\}$ and $\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}=\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}$; clearly, $\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\neq\emptyset$ by hypothesis. Then, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, the relation $\mathscr{Q}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{S}_{\mathfrak{g}}$ holds. Since, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, there exists a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\xi\}\subset\mathfrak{T}_{\mathfrak{g}}$ such that $\bigcap_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}=\{\xi\}$, it follows that $\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}=\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$. Since, for every $(\alpha,\beta)\in I_n^*\times I_n^*$, $\mathscr{Q}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{S}_{\mathfrak{g}}$ is a two-point $\mathfrak{T}_{\mathfrak{g}}$-set satisfying the relation $\mathscr{Q}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}\subseteq\mathscr{S}_{\mathfrak{g}}$, where $\mathscr{R}_{\mathfrak{g},(\alpha,\beta)}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$, it follows that $\mathscr{S}_{\mathfrak{g}}=\bigcup_{\sigma\in I_n^*}\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$. This proves the theorem. □

When a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, it is natural that we should attempt to obtain some information about the various $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets into which it can be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. The maximal $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ are particularly interesting.

Definition 3.17. If $\zeta\in\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ is a point of a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then

$$\text{(3.17)}\quad \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\stackrel{\text{def}}{=}\big\{\xi\in\mathscr{S}_{\mathfrak{g}}:\ \big(\exists\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\big[(\xi,\zeta)\in\mathscr{R}_{\mathfrak{g}}^2\subseteq\mathscr{S}_{\mathfrak{g}}^2\big]\big\}$$

is called the "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\mathscr{S}_{\mathfrak{g}}$ corresponding to ζ."

According to this definition, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component is nonempty, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, and is not a proper $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set of any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set of a $\mathscr{T}_{\mathfrak{g}}$-space. The theorem follows.

Theorem 3.18. *For each point $\zeta\in\mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ of $\mathscr{S}_{\mathfrak{g}}$ corresponding to ζ is the largest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in $\mathfrak{T}_{\mathfrak{g}}$ which contains the point ζ:*

$$\text{(3.18)}\qquad \big(\forall\zeta\in\mathfrak{T}_{\mathfrak{g}}\big)\big(\nexists\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\big[\mathscr{R}_{\mathfrak{g}}\supset\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big].$$

Proof. Let $(\xi,\zeta)\in\big(\mathscr{R}_{\mathfrak{g},\zeta}\setminus\{\zeta\}\big)\times\mathscr{R}_{\mathfrak{g},\zeta}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, where $\mathscr{R}_{\mathfrak{g},\zeta}\in\mathfrak{g}\text{-}\mathrm{Q}\,[\mathfrak{T}_{\mathfrak{g}}]$ is any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set which contains $\zeta\in\mathfrak{T}_{\mathfrak{g}}$, and $\xi\in\mathscr{R}_{\mathfrak{g},\zeta}$. Clearly, $\mathscr{Q}_{\mathfrak{g},(\xi,\zeta)}=\{\xi,\zeta\}\subseteq\mathscr{R}_{\mathfrak{g},\zeta}$ and, therefore, $\xi\in\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$, implying $\mathscr{R}_{\mathfrak{g},\zeta}\subseteq\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. To prove the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness of $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$, consider an arbitrary point $\eta\in$

$\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Since $\mathscr{Q}_{\mathfrak{g},(\eta,\zeta)} = \{\eta,\zeta\} \subseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{R}_{\mathfrak{g},\eta} \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\mathscr{Q}_{\mathfrak{g},(\eta,\zeta)} = \{\eta,\zeta\} \subseteq \mathscr{R}_{\mathfrak{g},\eta}$. Therefore, $\mathscr{R}_{\mathfrak{g},\eta} \subseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ and, consequently, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] = \bigcup_{\eta \in \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]} \mathscr{R}_{\mathfrak{g},\eta}$. But, this is the union of the collection $\{\mathscr{R}_{\mathfrak{g},\eta} :\ \eta \in \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\} \subseteq \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected sets with a common point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Hence, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$. ☐

In a $\mathscr{T}_{\mathfrak{g}}$-space, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components are $\mathfrak{g}$-$\mathfrak{T}$-closed sets, as demonstrated in the following theorem.

Theorem 3.19. *For each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \subset \mathfrak{T}_{\mathfrak{g}}$ of $\mathscr{S}_{\mathfrak{g}}$ corresponding to ζ is a $\mathfrak{g}$-$\mathfrak{T}$-closed set of $\mathfrak{T}_{\mathfrak{g}}$:*

$$(\forall \zeta \in \mathfrak{T}_{\mathfrak{g}})\big[\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\big]. \tag{3.19}$$

Proof. Let $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \subset \mathfrak{T}_{\mathfrak{g}}$ be the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\mathscr{S}_{\mathfrak{g}}$ corresponding to $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Then, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ is the largest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in $\mathfrak{T}_{\mathfrak{g}}$ containing the point ζ. Suppose that $\xi \in \neg\,\mathrm{op}_{\mathfrak{g}}(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta])$. Since $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]) \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set, and $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ is the largest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in $\mathfrak{T}_{\mathfrak{g}}$ which contains the point ζ, $\mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} = \{\xi,\zeta\} \subseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Hence, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta])$, meaning that $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ must be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set in $\mathfrak{T}_{\mathfrak{g}}$. This proves the theorem. ☐

A central fact about the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components of a $\mathscr{T}_{\mathfrak{g}}$-space is that, to each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ there corresponds a unique $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ of $\mathscr{S}_{\mathfrak{g}}$. This fact is contained in the following theorem.

Theorem 3.20. *The class $\{\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] :\ \zeta \in \mathfrak{T}_{\mathfrak{g}}\}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ forms a partition of $\mathfrak{T}_{\mathfrak{g}}$:*

$$\{\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] :\ \zeta \in \mathfrak{T}_{\mathfrak{g}}\} \implies \bigsqcup_{\zeta \in \mathfrak{T}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] = \Omega. \tag{3.20}$$

Proof. Let $\{\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] :\ \zeta \in \mathfrak{T}_{\mathfrak{g}}\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Clearly, $\Omega = \bigcup_{\zeta \in \mathfrak{T}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Let $\eta \in \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \cap \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$. Then, since $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta], \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi] \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}]$ and contain the point $\eta \in \mathfrak{T}_{\mathfrak{g}}$, it follows that, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\eta] \supseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ and $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\eta] \supseteq \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$. But $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta], \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$ are $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components and, hence, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] = \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\eta] = \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$. This shows that distinct $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components are disjoint or, equivalently, $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \cap \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi] \neq \emptyset$ implies $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] = \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\xi]$. ☐

By virtue of this theorem, it thus follows that, each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ belongs to a unique $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ of $\mathscr{S}_{\mathfrak{g}}$. The corollary follows.

Corollary 3.21. *For each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, there corresponds a unique $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ of $\mathscr{S}_{\mathfrak{g}}$ containing it:*

$$(\forall \zeta \in \mathfrak{T}_{\mathfrak{g}})(\exists!\, \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g}}])\big[\zeta \in \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big]. \tag{3.21}$$

A $\mathscr{T}_{\mathfrak{g}}$-space that is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected has at most one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component, as demonstrated in the following proposition.

Proposition 6. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big)$, then it has at most one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}$-$\mathrm{C}_{\Omega}[\zeta] = \Omega$:

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big) \iff \exists!\, \mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\zeta] = \Omega. \tag{3.22}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big)$, and let it be supposed that it has $\alpha \in I_{\infty}^{*}$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-components $\mathfrak{g}$-$\mathrm{C}_{\Omega}[\zeta_1]$, $\mathfrak{g}$-$\mathrm{C}_{\Omega}[\zeta_2]$, …, $\mathfrak{g}$-$\mathrm{C}_{\Omega}[\zeta_{\alpha}]$. Then, $\bigsqcup_{\mu \in I_{\alpha}^{*}} \mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\zeta_{\mu}] = \Omega$ because $\bigcap_{\mu \in I_{\alpha}^{*}} \mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\zeta_{\mu}] = \emptyset$. Hence, $\mathfrak{T}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, which contradicts the fact that $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big)$. □

The combination of an additional concept called *path* with the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness will bring forth a further refinement of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness called *pathwise* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-connectedness*.

Definition 3.22. A path from an initial point $\xi \in \mathfrak{T}_{\mathfrak{g}}$ to a terminal point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $([0,1], \mathfrak{T}_{\mathfrak{g}})$-continuous map $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ with $\big(\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)\big) = (\xi, \zeta)$. A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to be pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected if and only if, for every $(\xi, \zeta) \in \mathscr{S}_{\mathfrak{g}} \times \mathscr{S}_{\mathfrak{g}}$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}\big)\big[\mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{Q}_{\mathfrak{g}} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big]. \tag{3.23}$$

Evidently, $\mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)$ signifies the *image* of the $([0,1], \mathfrak{T}_{\mathfrak{g}})$-continuous map $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ from the initial point $\xi = \varphi_{\mathfrak{g},\zeta}(0)$ to the terminal point $\zeta = \varphi_{\mathfrak{g},\zeta}(1)$. The following theorem is an immediate consequence of the above definition.

Theorem 3.23. *A subset $\Gamma \subseteq \Omega$ of Ω of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$, with the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Omega} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, is said to be pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected if and only if, with the relative $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \big\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma : \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\big\}$, the $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.*

Proof. – Necessity. Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, and suppose that a subset $\Gamma \subseteq \Omega$ of Ω, with the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Omega} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Then, for every $(\xi, \zeta) \in \Gamma \times \Gamma \subseteq \Omega \times \Omega$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g},\Omega}\big)\big[\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\omega} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big].$$

Since $\mathscr{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ and $\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \supseteq \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$, it follows that $\mathscr{Q}_{\mathfrak{g},\gamma} = \mathscr{Q}_{\mathfrak{g},\omega} \cap \Gamma$ for every $\mathscr{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$. Since $\big\{\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)\big\} \subset \Gamma \times \Gamma$, it also follows that $\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\gamma} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)$. Therefore, for every $(\xi, \zeta) \in \Gamma \times \Gamma$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g},\Gamma}\big)\big[\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\gamma} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big].$$

Hence, with the relative $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \big\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma : \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\big\}$, the $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.

– *Sufficiency.* Conversely, suppose that, with the relative $\mathfrak{g}$-topology given by $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \big\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma : \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\big\}$, the $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Then, for every $(\xi, \zeta) \in \Gamma \times \Gamma$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g},\Gamma}\big)\big[\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\gamma} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big].$$

Since $\mathscr{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big]$ and $\mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Gamma}\big] \subseteq \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$, it follows that a $\mathscr{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ exists such that $\mathscr{Q}_{\mathfrak{g},\omega} \cap \Gamma = \mathscr{Q}_{\mathfrak{g},\gamma}$. Furthermore, since $\mathscr{Q}_{\mathfrak{g},\omega} \subseteq \Gamma$ and

$\mathscr{Q}_{\mathfrak{g},\omega} \supseteq \mathscr{Q}_{\mathfrak{g},\gamma}$, it follows that $\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\omega} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big)$. Therefore, for every $(\xi,\zeta) \in \Gamma \times \Gamma \subseteq \Omega \times \Omega$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g},\Omega}\big)\big[\Gamma \supseteq \mathscr{Q}_{\mathfrak{g},\omega} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big)\big].$$

Hence, the subset $\Gamma \subseteq \Omega$, with the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Omega} : \mathscr{P}(\Omega) \to \mathscr{P}(\Omega)$, is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. □

The relationship between the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness and pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness follows.

