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APPROXIMATING THE RIEMANN–STIELTJES INTEGRAL BY A THREE–POINT QUADRATURE RULE AND APPLICATIONS

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Abstract. In this paper, a three–point quadrature rule for the Riemann–Stieltjes integral is introduced. As application; an error estimate for the obtained quadrature rule is provided as well.

1. Introduction

The Riemann–Stieltjes integral $\int_a^b f(t)\,dg(t)$ is an important concept in Mathematics with multiple applications in several subfields including Probability Theory & Statistics, Complex Analysis, Functional Analysis, Operator Theory and others.

In 2008, Mercer [27] has introduced new midpoint and trapezoid type rules for the Riemann–Stieltjes integral which engender a natural generalization of Hadamard's integral inequality, as follows:

Theorem 1.1. *Let g be continuous and increasing on $[a,b]$, let $c \in [a,b]$ which satisfies*

$$\int_a^b g(t)\,dt = (c-a)\,g(a) + (b-c)\,g(b).$$

If $f'' \geq 0$, then we have

$$f(c)\,[g(b)-g(a)] \leq \int_a^b f dg \leq [G-g(a)]\,f(a) + [g(b)-G]\,f(b) \tag{1.1}$$

where, $G := \frac{1}{b-a}\int_a^b g(t)\,dt$.

In fact, Mercer established the following quadrature rule for the Riemann–Stieltjes integral.

$$\int_a^b f dg \cong [G-g(a)]\,f(a) + [g(b)-G]\,f(b), \tag{1.2}$$

and so that, he obtained the error as follows:

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Theorem 1.2. *Suppose that f'' and g' are continuous on $[a,b]$ and that g is monotonic there. Let $G := \frac{1}{b-a}\int_a^b g(t)\,dt$. Then there exist $\eta, \sigma \in (a,b)$ such that*

$$\int_a^b f dg - [G - g(a)] f(a) - [g(b) - G] f(b) = -f''(\eta) g'(\sigma) \frac{(b-a)^3}{12}. \tag{1.3}$$

Recently, Alomari and Dragomir [7], proved several new error bounds for the Mercer–Trapezoid quadrature rule (1.2) for the Riemann-Stieltjes integral under various assumptions for the integrand (and integrator) involved.

After that, and motivated by the method used in [27], Alomari and Dragomir [8] introduced the following quadrature formula:

Theorem 1.3. *Suppose that f'' and g' are continuous on $[a,b]$ and that g is monotonic on $[a,x]$ and $[x,b]$. Then there exist $\xi_1, \eta_1 \in (a,x)$ and $\xi_2, \eta_2 \in (x,b)$ such that*

$$\begin{aligned}\int_a^b f(t) g'(t)\,dt &= [G(a,x) - g(a)] f(a) + [G(x,b) - G(a,x)] f(x) \\ &\quad + [g(b) - G(x,b)] f(b) \\ &\quad - \frac{1}{12}\left[f''(\xi_1) g'(\eta_1)(x-a)^3 + f''(\xi_2) g'(\eta_2)(b-x)^3\right],\end{aligned} \tag{1.4}$$

for all $a < x < b$, where $G(\alpha,\beta) := \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\,dt$.

For other quadrature rules for Riemann–Stieltjes integral under various assumptions to the function involved the reader may refer to [1]–[6], [9]–[26] and [28].

In this work, we study the quadrature rule

$$\begin{aligned}\int_a^b f(t)\,dg(t) \cong [G(a,x) - g(a)] f(a) + [G(x,b) - G(a,x)] f(x)& \\ + [g(b) - G(x,b)] f(b)&\end{aligned}$$

for all $x \in (a,b)$, by relaxing the conditions in Theorem 1.3. Various error estimates for the above quadrature rule are proved. As application an error estimate for the new three–point quadrature rule for Riemann–Stieltjes integral is given.

2. The case when f is of bounded variation

Theorem 2.1. *Fix $x \in (a,b)$. Let $f, g : [a,b] \to \mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and g is continuous. If g is increasing on the both intervals $[a,x]$ and $[x,b]$, then*

$$|\mathcal{R}(f,g;x)| \le \left[\frac{g(b) - g(a)}{2} + \left|g(x) - \frac{g(a) + g(b)}{2}\right|\right] \cdot \bigvee_a^b (f) \tag{2.1}$$

for all $a < x < b$, where $G(\alpha,\beta) := \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\,dt$.