Theorem 3.24. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}}$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.*

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be an arbitrary pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}} = \emptyset$, then $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{D}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and, therefore, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Suppose $\mathscr{S}_{\mathfrak{g}} \neq \emptyset$, consider any point $\xi \in \mathscr{S}_{\mathfrak{g}}$. Since $\mathscr{S}_{\mathfrak{g}}$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, for every $\zeta \in \mathscr{S}_{\mathfrak{g}}$, there is a path $\varphi_{\mathfrak{g},\zeta} : [0,1] \to \mathscr{S}_{\mathfrak{g}}$ from the initial point $\xi \in \mathscr{S}_{\mathfrak{g}}$ to the terminal point $\zeta \in \mathscr{S}_{\mathfrak{g}}$, and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, containing $\xi, \zeta \in \mathscr{S}_{\mathfrak{g}}$, such that $\mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big)$. Clearly, $\mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big) \in \mathrm{C}\big[[0,1]\,;\mathfrak{T}_{\mathfrak{g}}\big]$. Moreover, $\mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} \supseteq \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big)$ and, consequently, $\mathscr{S}_{\mathfrak{g}} \supseteq \bigcup_{\zeta \in \mathscr{S}_{\mathfrak{g}}} \mathscr{Q}_{\mathfrak{g},(\xi,\zeta)} \supseteq \bigcup_{\zeta \in \mathscr{S}_{\mathfrak{g}}} \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big) = \mathscr{S}_{\mathfrak{g}}$, since $\xi \in \mathscr{S}_{\mathfrak{g}}$. But, $\xi \in \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big)$ for every $\zeta \in \mathscr{S}_{\mathfrak{g}}$ and, hence, $\bigcap_{\zeta \in \mathscr{S}_{\mathfrak{g}}} \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big) \neq \emptyset$. Furthermore, $\mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big) \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ for every $\zeta \in \mathscr{S}_{\mathfrak{g}}$, and by the relation $\mathscr{S}_{\mathfrak{g}} = \bigcup_{\zeta \in \mathscr{S}_{\mathfrak{g}}} \mathrm{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big)$, it follows, then, that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. This proves the theorem. □

Thus, pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a stronger form of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness. For this reason, we stated that pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a further refinement of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. An immediate consequence of such a statement is the following proposition.

Proposition 7. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space, then it is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected:

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{PC} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{PC} \rangle}\big) \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}\big). \tag{3.24}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space, and suppose it be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. Then, $\mathfrak{T}_{\mathfrak{g}}$ has a nonempty proper $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. There exists, then, $(\xi,\zeta) \in \mathscr{S}_{\mathfrak{g}} \times \complement(\mathscr{S}_{\mathfrak{g}})$. Let $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ be a path from ξ to ζ. Clearly, $[0,1] \supset \mathrm{im}\big(\varphi^{-1}_{\mathfrak{g},\zeta\,|\mathscr{S}_{\mathfrak{g}}}\big)$ for $0 \in \mathrm{im}\big(\varphi^{-1}_{\mathfrak{g},\zeta\,|\mathscr{S}_{\mathfrak{g}}}\big)$ and $1 \notin \mathrm{im}\big(\varphi^{-1}_{\mathfrak{g},\zeta\,|\mathscr{S}_{\mathfrak{g}}}\big)$, or for $0 \notin \mathrm{im}\big(\varphi^{-1}_{\mathfrak{g},\zeta\,|\mathscr{S}_{\mathfrak{g}}}\big)$ and $1 \in \mathrm{im}\big(\varphi^{-1}_{\mathfrak{g},\zeta\,|\mathscr{S}_{\mathfrak{g}}}\big)$. Since $\varphi_{\mathfrak{g},\zeta} \in \mathrm{C}\big[[0,1]\,;\mathfrak{T}_{\mathfrak{g}}\big]$, it follows that $\mathrm{im}\big(\varphi^{-1}_{\mathfrak{g},\zeta\,|\mathscr{S}_{\mathfrak{g}}}\big)$ is both open and closed. But, this contradicts the fact that $[0,1]$ is connected. Hence, the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. □

Definition 3.25. Let $\phi_{\mathfrak{g},\zeta}$, $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ be two paths in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ satisfying $\big(\phi_{\mathfrak{g},\zeta}(0), \phi_{\mathfrak{g},\zeta}(1)\big) = \big(\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)\big) = (\xi,\zeta)$. Then, $\phi_{\mathfrak{g},\zeta}$ is said to be "homotopic" to $\varphi_{\mathfrak{g},\zeta}$, written $\phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$, if there exists a $\big([0,1]^2, \mathfrak{T}_{\mathfrak{g}}\big)$-continuous map $\mathfrak{h}_{\mathfrak{g}} : [0,1]^2 \longrightarrow \mathfrak{T}_{\mathfrak{g}}$, called a "homotopy" from $\phi_{\mathfrak{g},\zeta}$ to $\varphi_{\mathfrak{g},\zeta}$, written

$\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$, satisfying,

$$\begin{aligned}\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu) &= (1-\mu)\,\phi_{\mathfrak{g},\zeta}(\lambda)+\mu\varphi_{\mathfrak{g},\zeta}(\lambda)\quad\forall\mu\in\{0,1\},\\ \mathfrak{h}_{\mathfrak{g}}(\lambda,\mu) &= (1-\lambda)\,\xi+\lambda\zeta\quad\forall\lambda\in\{0,1\}.\end{aligned}\tag{3.25}$$

The homotopy $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$ is said to establish a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy from $\phi_{\mathfrak{g},\zeta}$ to $\varphi_{\mathfrak{g},\zeta}$ in a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ if it belongs to the class:

$$\mathfrak{g}\text{-H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]\stackrel{\text{def}}{=}\big\{\mathfrak{h}_{\mathfrak{g}}:\ \big(\exists\phi_{\mathfrak{g},\zeta},\varphi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\mathscr{R}_{\mathfrak{g}}\big)\big[\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}\big]\big\}.\tag{3.26}$$

For any $\phi_{\mathfrak{g},\zeta},\varphi_{\mathfrak{g},\zeta},\psi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$, the statements $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\phi_{\mathfrak{g},\zeta}$, $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$ implies $\mathfrak{h}_{\mathfrak{g}}:\varphi_{\mathfrak{g},\zeta}\simeq\phi_{\mathfrak{g},\zeta}$ and, $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$ and $\mathfrak{h}_{\mathfrak{g}}:\varphi_{\mathfrak{g},\zeta}\simeq\psi_{\mathfrak{g},\zeta}$ implies $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\psi_{\mathfrak{g},\zeta}$ hold, as shown in the following theorem.

Theorem 3.26. *The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy relation is an equivalence relation in the collection of all paths in any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$.*

Proof. – *Reflexivity.* Let $\phi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\mathscr{R}_{\mathfrak{g}}$ be any path, where $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ is any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then the $\big([0,1]^2,\mathscr{R}_{\mathfrak{g}}\big)$-continuous map $\mathfrak{h}_{\mathfrak{g}}:[0,1]^2\longrightarrow\mathscr{R}_{\mathfrak{g}}$ defined, for every $(\lambda,\mu)\in[0,1]^2$, by $\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu)=\phi_{\mathfrak{g},\zeta}(\lambda)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy from $\phi_{\mathfrak{g},\zeta}$ to $\phi_{\mathfrak{g},\zeta}$, and that defined, for every $(\lambda,\mu)\in[0,1]^2$, by $\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu)=\varphi_{\mathfrak{g},\zeta}(\mu)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy from $\varphi_{\mathfrak{g},\zeta}$ to $\varphi_{\mathfrak{g},\zeta}$. Hence, $\mathfrak{h}_{\mathfrak{g}}\in\mathfrak{g}\text{-H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$, and $\simeq$ is reflexive.

– *Symmetry.* Let $\mathfrak{h}_{\mathfrak{g}}\in\mathfrak{g}\text{-H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$. Then the $\big([0,1]^2,\mathscr{R}_{\mathfrak{g}}\big)$-continuous map $\hat{\mathfrak{h}}_{\mathfrak{g}}:[0,1]^2\longrightarrow\mathscr{R}_{\mathfrak{g}}$ defined, for every $(\lambda,\mu)\in[0,1]^2$, by $\hat{\mathfrak{h}}_{\mathfrak{g}}(\lambda,\mu)=\mathfrak{h}_{\mathfrak{g}}(\lambda,1-\mu)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopy $\hat{\mathfrak{h}}_{\mathfrak{g}}:\varphi_{\mathfrak{g},\zeta}\simeq\phi_{\mathfrak{g},\zeta}$. Hence, $\hat{\mathfrak{h}}_{\mathfrak{g}}\in\mathfrak{g}\text{-H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$, and $\simeq$ is symmetric.

– *Transitivity.* Let $\mathfrak{h}_{\mathfrak{g},\alpha},\mathfrak{h}_{\mathfrak{g},\beta}\in\mathfrak{g}\text{-H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopies $\mathfrak{h}_{\mathfrak{g},\alpha}:\phi_{\mathfrak{g},\zeta}\simeq\varphi_{\mathfrak{g},\zeta}$ and $\mathfrak{h}_{\mathfrak{g},\beta}:\varphi_{\mathfrak{g},\zeta}\simeq\psi_{\mathfrak{g},\zeta}$, respectively. Consider the $\big([0,1]^2,\mathscr{R}_{\mathfrak{g}}\big)$-continuous map $\mathfrak{h}_{\mathfrak{g}}:[0,1]^2\longrightarrow\mathscr{R}_{\mathfrak{g}}$ defined, for every $(\lambda,\mu)\in[0,1]^2$, by $\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu)=\mathfrak{h}_{\mathfrak{g},\alpha}(\lambda,\eta\mu)$ if $\mu\in\left[0,\frac{1}{\eta}\right]$ and $\mathfrak{h}_{\mathfrak{g}}(\lambda,\mu)=\mathfrak{h}_{\mathfrak{g},\beta}(\lambda,\eta\mu-1)$ if $\mu\in\left[\frac{1}{\eta},1\right]$, where $\eta\in(1,\infty)$. Clearly, $\mathfrak{h}_{\mathfrak{g}}:\phi_{\mathfrak{g},\zeta}\simeq\psi_{\mathfrak{g},\zeta}$. Hence, it follows that, $\mathfrak{h}_{\mathfrak{g}}\in\mathfrak{g}\text{-H}\big[[0,1]^2;\mathscr{R}_{\mathfrak{g}}\big]$, and $\simeq$ is transitive. □

The concept of simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space is defined below.

Definition 3.27. Let $\varphi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\mathfrak{T}_{\mathfrak{g}}$ be a path from $\xi\in\mathfrak{T}_{\mathfrak{g}}$ to $\zeta\in\mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ with $\big(\varphi_{\mathfrak{g},\zeta}(0),\varphi_{\mathfrak{g},\zeta}(1)\big)=(\xi,\zeta)$. Then:

– I. If $\varphi_{\mathfrak{g},\zeta}:[0,1]\longrightarrow\{\zeta\}$, then $\varphi_{\mathfrak{g},\zeta}$ is called a "constant path" at $\zeta\in\mathfrak{T}_{\mathfrak{g}}$: $\varphi_{\mathfrak{g},\zeta}(\lambda)\stackrel{\text{def}}{=}\mathfrak{c}_{\mathfrak{g}}(\lambda)$ for all $\lambda\in[0,1]$.

– II. If $\varphi_{\mathfrak{g},\zeta}:\{0,1\}\longrightarrow\{\zeta\}$, then $\varphi_{\mathfrak{g},\zeta}$ is called a "closed path" at $\zeta\in\mathfrak{T}_{\mathfrak{g}}$: $\varphi_{\mathfrak{g},\zeta}(\lambda)\stackrel{\text{def}}{=}\mathfrak{k}_{\mathfrak{g}}(\lambda)$ for all $\lambda\in[0,1]$

– III. If, for every $\lambda\in[0,1]$, $\varphi_{\mathfrak{g},\zeta}(\lambda)=\mathfrak{k}_{\mathfrak{g}}(\lambda)$ and $\varphi_{\mathfrak{g},\zeta}(\lambda)\simeq\mathfrak{c}_{\mathfrak{g}}(\lambda)$, then $\varphi_{\mathfrak{g},\zeta}$ is said to be "contractable to the point $\zeta\in\mathfrak{T}_{\mathfrak{g}}$."