Proof. It is easy to observe that

$$\mathcal{R}(f,g;x) = \int_a^x [g(t) - G(a,x)]\,df(t) + \int_x^b [g(t) - G(x,b)]\,df(t). \tag{2.2}$$

Using the fact that for a continuous function $p : [c,d] \to \mathbb{R}$ and a function $\nu : [c,d] \to \mathbb{R}$ of bounded variation, then the Riemann–Stieltjes integral $\int_c^d p(t)\,d\nu(t)$ exists and one has the inequality

$$\left|\int_c^d p(t)\,d\nu(t)\right| \le \sup_{t\in[c,d]} |p(t)| \bigvee_c^d (\nu)\,. \tag{2.3}$$

As f is of bounded variation on $[a,b]$, by (2.3) we have

$$\begin{aligned}
|\mathcal{R}(f,g;x)| &\le \left|\int_a^x [g(t)-G(a,x)]\,df(t)\right| + \left|\int_x^b [g(t)-G(x,b)]\,df(t)\right| \\
&\le \sup_{t\in[a,x]} |g(t)-G(a,x)| \cdot \bigvee_a^x (f) + \sup_{t\in(x,b]} |g(t)-G(x,b)| \cdot \bigvee_x^b (f)\,, \qquad (2.4)
\end{aligned}$$

but since g is increasing on $[a,x]$ and $[x,b]$, then

$$\begin{aligned}
\sup_{t\in[a,x]} |g(t)-G(a,x)| &= \max\{g(x)-G(a,x)\,, G(a,x)-g(a)\} \\
&= \frac{1}{2}\left[g(x)-g(a)+|g(x)-2G(a,x)+g(a)|\right], \qquad (2.5)
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\in(x,b]} |g(t)-G(x,b)| &= \max\{g(b)-G(x,b)\,, G(x,b)-g(x)\} \\
&= \frac{1}{2}\left[g(b)-g(x)+|g(b)-2G(x,b)+g(x)|\right]. \qquad (2.6)
\end{aligned}$$

Also, since

$$g(a) \le G(a,x) \le g(x)\,,$$

and

$$g(x) \le G(x,b) \le g(b)\,,$$

so that from (2.5) and (2.6), we have

$$\sup_{t\in[a,x]} |g(t)-G(a,x)| \le g(x)-g(a)$$

and

$$\sup_{t\in(x,b]} |g(t)-G(x,b)| \le g(b)-g(x)$$

which gives by (2.4) that

$$\begin{aligned}
|\mathcal{R}(f,g;x)| &\le \sup_{t\in[a,x]} |g(t)-G(a,x)| \cdot \bigvee_a^x (f) + \sup_{t\in(x,b]} |g(t)-G(x,b)| \cdot \bigvee_x^b (f) \\
&\le [g(x)-g(a)] \cdot \bigvee_a^x (f) + [g(b)-g(x)] \cdot \bigvee_x^b (f) \\
&\le \left[\frac{g(b)-g(a)}{2} + \left|g(x)-\frac{g(a)+g(b)}{2}\right|\right] \cdot \bigvee_a^b (f)\,,
\end{aligned}$$

and thus the theorem is proved. □

Corollary 2.1. *In Theorem 2.1, choose $g(t) = t$, $t \in [a,b]$, then we have the inequality:*

$$\left|\frac{1}{2}\left[f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right]-\int_a^b f(t)\,dt\right| \leq \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\bigvee_a^b(f), \tag{2.7}$$

for all $a < x < b$. Moreover, if we choose $x = \frac{a+b}{2}$, then we get

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\int_a^b f(t)\,dt\right| \leq \frac{1}{2}(b-a)\bigvee_a^b(f). \tag{2.8}$$

Theorem 2.2. *Fix $x \in (a,b)$. Let $f, g : [a,b] \to \mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and g is continuous.*

(1) *If g is of bounded variation on $[a,b]$, then*

$$|\mathcal{R}(f,g;x)| \leq \left[\bigvee_a^b(g)+\left|\bigvee_a^x(g)-\bigvee_x^b(g)\right|\right]\cdot\bigvee_a^b(f). \tag{2.9}$$

(2) *If g is of L_g–Lipschitzian on $[a,b]$, then*

$$|\mathcal{R}(f,g;x)| \leq \frac{L_g}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\bigvee_a^b(f). \tag{2.10}$$

for all $a < x < b$.