A $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is "simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected" if and only if, at each point $\zeta\in\mathfrak{T}_{\mathfrak{g}}$, any closed path $\mathfrak{k}_{\mathfrak{g}}:[0,1]\longrightarrow\mathfrak{T}_{\mathfrak{g}}$ is contractable to ζ.

The necessary and sufficient conditions for a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space to be simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected are contained in the following theorem.

Theorem 3.28. *Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space. Then, $\mathfrak{T}_{\mathfrak{g}}$ is simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected if and only if, at each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, any closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ at ζ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopic to the constant path $\mathfrak{c}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ at $\zeta \in \mathfrak{T}_{\mathfrak{g}}$: $\mathfrak{h}_{\mathfrak{g}} : \mathfrak{k}_{\mathfrak{g}} \simeq \mathfrak{c}_{\mathfrak{g}}$ for each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$.*

Proof. – *Necessity.* Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space, and suppose it be simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Since $\mathfrak{T}_{\mathfrak{g}}$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, for every $(\xi,\zeta) \in \Omega \times \Omega$,

$$\big(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big)\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}\big)\big[\Omega \supseteq \mathscr{Q}_{\mathfrak{g}} \supseteq \operatorname{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big)\big].$$

If $\varphi_{\mathfrak{g},\zeta} : \{0,1\} \longrightarrow \{\zeta\}$, then $\varphi_{\mathfrak{g},\zeta}$ is a closed path at $\zeta \in \mathfrak{T}_{\mathfrak{g}}$: $\varphi_{\mathfrak{g},\zeta}(\lambda) = \mathfrak{k}_{\mathfrak{g}}(\lambda)$ for all $\lambda \in [0,1]$. Since $\mathfrak{T}_{\mathfrak{g}}$ is simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, it follows that, at each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, the closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$ is contractable to ζ. Thus, at each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, the closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ at ζ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopic to the constant path $\mathfrak{c}_{\mathfrak{g}} : [0,1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$ at $\zeta \in \mathfrak{T}_{\mathfrak{g}}$: $\mathfrak{h}_{\mathfrak{g}} : \mathfrak{k}_{\mathfrak{g}} \simeq \mathfrak{c}_{\mathfrak{g}}$ for each $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. The condition of the theorem is, therefore, necessary.

– *Sufficiency.* Conversely, suppose that, at every point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ in a pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, any closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ at ζ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-homotopic to the constant path $\mathfrak{c}_{\mathfrak{g}} : [0,1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$ at $\zeta \in \mathfrak{T}_{\mathfrak{g}}$: $\mathfrak{h}_{\mathfrak{g}} : \mathfrak{k}_{\mathfrak{g}} \simeq \mathfrak{c}_{\mathfrak{g}}$ for every $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Then, there exists a path $\varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ satisfying $\varphi_{\mathfrak{g},\zeta} : \{0,1\} \longrightarrow \{\zeta\}$ and, therefore, contractable to $\zeta \in \mathfrak{T}_{\mathfrak{g}}$. Thus, at each point $\zeta \in \mathfrak{T}_{\mathfrak{g}}$, any closed path $\mathfrak{k}_{\mathfrak{g}} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}$ is contractable to ζ. The $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is, then, simply $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. The condition of the theorem is, therefore, sufficient. □

The definition of local $\mathfrak{g}$-$\mathfrak{T}$-connectedness at a point $\xi \in \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is now given.

Definition 3.29. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then:

– I. $\mathfrak{T}_{\mathfrak{g}}$ is said to be "locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected at a point $\xi \in \mathfrak{T}_{\mathfrak{g}}$" if and only if,

$$\big(\forall \mathscr{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\xi \in \mathscr{Q}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g}}\big]. \tag{3.27}$$

– II. $\mathfrak{T}_{\mathfrak{g}}$ is said to be "locally pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected" if and only if, given any $(\xi, \mathscr{U}_{\mathfrak{g},\xi}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, there exists $(\xi, \mathscr{Q}_{\mathfrak{g},\xi}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ such that $(\zeta,\eta) \in \mathscr{Q}_{\mathfrak{g}} \times \mathscr{Q}_{\mathfrak{g}}$, with $\zeta \neq \eta$, implies that,

$$\big(\exists \varphi_{\mathfrak{g},\zeta} : [0,1] \longrightarrow \mathfrak{T}_{\mathfrak{g}}\big)\big[\{\zeta,\eta\} \subseteq \operatorname{im}\big(\varphi_{\mathfrak{g},\zeta_{|[0,1]}}\big) \subseteq \mathscr{Q}_{\mathfrak{g},\xi} \subseteq \mathscr{U}_{\mathfrak{g},\xi}\big]. \tag{3.28}$$

The $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to be "locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected" if and only if it is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected at every point $\xi \in \mathfrak{T}_{\mathfrak{g}}$.

As an immediate consequence of the above definition, it is shown below that local pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness implies locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.

Theorem 3.30. *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a locally pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space, then it is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected:*

$$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LPC} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LPC} \rangle}\big) \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LC} \rangle} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LC} \rangle}\big). \tag{3.29}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a locally pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space. Then, for any given $(\xi,\mathscr{U}_{\mathfrak{g},\xi})\in\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, there exists $(\xi,\mathscr{Q}_{\mathfrak{g},\xi})\in\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ such that $(\zeta,\xi)\in\mathscr{Q}_{\mathfrak{g}}\times\mathscr{Q}_{\mathfrak{g}}$, with $\zeta\neq\xi$, implies that,

$$\big(\exists\varphi_{\mathfrak{g},\zeta}:[0,1]\rightarrow\mathfrak{T}_{\mathfrak{g}}\big)\big[\{\zeta,\xi\}\subseteq\operatorname{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big)\subseteq\mathscr{Q}_{\mathfrak{g},\xi}\subseteq\mathscr{U}_{\mathfrak{g},\xi}\big].$$

Consequently, $\xi\in\operatorname{im}\big(\varphi_{\mathfrak{g},\zeta\,|[0,1]}\big)\subseteq\mathscr{Q}_{\mathfrak{g},\xi}\subseteq\mathscr{U}_{\mathfrak{g},\xi}$ and, therefore, $\xi\in\mathscr{Q}_{\mathfrak{g},\xi}\subseteq\mathscr{U}_{\mathfrak{g},\xi}$. Hence, it follows that

$$\big(\forall\mathscr{U}_{\mathfrak{g}},\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big(\exists\mathscr{Q}_{\mathfrak{g}},\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\xi\in\mathscr{Q}_{\mathfrak{g},\xi}\subseteq\mathscr{U}_{\mathfrak{g},\xi}\big].$$

The $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is therefore locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. □

In a locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set, as demonstrated in the following theorem.

Theorem 3.31. *If* $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}\left[\zeta\right]$ *be the* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-component of* $\mathscr{S}_{\mathfrak{g}}$ *corresponding to* ζ *in a locally* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-connected* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then* $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}\left[\zeta\right]\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$*:*

$$\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}\left[\zeta\right]\subseteq\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{LC}\rangle}=\big(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{LC}\rangle}\big)\implies\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}\left[\zeta\right]\in\mathfrak{g}\text{-O}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{LC}\rangle}\big].\tag{3.30}$$

Proof. Let $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}\left[\zeta\right]$ be the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\mathscr{S}_{\mathfrak{g}}$ corresponding to ζ in a locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, local $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness at $\zeta\in\mathfrak{T}_{\mathfrak{g}}$ implies

$$\big(\forall\mathscr{U}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big(\exists\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\zeta\in\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{U}_{\mathfrak{g}}\big].$$

Consequently, $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}\left[\zeta\right]=\bigcup_{\mathscr{Q}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}}\mathscr{Q}_{\mathfrak{g}}\subseteq\bigcup_{\mathscr{U}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}}\mathscr{U}_{\mathfrak{g}}$. But, since every $\mathscr{U}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ satisfies $\mathscr{U}_{\mathfrak{g}}\subseteq\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g}}\big)$ for some $\mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g}}$, it follows that the relation $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}\left[\zeta\right]\subseteq\bigcup_{\mathscr{O}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}}\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g}}\big)=\operatorname{op}_{\mathfrak{g}}\big(\bigcup_{\mathscr{O}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}}\mathscr{O}_{\mathfrak{g}}\big)$ holds. Thus, $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}\left[\zeta\right]\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. □

The necessary and sufficient conditions for a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ to be locally $\mathfrak{g}$-$\mathfrak{T}$-connected at a point $\xi\in\mathfrak{T}_{\mathfrak{g}}$ is contained in the following theorem.

Theorem 3.32. *A* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *is locally* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-connected at a point* $\xi\in\mathfrak{T}_{\mathfrak{g}}$ *if and only if,*

$$\big(\forall\mathscr{O}_{\mathfrak{g},\xi}\in\mathscr{T}_{\mathfrak{g}}\big)\big(\exists\mathscr{Q}_{\mathfrak{g},\xi}\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\xi\in\mathscr{Q}_{\mathfrak{g},\xi}\subseteq\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)\big].\tag{3.31}$$

Proof. – *Necessity.* Let it be assumed that the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected at $\xi\in\mathfrak{T}_{\mathfrak{g}}$, and let $\mathscr{O}_{\mathfrak{g},\xi}\in\mathscr{T}_{\mathfrak{g}}$ be an arbitrary $\mathscr{T}_{\mathfrak{g}}$-open neighbourhood of ξ. There exists, then, a $\mathscr{T}_{\mathfrak{g}}$-open neighbourhood $\hat{\mathscr{O}}_{\mathfrak{g},\xi}\in\mathscr{T}_{\mathfrak{g}}$ of ξ such that $\xi\in\operatorname{op}_{\mathfrak{g}}\big(\hat{\mathscr{O}}_{\mathfrak{g},\xi}\big)\subseteq\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)$ and, for every $\{\zeta,\eta\}\subseteq\hat{\mathscr{O}}_{\mathfrak{g},\xi}$,

$$\big(\exists\mathscr{Q}_{\mathfrak{g},(\zeta,\eta)}\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\{\zeta,\eta\}\subseteq\mathscr{Q}_{\mathfrak{g},(\zeta,\eta)}\subseteq\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)\big].$$

Suppose $\eta\in\hat{\mathscr{O}}_{\mathfrak{g},\xi}$ be the arbitrary point. Then, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected set $\mathscr{Q}_{\mathfrak{g},(\xi,\eta)}\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ satisfying $\{\xi,\eta\}\subseteq\mathscr{Q}_{\mathfrak{g},(\xi,\eta)}\subseteq\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)$. Let $\mathscr{Q}_{\mathfrak{g},\xi}=\bigcup_{\eta\in\hat{\mathscr{O}}_{\mathfrak{g},\xi}}\mathscr{Q}_{\mathfrak{g},(\xi,\eta)}\subseteq\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)$. Since $\mathscr{Q}_{\mathfrak{g},\xi}\supseteq\hat{\mathscr{O}}_{\mathfrak{g},\xi}$ and $\bigcup_{\eta\in\hat{\mathscr{O}}_{\mathfrak{g},\xi}}\mathscr{Q}_{\mathfrak{g},(\xi,\eta)}\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it follows that $\mathscr{Q}_{\mathfrak{g},\xi}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected neighbourhood of ξ contained in $\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)$. The condition of the theorem is, therefore, necessary.

– *Sufficiency.* Conversely, suppose the following condition holds:

$$\big(\forall\mathscr{O}_{\mathfrak{g},\xi}\in\mathscr{T}_{\mathfrak{g}}\big)\big(\exists\mathscr{Q}_{\mathfrak{g},\xi}\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\xi\in\mathscr{Q}_{\mathfrak{g},\xi}\subseteq\operatorname{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\xi}\big)\big].$$

Let $\mathscr{O}_{\mathfrak{g},\xi} \in \mathscr{T}_{\mathfrak{g}}$ be an arbitrary $\mathscr{T}_{\mathfrak{g}}$-open neighbourhood of ξ. Then, $\mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\xi})$ contains a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected neighbourhood $\mathscr{Q}_{\mathfrak{g},\xi}$ of ξ. Since $\mathscr{Q}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$, for any $\{\zeta,\eta\} \subseteq \mathscr{Q}_{\mathfrak{g},\xi}$, there exists $\mathscr{Q}_{\mathfrak{g},(\xi,\eta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\{\zeta,\eta\} \subseteq \mathscr{Q}_{\mathfrak{g},(\zeta,\eta)}$. But $\mathscr{Q}_{\mathfrak{g},\xi} \supseteq \mathscr{Q}_{\mathfrak{g},(\zeta,\eta)}$ and, consequently, $\{\zeta,\eta\} \subseteq \mathscr{Q}_{\mathfrak{g},(\zeta,\eta)} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\xi})$. Hence, the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected at $\xi \in \mathfrak{T}_{\mathfrak{g}}$. The condition of the theorem is, therefore, sufficient. □

The notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness between any $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ and the relevant basic theorems are now discussed.

Theorem 3.33. *Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, let $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, …, $\mathscr{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Lambda}]$ be $n \geq 1$ mutually disjoint $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathfrak{T}_{\mathfrak{g},\Lambda}$, where $\Lambda \in \{\Omega,\Sigma\}$, and let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-B}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-bijective map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$. Then*

– I. $\pi_{\mathfrak{g}}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big) = \bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$,
– II. $\pi_{\mathfrak{g}}^{-1}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big) = \bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$.

Proof. – I. Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, let $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, …, $\mathscr{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega}]$, and let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-B}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$. If $\zeta \in \pi_{\mathfrak{g}}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big)$, then, since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-B}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$, there exists $\xi \in \bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}$ such that, $\pi_{\mathfrak{g}}^{-1}(\zeta) = \xi \in \bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}$. Consequently,

$$\begin{aligned}\pi_{\mathfrak{g}}^{-1}(\zeta) \in \bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha} \quad &\Rightarrow \quad \bigvee_{\alpha\in I_n^*}\big(\pi_{\mathfrak{g}}^{-1}(\zeta) \in \mathscr{S}_{\mathfrak{g},\alpha}\big)\\ &\Rightarrow \quad \bigvee_{\alpha\in I_n^*}\big(\zeta \in \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\alpha})\big) \;\Rightarrow\; \zeta \in \bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\alpha}).\end{aligned}$$

Hence, $\pi_{\mathfrak{g}}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big) \subseteq \bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$. Conversely, if it be assumed that $\zeta \in \bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$ then,

$$\begin{aligned}\bigvee_{\alpha\in I_n^*}\big(\zeta \in \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\alpha})\big) \quad &\Rightarrow \quad \bigvee_{\alpha\in I_n^*}\big(\pi_{\mathfrak{g}}^{-1}(\zeta) \in \mathscr{S}_{\mathfrak{g},\alpha}\big)\\ &\Rightarrow \quad \pi_{\mathfrak{g}}^{-1}(\zeta) \in \bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha} \;\Rightarrow\; \zeta \in \pi_{\mathfrak{g}}\big(\textstyle\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big).\end{aligned}$$

Hence, $\pi_{\mathfrak{g}}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big) \supseteq \bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$.

– II. If $\xi \in \pi_{\mathfrak{g}}^{-1}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big)$, where $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, …, $\mathscr{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$, then,

$$\begin{aligned}\pi_{\mathfrak{g}}(\xi) \in \bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha} \quad &\Rightarrow \quad \bigvee_{\alpha\in I_n^*}\big(\pi_{\mathfrak{g}}(\xi) \in \mathscr{S}_{\mathfrak{g},\alpha}\big)\\ &\Rightarrow \quad \bigvee_{\alpha\in I_n^*}\big(\xi \in \pi_{\mathfrak{g}}^{-1}(\mathscr{S}_{\mathfrak{g},\alpha})\big) \;\Rightarrow\; \xi \in \bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}^{-1}(\mathscr{S}_{\mathfrak{g},\alpha}).\end{aligned}$$

Hence, $\pi_{\mathfrak{g}}^{-1}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big)\subseteq\bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$. Conversely, if it be supposed that $\xi\in\bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$ then,

$$\begin{aligned}\bigvee_{\alpha\in I_n^*}\left(\xi\in\pi_{\mathfrak{g}}^{-1}\left(\mathscr{S}_{\mathfrak{g},\alpha}\right)\right) \quad&\Rightarrow\quad \bigvee_{\alpha\in I_n^*}\left(\pi_{\mathfrak{g}}\left(\xi\right)\in\mathscr{S}_{\mathfrak{g},\alpha}\right)\\ &\Rightarrow\quad \pi_{\mathfrak{g}}\left(\xi\right)\in\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\ \Rightarrow\ \xi\in\pi_{\mathfrak{g}}^{-1}\big(\textstyle\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big).\end{aligned}$$

Hence, $\pi_{\mathfrak{g}}^{-1}\big(\bigsqcup_{\alpha\in I_n^*}\mathscr{S}_{\mathfrak{g},\alpha}\big)\supseteq\bigsqcup_{\alpha\in I_n^*}\pi_{\mathfrak{g}}^{-1}\big(\mathscr{S}_{\mathfrak{g},\alpha}\big)$. □

The following theorem shows, among others, that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a $\mathscr{T}_{\mathfrak{g}}$-property.

Theorem 3.34. *Let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\longrightarrow\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map and let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$, then $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$:*

$$\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]\implies\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]. \tag{3.32}$$

Proof. Let $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, $\mathscr{S}_{\mathfrak{g},\omega}=\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, and suppose that $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, that is, $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\notin\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. There exists, therefore, $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\sigma,\zeta_\sigma}\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$ such that,

$$\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{U}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big)\bigvee\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{V}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big).$$

Set $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)=\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)$, where $\mathscr{S}_{\mathfrak{g},\omega}\supseteq\bigcup_{\lambda=\xi_\omega,\zeta_\omega}\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}$ and, for every $(\lambda,\mu)\in\{(\xi_\sigma,\xi_\omega),(\zeta_\sigma,\zeta_\omega)\}$, set

$$\big[\mathscr{U}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\mu)}}\big)\big]\vee\big[\mathscr{V}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\mu)}}\big)\big].$$

In other words, $\mathscr{S}_{\mathfrak{g},(\omega,\xi_\omega)}\subseteq\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ denotes the $\mathfrak{T}_{\mathfrak{g}}$-set of all $\xi\in\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ for which $\pi_{\mathfrak{g}}\left(\xi\right)\in\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\xi_\omega)}}\big)$, and $\mathscr{S}_{\mathfrak{g},(\omega,\zeta_\omega)}\subseteq\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ denotes the $\mathfrak{T}_{\mathfrak{g}}$-set of all $\zeta\in\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ for which $\pi_{\mathfrak{g}}\left(\zeta\right)\in\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\zeta_\omega)}}\big)$. Since the inequality $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)\neq\emptyset$ holds for every $\lambda\in\{\xi_\omega,\zeta_\omega\}$, and both the relations $\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ and $\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)=\emptyset$ hold, it follows that, $\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}\neq\emptyset$ for every $\lambda\in\{\xi_\omega,\zeta_\omega\}$, $\bigcup_{\lambda=\xi_\omega,\zeta_\omega}\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}=\mathscr{S}_{\mathfrak{g},\omega}$ and $\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}=\emptyset$. Since $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, for any $\lambda\in\{\xi_\omega,\zeta_\omega\}$, there exists, for every $\big(\mathscr{O}_{\mathfrak{g},(\sigma,\lambda)},\mathscr{K}_{\mathfrak{g},(\sigma,\lambda)}\big)\in\mathscr{T}_{\mathfrak{g},\Sigma}\times\neg\mathscr{T}_{\mathfrak{g},\Sigma}$, $\big(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)},\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\big)\in\mathscr{T}_{\mathfrak{g},\Omega}\times\neg\mathscr{T}_{\mathfrak{g},\Omega}$, with $\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}$, $\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\subset\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}$ and $\mathscr{O}_{\mathfrak{g},(\sigma,\lambda)}$, $\mathscr{K}_{\mathfrak{g},(\sigma,\lambda)}\subset\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)$, such that,

$$\left[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{O}_{\mathfrak{g},(\sigma,\lambda)}\right)\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\right]\vee\left[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{K}_{\mathfrak{g},(\sigma,\lambda)}\right)\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\right].$$

Since $\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)=\emptyset$ implies $\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}=\emptyset$, it follows, evidently, that,

$$\Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\pi_{\mathfrak{g}}^{-1}\left(\mathscr{O}_{\mathfrak{g},(\sigma,\lambda)}\right)=\emptyset\Big)\bigwedge\Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega}\pi_{\mathfrak{g}}^{-1}\left(\mathscr{K}_{\mathfrak{g},(\sigma,\lambda)}\right)=\emptyset\Big).$$

Therefore, the setting $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) = \bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)$ holds. It now remains to prove that it is the case and the supposition that $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ is a contradiction. Since $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ it follows that, for all pair $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\omega,\zeta_\omega} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$,

$$\begin{aligned}
&\neg\Big(\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} = \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} = \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big)\\
&\Leftrightarrow \neg\Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} = \emptyset\Big) \bigwedge \neg\Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} = \emptyset\Big)\\
&\Rightarrow \Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} \neq \emptyset\Big) \bigwedge \Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} \neq \emptyset\Big).
\end{aligned}$$

There exists, then, a unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\eta_\omega\} \subset \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ such that,

$$\Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} \supseteq \{\eta_\omega\}\Big) \bigwedge \Big(\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} \supseteq \{\eta_\omega\}\Big).$$

Since $\{\eta_\omega\} \subset \bigcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{S}_{\mathfrak{g},(\omega,\lambda)} = \mathscr{S}_{\mathfrak{g},\omega}$ and $\bigcap_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{S}_{\mathfrak{g},(\omega,\lambda)} = \emptyset$, it results that,

$$\big[\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \supset \mathscr{S}_{\mathfrak{g},(\omega,\xi_\omega)} \supset \{\eta_\omega\}\big] \vee \big[\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \supset \mathscr{S}_{\mathfrak{g},(\omega,\zeta_\omega)} \supset \{\eta_\omega\}\big].$$

On the other hand, since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, it follows that, for every unit $\mathfrak{T}_{\mathfrak{g}}$-set $\{\eta_\sigma\} \subset \bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big) = \mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$,

$$\bigvee_{\lambda=\xi_\omega,\zeta_\omega} \big[\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \supset \mathscr{S}_{\mathfrak{g},(\omega,\lambda)} \supset \pi_{\mathfrak{g}}^{-1}\left(\{\eta_\sigma\}\right)\big].$$

In particular, if $\pi_{\mathfrak{g}}^{-1}\left(\{\eta_\sigma\}\right) = \{\eta_\omega\}$, then $\{\eta_\sigma\} = \pi_{\mathfrak{g}}\left(\{\eta_\omega\}\right)$, leading to a contradiction. There exists, therefore, $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\omega,\zeta_\omega} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ such that,

$$\Big(\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\lambda} = \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\lambda} = \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big).$$

This proves that the supposition $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ is a contradiction and, hence, $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. □

The following corollary is another way of saying that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectivity of $\mathfrak{T}_{\mathfrak{g},\Omega}$ implies the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectivity of $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Corollary 3.35. *Let* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *be a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map and let* $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set. If* $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ *is* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-separated in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$*, then* $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ *is* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-separated in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*:*

$$\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \implies \mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]. \tag{3.33}$$

If the image of a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then it is also $\mathfrak{T}_{\mathfrak{g}}$-connected, as proved in the following proposition.

Proposition 8. Let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\longrightarrow\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map and let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, then $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$:

$$\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\implies\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]. \tag{3.34}$$

Proof. Let $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, $\mathscr{S}_{\mathfrak{g},\omega}=\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ and, suppose that $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, that is, $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\notin\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. There exists, then, $\big(\mathscr{R}_{\mathfrak{g},\lambda},\mathscr{S}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\sigma,\zeta_\sigma}\in\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]\times\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$ such that,

$$\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{R}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big)\bigvee\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathscr{S}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\Big).$$

Since $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, set $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)$ and, for every $\lambda\in\{\xi_\sigma,\zeta_\sigma\}$, let

$$\big[\mathscr{R}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)\big]\vee\big[\mathscr{S}_{\mathfrak{g},\lambda}=\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)\big].$$

On the other hand, since $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, there exists, for any $\lambda\in\{\xi_\sigma,\zeta_\sigma\}$, $\big(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)},\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\big)\in\mathscr{T}_{\mathfrak{g},\Omega}\times\neg\mathscr{T}_{\mathfrak{g},\Omega}$, satisfying $\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}$, $\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\subset\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$, such that,

$$\begin{aligned}&\big[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{R}_{\mathfrak{g},\lambda}\right)\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\big]\vee\big[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{S}_{\mathfrak{g},\lambda}\right)\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\big]\\ \Rightarrow\quad&\big[\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)\subseteq\pi_{\mathfrak{g}}\big(\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\big)\big]\vee\big[\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)\\&\qquad\supseteq\pi_{\mathfrak{g}}\big(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\big)\big].\end{aligned}$$

Since $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\big)$, it is plain that $\pi_{\mathfrak{g}}\big(\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\big)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}\big(\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\big)$, and also $\emptyset=\bigcap_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}\big(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\big)$, implying $\pi_{\mathfrak{g}}\big(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\big)=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}\big(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\big)$, for some $\big(\mathscr{O}_{\mathfrak{g},\omega},\mathscr{K}_{\mathfrak{g},\omega}\big)\in\mathscr{T}_{\mathfrak{g},\Omega}\times\neg\mathscr{T}_{\mathfrak{g},\Omega}$. But, clearly the relation $\big(\pi_{\mathfrak{g}}\big(\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\big),\pi_{\mathfrak{g}}\big(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\big)\big)\in\mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]\times\mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$ holds. Thus, $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ implies $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, or equivalently, $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ implies $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. This proves the proposition. □

The following corollary is another way of saying that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectivity of $\mathfrak{T}_{\mathfrak{g},\Omega}$ implies the $\mathfrak{T}_{\mathfrak{g}}$-connectivity of $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Corollary 3.36. *Let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\longrightarrow\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map and let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{T}_{\mathfrak{g}}$-separated in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, then $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, then:*

$$\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\implies\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]. \tag{3.35}$$

Theorem 3.37. *Let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\longrightarrow\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute map and let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$, then $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$:*

$$\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]\implies\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)\in\mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]. \tag{3.36}$$

Proof. Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ be a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-irresolute map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$, let $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set, and suppose $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. Since $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \notin \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, or equivalently $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, by hypothesis, there exists $\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi_\sigma,\zeta_\sigma} \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \times \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ such that,

$$\Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)\Bigg) \bigvee \Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)\Bigg).$$

On the other hand, since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, there exists, for any $\lambda \in \{\xi_\sigma,\zeta_\sigma\}$, $\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)},\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$, satisfying $\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}$, $\mathscr{K}_{\mathfrak{g},(\omega,\lambda)} \subset \operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)$, such that,

$$\left[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{U}_{\mathfrak{g},\lambda}\right) \subseteq \operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},(\omega,\lambda)}\right)\right] \vee \left[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{V}_{\mathfrak{g},\lambda}\right) \supseteq \neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},(\omega,\lambda)}\right)\right].$$

Since both the relation $\pi_{\mathfrak{g}}^{-1}\left(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda}\right) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \pi_{\mathfrak{g}}^{-1}\left(\mathscr{U}_{\mathfrak{g},\lambda}\right)$ and the relation $\pi_{\mathfrak{g}}^{-1}\left(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda}\right) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \pi_{\mathfrak{g}}^{-1}\left(\mathscr{V}_{\mathfrak{g},\lambda}\right)$. Evidently, $\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \pi_{\mathfrak{g}}^{-1}\left(\mathscr{U}_{\mathfrak{g},\lambda}\right) \subset \operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)$, and also $\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \pi_{\mathfrak{g}}^{-1}\left(\mathscr{V}_{\mathfrak{g},\lambda}\right) \subset \operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right)$, and from which it follows that a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation $\operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},(\omega,\lambda)}}\right)$ is realised in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Consequently, $\operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. Therefore, $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \text{D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ implies $\operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, or equivalently, $\operatorname{dom}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ implies $\operatorname{im}\left(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\right) \in \text{Q}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. This proves proves the proposition. ☐

In actual fact, between any two such $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, being a $\mathscr{T}_{\mathfrak{g}}$-property, is preserved by a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-homeomorphism $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$.

Theorem 3.38. *Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, and let $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-homeomorphism. If $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected:*

$$\mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma} : \quad \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{\langle \text{C}\rangle} = \left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Omega}^{\langle \text{C}\rangle}\right) \implies \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{\langle \text{C}\rangle} = \left(\Sigma,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Sigma}^{\langle \text{C}\rangle}\right). \tag{3.37}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, let $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}\text{-}\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-homeomorphism, and suppose that $\mathfrak{T}_{\mathfrak{g},\Sigma}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. There exists, then, $\left(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\right)_{\lambda=\xi_\sigma,\zeta_\sigma} \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \times \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ such that,

$$\Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}\right)\Bigg) \bigvee \Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}\right)\Bigg).$$

Clearly, $\operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}\right) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and, with no loss of generality, consider the setting $\operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma}\right) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\right)$ so that, for every $\lambda \in \{\xi_\sigma,\zeta_\sigma\}$, either $\mathscr{U}_{\mathfrak{g},\lambda} = \operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\right)$ or $\mathscr{V}_{\mathfrak{g},\lambda} = \operatorname{dom}\left(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\right)$. Since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-Hom}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, $\pi_{\mathfrak{g}}^{-1} : \mathfrak{T}_{\mathfrak{g},\Sigma} \cong \mathfrak{T}_{\mathfrak{g},\Omega}$ and, for any $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right] \times \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right]$, $\pi_{\mathfrak{g}}^{-1}\left(\bigsqcup_{\lambda=\alpha,\beta} \mathscr{S}_{\mathfrak{g},\lambda}\right) =$

$\bigsqcup_{\lambda=\alpha,\beta}\pi_{\mathfrak{g}}^{-1}(\mathscr{S}_{\mathfrak{g},\lambda})$, where $\Lambda\in\{\Omega,\Sigma\}$, it results that,

$$\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\lambda})=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathrm{im}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda})\Big)$$
$$\bigvee\Big(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}(\mathscr{V}_{\mathfrak{g},\lambda})=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathrm{im}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda})\Big),$$

where $\mathrm{im}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma})=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathrm{im}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda})$. On the other hand, since $\pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\lambda})\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ and $\pi_{\mathfrak{g}}^{-1}(\mathscr{V}_{\mathfrak{g},\lambda})\in\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ hold for every $\lambda\in\{\xi_\sigma,\zeta_\sigma\}$, there exist, therefore, $(\mathscr{U}_{\mathfrak{g},\eta},\mathscr{V}_{\mathfrak{g},\eta})_{\eta=\xi_\omega,\zeta_\omega}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ such that,

$$\begin{aligned}\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\lambda})&=\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{U}_{\mathfrak{g},\eta},\\ \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\pi_{\mathfrak{g}}^{-1}(\mathscr{V}_{\mathfrak{g},\lambda})&=\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{V}_{\mathfrak{g},\eta}.\end{aligned}$$

By substitution, then, it follows that,

$$\Big(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{U}_{\mathfrak{g},\eta}=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathrm{im}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda})\Big)$$
$$\bigvee\Big(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{V}_{\mathfrak{g},\eta}=\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma}\mathrm{im}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda})\Big).$$

Since $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-Hom}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ and $\pi_{\mathfrak{g}}^{-1}\in\mathfrak{g}\text{-Hom}[\mathfrak{T}_{\mathfrak{g},\Sigma};\mathfrak{T}_{\mathfrak{g},\Omega}]$, for each $\mathrm{im}(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda})$, there exists a unique $\mathrm{dom}(\pi_{\mathfrak{g}|\Omega_\eta})$, with $\mathrm{dom}(\pi_{\mathfrak{g}|\Omega})=\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathrm{dom}(\pi_{\mathfrak{g}|\Omega_\eta})$. Thus, there exists $(\mathscr{U}_{\mathfrak{g},\eta},\mathscr{V}_{\mathfrak{g},\eta})_{\eta=\xi_\omega,\zeta_\omega}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ such that,

$$\Big(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{U}_{\mathfrak{g},\eta}=\mathrm{dom}(\pi_{\mathfrak{g}|\Omega})\Big)\bigvee\Big(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega}\mathscr{V}_{\mathfrak{g},\eta}=\mathrm{dom}(\pi_{\mathfrak{g}|\Omega})\Big).$$

Hence, $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated. □

An immediate consequence of the above theorem is the following corollary.

Corollary 3.39. *Let $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma}=(\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, and let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\cong\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-homeomorphism. If $\mathfrak{T}_{\mathfrak{g},\Sigma}=(\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, then $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated:*

$$(3.38)\ \mathfrak{T}_{\mathfrak{g},\Omega}\cong\mathfrak{T}_{\mathfrak{g},\Sigma}:\quad \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{\langle\mathrm{D}\rangle}=\big(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Omega}^{\langle\mathrm{D}\rangle}\big)\ \Leftarrow\ \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{\langle\mathrm{D}\rangle}=\big(\Sigma,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Sigma}^{\langle\mathrm{D}\rangle}\big).$$

For every $\mu\in I_n^*$, let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle\mathrm{C}\rangle}=\big(\Omega_\mu,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle\mathrm{C}\rangle}\big)$ stand for the shortened form of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}(\Omega_\mu)=\big(\Omega_\mu,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}(\Omega_\mu)\big)$. In the following lemma, it is proved that the Cartesian product of two $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-spaces is also a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-space.

Lemma 3.40. *If $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle\mathrm{C}\rangle}=\big(\Omega_\mu,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\mu}^{\langle\mathrm{C}\rangle}\big)$, $\mu\in\{\alpha,\beta\}$, be two $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-spaces, then $\bigtimes_{\mu=\alpha,\beta}\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle\mathrm{C}\rangle}$ is also a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle\mathrm{C}\rangle}$-space.*

Proof. Let $\mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu} = \big(\Omega_\mu, \mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}\big)$, $\mu \in \{\alpha,\beta\}$, be two $\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g}}$-spaces, and suppose $\xi = (\xi_\alpha, \xi_\beta) \in \bigtimes_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$ and $\zeta = (\zeta_\alpha, \zeta_\beta) \in \bigtimes_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$ be any two pairs of points in $\bigtimes_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$. Then,

$$\big[\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\beta} \cong \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\beta}\big] \wedge \big[\mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\alpha} \times \{\zeta_\beta\} \cong \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\alpha}\big].$$

Consequently, $\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\beta}$ and $\mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\alpha} \times \{\zeta_\beta\}$ are both $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. But,

$$\big(\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\beta}\big) \cap \big(\mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\alpha} \times \{\zeta_\beta\}\big) = \{(\xi_\alpha, \zeta_\beta)\} \neq \emptyset.$$

Hence, $\big(\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\beta}\big) \cup \big(\mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\alpha} \times \{\zeta_\beta\}\big)$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Accordingly, $\xi,\ \zeta \in \bigtimes_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$ belong to the same $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component. That is, $\xi,\ \zeta \in \mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\eta] \subseteq \bigtimes_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\Omega = \bigtimes_{\mu=\alpha,\beta} \Omega_\mu$ corresponding to the point $\eta \in \bigtimes_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$. But $\xi,\ \zeta \in \bigtimes_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$ were arbitrary. Hence the Cartesian product $\bigtimes_{\mu=\alpha,\beta} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$ has one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-}\mathrm{C}_{\Omega}[\eta] = \bigtimes_{\mu=\alpha,\beta} \Omega_\mu$, and is therefore a $\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g}}$-space. □

More generally, the Cartesian product of $\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g}}$-spaces is also a $\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g}}$-space; that is, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is a product invariant $\mathscr{T}_{\mathfrak{g}}$-property. The theorem follows.

Theorem 3.41. *If* $\big\{\mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu} = \big(\Omega_\mu, \mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}\big) :\ \mu \in I^*_n\big\}$ *be a collection of* $n \geq 1$ $\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g}}$*-spaces, then* $\bigtimes_{\mu \in I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$ *is also a* $\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g}}$*-space:*

$$\big\{\mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu} = \big(\Omega_\mu, \mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}\big) :\ \mu \in I^*_n\big\} \ \Rightarrow\ \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g}} = \bigtimes_{\mu \in I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}\,. \tag{3.39}$$

Proof. Let $\big\{\mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu} = \big(\Omega_\mu, \mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}\big) :\ \mu \in I^*_n\big\}$ be a collection of $n \geq 1$ $\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g}}$-spaces, and let $\bigtimes_{\mu \in I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$ be the Cartesian product of these $\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g}}$-spaces. Moreover, let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \bigtimes_{\mu \in I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$, and let $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \subseteq \bigtimes_{\mu \in I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$ be the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component of $\mathscr{S}_{\mathfrak{g}} \subseteq \bigtimes_{\mu \in I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$ corresponding to $\zeta \in \bigtimes_{\mu \in I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$. By hypothesis, let it be claimed that, for every $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \bigtimes_{\mu \in I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$, $\xi \in \neg\,\mathrm{op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big)$ and, thus, $\xi \in \mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ since $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big)$, meaning that $\mathfrak{g}\text{-}\mathrm{C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$ must be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set in $\bigtimes_{\mu \in I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$. For every $\big(\mu, \sigma(\mu), \mathscr{O}_{\mathfrak{g},\sigma(\mu)}\big) \in \big\{\mu\big\} \times I^*_\infty \times \mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$, there exists $I_{\sigma(\mu)} \subseteq I^*_\infty$ such that $\mathscr{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup_{\nu \in I^*_{\sigma(\mu)}} \mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}$. Thus, the class $\mathscr{B}\big[\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}\big] \stackrel{\mathrm{def}}{=} \big\{\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)} :\ (\nu, \mu, \sigma(\nu,\mu)) \in I^*_\infty \times \big\{\mu\big\} \times I^*_\infty\big\}$ is a $\mathscr{T}_{\mathfrak{g}}$-basis for $\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu} : \mathscr{P}(\Omega_\mu) \to \mathscr{P}(\Omega_\mu)$. Therefore, for any $\xi \in \mathscr{O}_{\mathfrak{g},\sigma(\mu)} \in \mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$, there exists $\mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathscr{B}\big[\mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}\big]$ with $\xi \in \mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)} \subseteq \mathscr{O}_{\mathfrak{g},\sigma(\mu)} \in \mathfrak{g}\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}$. Now let

$$\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathscr{R}_{\mathfrak{g}} = \Bigg(\bigtimes_{\mu \in I^*_n \setminus J_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}\Bigg) \times \Bigg(\bigtimes_{\mu \in J_n \subset I^*_n} \mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)}\Bigg).$$

Now the following relation holds,

$$\mathscr{S}_{\mathfrak{g}} = \Bigg(\bigtimes_{\mu \in I^*_n \setminus J_n} \{\zeta_\mu\}\Bigg) \times \Bigg(\bigtimes_{\mu \in J_n \subset I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu}\Bigg) \cong \bigtimes_{\mu \in J_n \subset I^*_n} \mathfrak{g}\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle}_{\mathfrak{g},\mu},$$

and, hence, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Furthermore, $\zeta \in \mathscr{S}_{\mathfrak{g}}$ and, consequently, it follows that $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta] \supseteq \mathscr{S}_{\mathfrak{g}}$. But, by the property of the intersection of Cartesian products,

$$\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}} = \Bigg(\bigtimes_{\mu \in I_n^* \setminus J_n} \{\zeta_\mu\} \Bigg) \times \Bigg(\bigtimes_{\mu \in J_n \subset I_n^*} \mathscr{O}_{\mathfrak{g},\sigma(\nu,\mu)} \Bigg) \neq \emptyset.$$

Therefore, $\mathscr{R}_{\mathfrak{g}} \subset \bigtimes_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C} \rangle}$ contains a point of $\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Accordingly, $\xi \in \neg\,\mathrm{op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]\big) \subseteq \mathfrak{g}\text{-C}_{\mathscr{S}_{\mathfrak{g}}}[\zeta]$. Hence the Cartesian product $\bigtimes_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{\langle \mathrm{C} \rangle}$ has one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component $\mathfrak{g}\text{-C}_{\Omega}[\zeta] = \bigtimes_{\mu \in I_n^*} \Omega_\mu$, and is therefore a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C} \rangle}$-space. □

The concept of $(\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma})$-surjective map between any such $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ is now defined.

Definition 3.42 ($(\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma})$-Surjective Map)**.** A $(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is said to be surjective if and only if it belongs the following class:

$$\mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \stackrel{\mathrm{def}}{=} \big\{\pi_{\mathfrak{g}} :\ (\forall \zeta \in \mathfrak{T}_{\mathfrak{g},\Sigma})\,(\exists \xi \in \mathfrak{T}_{\mathfrak{g},\Omega})\,\big[\pi_{\mathfrak{g}}(\xi) = \zeta\big]\big\}. \tag{3.40}$$

If the domain of a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute surjective map is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then its codomain is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, as demonstrated in the theorem below.

Theorem 3.43. *Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces. If $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute surjective map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \to \mathfrak{T}_{\mathfrak{g},\Sigma}$ and $\mathfrak{T}_{\mathfrak{g},\Omega}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then $\mathfrak{T}_{\mathfrak{g},\Sigma}$ is also $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.*

Proof. Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces of which $\mathfrak{T}_{\mathfrak{g},\Sigma}$ is assumed to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, and let $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \to \mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute surjective map. Since $\mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma}\big) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$, there exists $\big(\mathscr{U}_{\mathfrak{g},\lambda}, \mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\sigma,\zeta_\sigma} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ such that,

$$\Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{U}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma_\lambda}\big) \Bigg)$$
$$\bigvee \Bigg(\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathscr{V}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma_\lambda}\big) \Bigg),$$

where $\bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma_\lambda}\big) = \mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma}\big) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ so that, for every $\lambda \in \{\xi_\sigma, \zeta_\sigma\}$, either $\mathscr{U}_{\mathfrak{g},\lambda} = \mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma_\lambda}\big)$ or $\mathscr{V}_{\mathfrak{g},\lambda} = \mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma_\lambda}\big)$. Since the relation $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ holds, there exists $(\mathscr{O}_{\mathfrak{g},\eta}, \mathscr{K}_{\mathfrak{g},\eta}) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$, $\eta \in \{\xi_\omega, \zeta_\omega\}$ with $(\lambda, \eta) \in \{(\xi_\sigma, \xi_\omega), (\zeta_\sigma, \zeta_\omega)\}$, such that,

$$\big[\pi_{\mathfrak{g}}^{-1}\big(\mathscr{U}_{\mathfrak{g},\lambda}\big) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta})\big] \vee \big[\pi_{\mathfrak{g}}^{-1}\big(\mathscr{V}_{\mathfrak{g},\lambda}\big) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta})\big].$$

Evidently, $\mathrm{dom}\big(\pi_{\mathfrak{g}|\Omega}\big) = \mathrm{im}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma}\big)$ and $\mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma}\big) = \mathrm{im}\big(\pi_{\mathfrak{g}|\Omega}\big)$. Since $\mathrm{im}\big(\pi_{\mathfrak{g}|\Omega}\big)$, $\mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma}\big) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$, set $\mathrm{im}\big(\pi_{\mathfrak{g}|\Omega}\big) = \bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \mathrm{im}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big)$ and $\mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma}\big) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma_\lambda}\big)$, and for any $(\lambda, \eta) \in \{(\xi_\sigma, \xi_\omega), (\zeta_\sigma, \zeta_\omega)\}$ set

$$\big[\mathrm{im}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big) = \mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma_\lambda}\big) = \mathscr{U}_{\mathfrak{g},\lambda}\big]$$
$$\vee \big[\mathrm{im}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big) = \mathrm{dom}\big({\pi_{\mathfrak{g}}^{-1}}_{|\Sigma_\lambda}\big) = \mathscr{V}_{\mathfrak{g},\lambda}\big].$$

Since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, for every $(\lambda,\eta) \in \{(\xi_\sigma,\xi_\omega),(\zeta_\sigma,\zeta_\omega)\}$,

$$\left(\bigcap_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \pi_{\mathfrak{g}}^{-1}\big(\mathscr{U}_{\mathfrak{g},\lambda}\big) = \emptyset\right)$$
$$\bigvee\left(\bigcap_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \pi_{\mathfrak{g}}^{-1}\big(\mathscr{V}_{\mathfrak{g},\lambda}\big) = \emptyset\right),$$

Thus, $\operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big) = \bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big)$. Since the relation $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \cap \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ holds, it follows that

$$\bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big) = \bigsqcup_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big) \subseteq \bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\eta}\right),$$
$$\bigcap_{\eta=\xi_\omega,\zeta_\omega} \operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega_\eta}\big) = \bigcap_{\lambda=\xi_\sigma,\zeta_\sigma} \operatorname{im}\big(\pi_{\mathfrak{g}}^{-1}{}_{|\Sigma_\lambda}\big) \supseteq \bigcap_{\eta=\xi_\omega,\zeta_\omega} \neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\eta}\right).$$

Thus, $\bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \operatorname{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\eta}\right) = \operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big)$ and $\bigcap_{\eta=\xi_\omega,\zeta_\omega} \neg\operatorname{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\eta}\right) = \emptyset$. There exists, then, $\big(\mathscr{U}_{\mathfrak{g},\lambda},\mathscr{V}_{\mathfrak{g},\lambda}\big)_{\lambda=\xi_\omega,\zeta_\omega} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big] \times \mathfrak{g}\text{-K}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ such that,

$$\left(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \mathscr{U}_{\mathfrak{g},\eta} = \operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big)\right) \bigvee \left(\bigsqcup_{\eta=\xi_\omega,\zeta_\omega} \mathscr{V}_{\mathfrak{g},\eta} = \operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big)\right).$$

Thus, $\operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big) \in \mathfrak{g}\text{-D}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, or equivalently $\operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big) \notin \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ which contradicts the assumption that $\operatorname{dom}\big(\pi_{\mathfrak{g}|\Omega}\big) \in \mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. Hence, $\mathfrak{T}_{\mathfrak{g},\Sigma}$ must be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. □

Pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness is also preserved under a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map, as proved below.

Theorem 3.44. *Let $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map and let $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set. If $\operatorname{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$, then $\operatorname{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is also pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$.*

Proof. Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, $\mathscr{S}_{\mathfrak{g},\omega} = \operatorname{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$, and suppose ξ_σ, $\zeta_\sigma \in \operatorname{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$. Then, there exist ξ_ω, $\zeta_\omega \in \mathfrak{T}_{\mathfrak{g},\Omega}$ such that $\big(\pi_{\mathfrak{g}}(\xi_\omega),\pi_{\mathfrak{g}}(\zeta_\omega)\big) = (\xi_\sigma,\zeta_\sigma)$. But $\operatorname{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Omega}$ and, therefore, there exists a path $\varphi_{\mathfrak{g},\zeta} : [0,1] \rightarrow \mathfrak{T}_{\mathfrak{g},\Omega}$ such that $\varphi_{\mathfrak{g},\zeta}(0) = \xi_\omega$, $\varphi_{\mathfrak{g},\zeta}(1) = \zeta_\omega$, and $\operatorname{im}\big(\varphi_{\mathfrak{g},\zeta|[0,1]}\big) \subseteq \operatorname{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$. Since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and $\varphi_{\mathfrak{g},\zeta} \in \text{C}\big[[0,1];\mathfrak{T}_{\mathfrak{g},\Omega}\big]$, it follows that $\pi_{\mathfrak{g}} \circ \varphi_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-C}\big[[0,1];\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$. Moreover, $\pi_{\mathfrak{g}}\circ\varphi_{\mathfrak{g},\zeta}(0) = \pi_{\mathfrak{g}}(\xi_\omega) = \xi_\sigma$, $\pi_{\mathfrak{g}}\circ\varphi_{\mathfrak{g},\zeta}(1) = \pi_{\mathfrak{g}}(\zeta_\omega) = \zeta_\sigma$, and $\operatorname{im}\big(\pi_{\mathfrak{g}}\circ\varphi_{\mathfrak{g},\zeta}\big) \subseteq \operatorname{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$. Hence, $\operatorname{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$ is pathwise $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. □

In the discussion section, categorical classifications of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-disconnectedness are presented. Thereafter, a nice application is given and, finally, the work is terminated with a concluding remarks section.

4. Discussion

4.1. **Categorical Classifications.** Having adopted a categorical approach in the classifications of the $\mathscr{T}_{\mathfrak{g}}$-properties in $\mathscr{T}_{\mathfrak{g}}$-spaces, called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-disconnectedness, the dual aims of the present section are, to establish the various relations amongst the elements of the sequence $\big\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}\big)\big\rangle_{\nu\in\ I_3^0}$ of $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces and the elements of the sequence $\big\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}^{\langle \mathrm{C}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}^{\langle \mathrm{C}\rangle}\big)\big\rangle_{\nu\in\ I_3^0}$ of $\mathfrak{g}$-$\mathscr{T}^{\langle \mathrm{C}\rangle}$-spaces, and to illustrate them through diagrams.

If a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-separated, then $\mathfrak{T}_{\mathfrak{g}}$ has a nonempty proper $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, where $\nu \in I_3^0$. But, for every $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relation $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds; for every $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relation given by $\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds. Consequently,

$$\mathrm{op}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{op}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}};$$
$$\neg\,\mathrm{op}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \subseteq \neg\,\mathrm{op}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \ \forall \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}.$$

Therefore, if $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is a nonempty proper $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set then,

$$\begin{array}{ccc}
\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}0\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}0\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] & \Longrightarrow & \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}1\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}1\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \\
 & & \Downarrow \\
\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}2\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}2\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] & \Longrightarrow & \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}3\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \cap \mathfrak{g}\text{-}3\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big].
\end{array}$$

In other words, $\mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}$-separation implies $\mathfrak{g}$-1-$\mathfrak{T}_{\mathfrak{g}}$-separation and the latter in turn implies $\mathfrak{g}$-0-$\mathfrak{T}_{\mathfrak{g}}$-separation. On the other hand, $\mathfrak{g}$-2-$\mathfrak{T}_{\mathfrak{g}}$-separation is implied by $\mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}$-separation. Similar implications also hold for $\mathfrak{g}$-$\mathfrak{T}$-separateness in a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$. For, if $\mathscr{S} \subset \mathfrak{T}$ is a nonempty proper $\mathfrak{g}$-$\mathfrak{T}$-open-closed set then,

$$\begin{array}{ccc}
\mathscr{S} \in \mathfrak{g}\text{-}0\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cap \mathfrak{g}\text{-}0\text{-}\mathrm{K}\big[\mathfrak{T}\big] & \Longrightarrow & \mathscr{S} \in \mathfrak{g}\text{-}1\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cap \mathfrak{g}\text{-}1\text{-}\mathrm{K}\big[\mathfrak{T}\big] \\
 & & \Downarrow \\
\mathscr{S} \in \mathfrak{g}\text{-}2\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cap \mathfrak{g}\text{-}2\text{-}\mathrm{K}\big[\mathfrak{T}\big] & \Longrightarrow & \mathscr{S} \in \mathfrak{g}\text{-}3\text{-}\mathrm{O}\big[\mathfrak{T}\big] \cap \mathfrak{g}\text{-}3\text{-}\mathrm{K}\big[\mathfrak{T}\big].
\end{array}$$

As above, $\mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}$-separation implies $\mathfrak{g}$-1-$\mathfrak{T}_{\mathfrak{g}}$-separation and the latter in turn implies $\mathfrak{g}$-0-$\mathfrak{T}_{\mathfrak{g}}$-separation. On the other hand, $\mathfrak{g}$-2-$\mathfrak{T}_{\mathfrak{g}}$-separation is implied by $\mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}$-separation.

For visualization, a so-called *categorical connectedness diagram*, expressing the various relations amongst $\mathfrak{g}$-$\mathfrak{T}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected spaces, is presented in Fig. **??** and that, expressing the various relationships amongst $\mathfrak{g}$-$\mathfrak{T}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected spaces, so-called *categorical disconnectedness diagram*, is presented in Fig. **??**. The categorical classifications of $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle}\big)$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle}\big)$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LPC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LPC}\rangle}\big)$, and $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{SC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{SC}\rangle}\big)$ called, respectively, *locally*, *pathwise*, *locally pathwise*, and *simply* $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces can be diagrammed in an analogous manner. The following implications concern the transformations of $\mathfrak{g}$-$\mathfrak{T}$-connected sets under some types of $\mathfrak{g}$-$\mathfrak{T}$-maps.

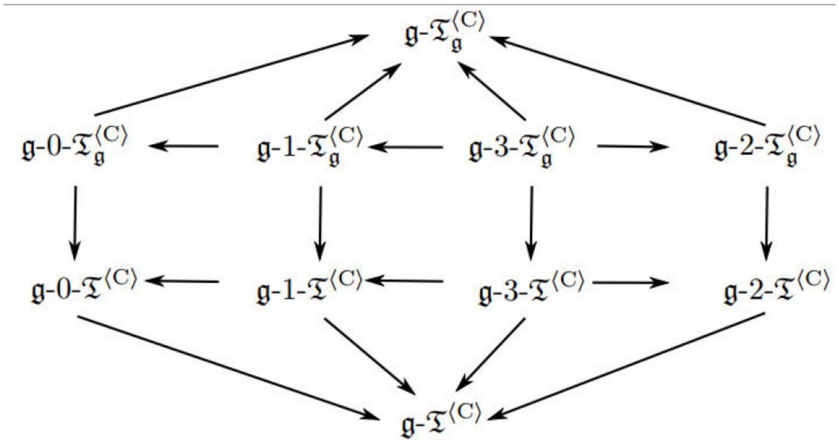

FIGURE 1. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected spaces.

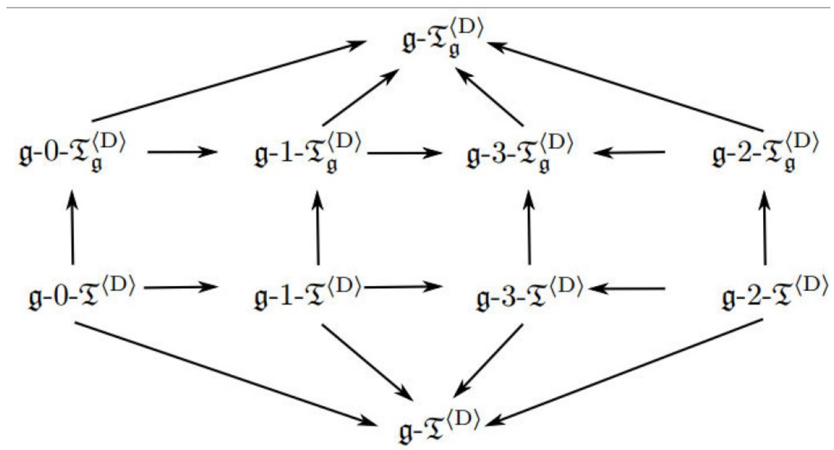

FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-separated and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated spaces.

For every $\nu \in I_3^0$, if $\pi_{\mathfrak{g}} \in \mathfrak{g}$-$\nu$-$\mathrm{I}\,[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}$-$\nu$-$\mathrm{S}\,[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ holds, then $\mathrm{dom}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}) \in \mathfrak{g}$-$\nu$-$\mathrm{Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ implies $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}) \in \mathfrak{g}$-$\nu$-$\mathrm{Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$, and hence the following implication:

$$\begin{gathered}\big(\mathrm{dom}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]\big) \wedge \big(\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{I}\,[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\,[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]\big) \\ \Downarrow \\ \mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]. \end{gathered} \tag{4.1}$$

For every $\nu \in I_3^0$, if $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ holds, then $\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ implies $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$, and hence the following implication:

$$\big(\mathrm{dom}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]\big) \wedge \big(\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\big)$$
$$\Downarrow$$
$$\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g}}}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]. \tag{4.2}$$

In the following section a nice application comprising of some interesting cases is discussed.

4.2. **A Nice Application.** Focusing on basic concepts from the point of view of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness, we shall now present a nice application comprising of some interesting cases. Let $\Omega_\sigma = \{\xi_\nu : \ \nu \in I_\sigma^*\}$ denote the underlying set, conditioned by the parameter $\sigma \in I_\infty^*$, and consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\sigma} = (\Omega_\sigma, \mathscr{T}_{\mathfrak{g},\sigma})$, where $\mathscr{T}_{\mathfrak{g},\sigma} : \mathscr{P}(\Omega_\sigma) \longrightarrow \mathscr{P}(\Omega_\sigma)$ will be defined in the following cases.

– CASE I. Set $\sigma = 1$. Then, $\Omega_1 = \{\xi_1\}$, $\mathscr{T}_{\mathfrak{g},1} = \{\emptyset, \Omega_1\} = \{\mathscr{O}_{\mathfrak{g},1}, \mathscr{O}_{\mathfrak{g},2}\}$, $\neg\mathscr{T}_{\mathfrak{g},1} = \{\Omega_1, \emptyset\} = \{\mathscr{K}_{\mathfrak{g},1}, \mathscr{K}_{\mathfrak{g},2}\}$, and, for every $(\mu,\nu) \in I_2^* \times I_3^0$ it results that $\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g},\mu})$, $\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g},\mu}) \in \{\mathscr{O}_{\mathfrak{g},1}, \mathscr{K}_{\mathfrak{g},1}, \mathscr{O}_{\mathfrak{g},2}, \mathscr{K}_{\mathfrak{g},2}\} = \{\emptyset, \Omega_1\}$. Therefore, for every $\nu \in I_3^0$, $\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},1}\big] = \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},1}\big] = \{\emptyset, \Omega_1\}$. Thus, for every $\nu \in I_3^0$, there exists neither a pair $(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{U}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},1}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},1}\big]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets nor a pair $(\mathscr{V}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},1}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},1}\big]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that:

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big).$$

Evidently, the $\mathfrak{T}_{\mathfrak{g}}$-sets $\emptyset$, $\Omega_1 \subseteq \mathfrak{U}$ are the only $\mathfrak{T}_{\mathfrak{g}}$-open-closed sets, and $\mathfrak{g}\text{-}\mathrm{C}_{\Omega_1}[\xi_1] = \{\xi_1\} = \Omega_1$ is the unique $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-component in $\mathfrak{T}_{\mathfrak{g},1}$. Thus, the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},1} = (\Omega_1, \mathscr{T}_{\mathfrak{g},1})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},1}^{\langle \mathrm{C}\rangle} = \big(\Omega_1, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},1}^{\langle \mathrm{C}\rangle}\big)$, and the latter in turn implies that it is also a $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{T}_{\mathfrak{g},1}^{\langle \mathrm{C}\rangle} = \big(\Omega_1, \mathscr{T}_{\mathfrak{g},1}^{\langle \mathrm{C}\rangle}\big)$. Hence, *every indiscrete $\mathscr{T}_{\mathfrak{g}}$-space which is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected is also $\mathfrak{T}_{\mathfrak{g}}$-connected.* Furthermore, the underlying set $\Omega_1 = \{\xi_1\}$ being a 1-point set, it also follows that, *every discrete $\mathscr{T}_{\mathfrak{g}}$-space that has at most one point is both $\mathfrak{T}_{\mathfrak{g}}$-connected and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected.*

– CASE II. Set $\sigma = 2$. Then, $\Omega_2 = \{\xi_1, \xi_2\}$. Choose $\mathscr{T}_{\mathfrak{g},2} = \{\emptyset, \Omega_2\} = \{\mathscr{O}_{\mathfrak{g},1}, \mathscr{O}_{\mathfrak{g},2}\}$ so that, $\neg\mathscr{T}_{\mathfrak{g},2} = \{\Omega_2, \emptyset\} = \{\mathscr{K}_{\mathfrak{g},1}, \mathscr{K}_{\mathfrak{g},2}\}$. Then, the collection of $\mathfrak{T}_{\mathfrak{g}}$-open sets is $\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] = \{\emptyset, \Omega_2\}$, and $\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big] = \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$ stands for the collection of $\mathfrak{T}_{\mathfrak{g}}$-closed sets. On the other hand, for every $\nu \in I_3^0$, $\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] = \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \cup \big\{\{\xi_1\}, \{\xi_2\}\big\} = \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \cup \big\{\{\xi_1\}, \{\xi_2\}\big\} = \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$. Clearly, there exists a pair $(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{U}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets or a pair $(\mathscr{V}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$ of nonempty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that:

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{U}_{\mathfrak{g},\lambda} = \Omega\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{V}_{\mathfrak{g},\lambda} = \Omega\Big).$$

This description is realised when either $(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{U}_{\mathfrak{g},\zeta}) = \big(\{\xi_1\}, \{\xi_2\}\big)$ or $(\mathscr{V}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}) = \big(\{\xi_2\}, \{\xi_1\}\big)$. On the other hand, there exists neither a pair $(\mathscr{U}_{\mathfrak{g},\xi}, \mathscr{U}_{\mathfrak{g},\zeta}) \in \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \times \mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$ of nonempty $\mathfrak{T}_{\mathfrak{g}}$-open sets nor a pair $(\mathscr{V}_{\mathfrak{g},\xi}, \mathscr{V}_{\mathfrak{g},\zeta}) \in \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big] \times \mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},2}\big]$

of nonempty $\mathfrak{T}_{\mathfrak{g}}$-closed sets such that the above statement holds. Thus, the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},2} = (\Omega_2, \mathscr{T}_{\mathfrak{g},2})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},2}^{\langle \mathrm{D}\rangle} = \big(\Omega_2, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},2}^{\langle \mathrm{D}\rangle}\big)$ but not a $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$-space $\mathfrak{T}_{\mathfrak{g},2}^{\langle \mathrm{D}\rangle} = \big(\Omega_2, \mathscr{T}_{\mathfrak{g},2}^{\langle \mathrm{D}\rangle}\big)$. Alternatively said, *every* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$*-space is a* $\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$*-space but the converse need not be true in general.* Moreover, the underlying set $\Omega_2 = \{\xi_1, \xi_2\}$ being a 2-point set, it follows that *every discrete* $\mathscr{T}_{\mathfrak{g}}$*-space that has at least two points is* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-separated.* It is plain that every $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \big\{\big(\{\xi_1\}, \{\xi_1, \xi_2\}\big), \big(\{\xi_2\}, \{\xi_1, \xi_2\}\big)\big\}$ is a pair of nonempty $\mathfrak{T}_{\mathfrak{g}}$-sets which are not $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separated, for, $\{\xi_1\} = \{\xi_1\} \cap \{\xi_1, \xi_2\} = \complement(\{\xi_2\}) = \complement(\{\xi_2\} \cap \{\xi_1, \xi_2\}) \neq \emptyset$, and $\mathscr{S}_{\mathfrak{g}} = \{\xi_1, \xi_2\}$ is the only $\mathfrak{T}_{\mathfrak{g}}$-set satisfying $\mathscr{S}_{\mathfrak{g}} = \{\xi_1\} \sqcup \{\xi_2\}$. Accordingly, $\mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g},2}] = \big\{\{\xi_1\}, \{\xi_2\}\big\}$ and $\mathfrak{g}\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{g},2}] = \big\{\{\xi_1, \xi_2\}\big\}$. Observe in passing that, $\Omega_2 = \bigsqcup_{\zeta=\xi_1,\xi_2} \mathfrak{g}\text{-}\mathrm{C}_{\Omega_2}[\zeta]$. Thus, *if a* $\mathscr{T}_{\mathfrak{g}}$*-space has more than one* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-component, then it is a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{D}\rangle}$*-space.*

– Case iii. Set $\sigma > 2$. Then, $\Omega_{>2} = \{\xi_\alpha : \alpha \in I_{\sigma>2}^{*}\}$. Let $\mathscr{T}_{\mathfrak{g},>2} : \mathscr{P}(\Omega_{>2}) \longrightarrow \mathscr{P}(\Omega_{>2})$ generate the elements of $\mathscr{T}_{\mathfrak{g},>2} = \big\{\mathscr{O}_{\mathfrak{g},(\alpha,\beta)} : (\alpha,\beta) \in I_\infty^0 \times I_\infty^0\big\}$ and $\neg\mathscr{T}_{\mathfrak{g},>2} = \big\{\mathscr{K}_{\mathfrak{g},(\alpha,\beta)} = \complement\big(\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\big) : (\alpha,\beta) \in I_\infty^0 \times I_\infty^0\big\}$ as thus:

$$\mathscr{O}_{\mathfrak{g},(\alpha,\beta)} = \begin{cases} \emptyset \quad \forall(\alpha,\beta) \in \{0\} \times \{0\}; \\ \big\{\xi_{\alpha+\mu} : \mu \in I_\beta^0\big\} \quad \forall(\alpha,\beta) \in I_\infty^{*} \times I_\infty^0; \\ \Omega_{>2} \quad \forall(\alpha,\beta) \in \{1\} \times \{\infty\}. \end{cases} \tag{4.3}$$

Clearly, $\Omega_{>2} \subseteq \mathfrak{U}$ is an ∞-point set. Furthermore, it is easily verified that, $\mathscr{T}_{\mathfrak{g},>2}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g},>2}\big(\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\big) \subseteq \mathscr{O}_{\mathfrak{g},(\alpha,\beta)}$ for every $(\alpha,\beta) \in I_\infty^0 \times I_\infty^0$, and, finally, $\mathscr{T}_{\mathfrak{g},>2}\big(\bigcup_{(\alpha,\beta)\in I_\infty^0 \times I_\infty^0} \mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\big) = \bigcup_{(\alpha,\beta)\in I_\infty^0 \times I_\infty^0} \mathscr{T}_{\mathfrak{g},>2}\big(\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\big)$. Hence, it follows that $\mathscr{T}_{\mathfrak{g},>2} : \mathscr{P}(\Omega_{>2}) \longrightarrow \mathscr{P}(\Omega_{>2})$ is a $\mathfrak{g}$-topology on the ∞-point set $\Omega_{>2}$. On the other hand, it can be shown that, for every $(\alpha,\beta,\nu) \in I_\infty^{*} \times I_\infty^0 \times I_3^0$,

$$\mathscr{O}_{\mathfrak{g},(\alpha,0)} \cap \mathscr{O}_{\mathfrak{g},(\alpha,\beta)} \subseteq \mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{O}_{\mathfrak{g},(\alpha,0)}\big) \cap \mathrm{op}_{\mathfrak{g},\nu}\big(\mathscr{O}_{\mathfrak{g},(\alpha,\beta)}\big) = \{\xi_\alpha\} \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{g},>2}].$$

This implies that the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},>2} = (\Omega_{>2}, \mathscr{T}_{\mathfrak{g},>2})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},>2}^{\langle \mathrm{LC}\rangle} = \big(\Omega_{>2}, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},>2}^{\langle \mathrm{LC}\rangle}\big)$, and hence, it is also a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},>2}^{\langle \mathrm{C}\rangle} = \big(\Omega_{>2}, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},>2}^{\langle \mathrm{C}\rangle}\big)$.

Moreover, $\mathscr{T}_{\mathfrak{g}}$-properties relative to such $\mathscr{T}_{\mathfrak{g}}$-spaces $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LC}\rangle}\big)$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{PC}\rangle}\big)$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{LPC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{LPC}\rangle}\big)$, and, also, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{\langle \mathrm{SC}\rangle} = \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{SC}\rangle}\big)$ called, respectively, *locally*, *pathwise*, *locally pathwise*, and *simply* $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{\langle \mathrm{C}\rangle}$-spaces can be discussed in an analogous manner by slight modifications of some $\mathscr{T}_{\mathfrak{g}}$-properties found in those cases. The next section provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness discussed in the preceding sections.

5. Conclusion

In this paper, a new theory called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Connectedness* has been developed, the foundation of which was based on the theories of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-maps. A careful perusal of the Mathematical developments of the earlier sections will show that the proposed theory has, in its own rights, several advantages. The very first advantage is that the theory holds equally well when $(\Lambda, \mathscr{T}_{\mathfrak{g},\Lambda}) = (\Lambda, \mathscr{T}_\Lambda)$, where $\Lambda \in \big\{\Omega, \Sigma\big\}$, and other characteristics adapted on this basis, in which case it might be called *Theory of* $\mathfrak{g}$-$\mathfrak{T}$*-Connectedness.*

Hence, in a $\mathscr{T}_{\mathfrak{g}}$-space the theoretical framework categorises such pairs of concepts as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected closed, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected semi-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected semi-closed, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected preopen and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected preclosed, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected semi-preopen and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected semi-preclosed as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way; in a $\mathscr{T}$-space it categorises such pairs of concepts as $\mathfrak{g}$-$\mathfrak{T}$-connected open and $\mathfrak{g}$-$\mathfrak{T}$-connected closed, $\mathfrak{g}$-$\mathfrak{T}$-connected semi-open and $\mathfrak{g}$-$\mathfrak{T}$-connected semi-closed, $\mathfrak{g}$-$\mathfrak{T}$-connected preopen and $\mathfrak{g}$-$\mathfrak{T}$-connected preclosed, and $\mathfrak{g}$-$\mathfrak{T}$-connected semi-preopen and $\mathfrak{g}$-$\mathfrak{T}$-connected semi-preclosed as $\mathfrak{g}$-$\mathfrak{T}$-connected of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way.

It is an interesting topic for future research to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness of mixed categories. More precisely, for some pair $(\nu,\mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness with respect to the elements of the classes $\{(\mathscr{U}_{\mathfrak{g},\nu},\mathscr{U}_{\mathfrak{g},\mu}) : (\mathscr{U}_{\mathfrak{g},\nu},\mathscr{U}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\}$ and $\{(\mathscr{V}_{\mathfrak{g},\nu},\mathscr{V}_{\mathfrak{g},\mu}) : (\mathscr{V}_{\mathfrak{g},\nu},\mathscr{V}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, as [31] and [32] developed the theory of b-open and b-closed sets in a $\mathscr{T}$-space $\mathfrak{T}$. Such a theory is what we thought would certainly be worth considering, and the discussion of this paper terminates here.

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