Proof. (1) Since f is of bounded variation on $[a,b]$, then by (2.4) we have

$$|\mathcal{R}(f,g;x)| \leq \sup_{t\in[a,x]}|g(t)-G(a,x)|\cdot\bigvee_a^x(f)+\sup_{t\in(x,b]}|g(t)-G(x,b)|\cdot\bigvee_x^b(f). \tag{2.11}$$

In [17], the author proved the following Ostrowski type inequality for functions of bounded variation

$$|g(t)-G(a,x)| = \left|g(t)-\frac{1}{x-a}\int_a^x g(s)\,ds\right| \leq \left[\frac{1}{2}+\left|\frac{t-\frac{a+x}{2}}{x-a}\right|\right]\bigvee_a^x(g),$$

it follows that,

$$\sup_{t\in[a,x]}|g(t)-G(a,x)| \leq \sup_{t\in[a,x]}\left[\frac{1}{2}+\left|\frac{t-\frac{a+x}{2}}{x-a}\right|\right]\bigvee_a^x(g) = \bigvee_a^x(g).$$

Similarly, one may observe that

$$\sup_{t\in[x,b]}|g(t)-G(x,b)| \leq \sup_{t\in[x,b]}\left[\frac{1}{2}+\left|\frac{t-\frac{x+b}{2}}{b-x}\right|\right]\bigvee_x^b(g) = \bigvee_x^b(g).$$

Combining the above two inequalities with (2.11), we get

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\le \bigvee_a^x (g)\cdot\bigvee_a^x (f)+\bigvee_x^b (g)\cdot\bigvee_x^b (f)\\ &\le \left[\bigvee_a^b (g)+\left|\bigvee_a^x (g)-\bigvee_x^b (g)\right|\right]\cdot\bigvee_a^b (f)\end{aligned}$$

which proves (2.9).

(2) In [25], the author proved the following Ostrowski type inequality for Lipschitzian functions

$$\begin{aligned}|g(t)-G(a,x)| &= \left|g(t)-\frac{1}{x-a}\int_a^x g(s)\,ds\right|\\ &\le L_g\left[\frac{1}{4}+\left(\frac{t-\frac{a+x}{2}}{x-a}\right)^2\right](x-a),\end{aligned}$$

it follows that,

$$\begin{aligned}\sup_{t\in[a,x]}|g(t)-G(a,x)| &\le L_g\sup_{t\in[a,x]}\left|g(t)-\frac{1}{x-a}\int_a^x g(s)\,ds\right|\\ &\le L_g\sup_{t\in[a,x]}\left[\frac{1}{4}+\left(\frac{t-\frac{a+x}{2}}{x-a}\right)^2\right](x-a)=\frac{1}{2}L_g(x-a).\end{aligned}$$

Similarly, one may observe that

$$\begin{aligned}\sup_{t\in[x,b]}|g(t)-G(x,b)| &\le L_g\sup_{t\in[x,b]}\left[\frac{1}{4}+\left(\frac{t-\frac{x+b}{2}}{b-x}\right)^2\right](b-x)\\ &=\frac{1}{2}L_g(b-x).\end{aligned}$$

Combining the above two inequalities with (2.11), we get

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\le \frac{1}{2}L_g(x-a)\cdot\bigvee_a^x (f)+\frac{1}{2}L_g(b-x)\cdot\bigvee_x^b (f)\\ &\le \frac{L_g}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\bigvee_a^b (f).\end{aligned}$$

which proves (2.10).

Thus the theorem is completely proved. □

3. The case when f is of Lipschitz type

Theorem 3.1. *Fix $x\in(a,b)$. Let $f,g:[a,b]\to\mathbb{R}$ be such that f is L_f–Lipschitzian on $[a,b]$and g is a Riemann integrable on $[a,b]$. If there exists positive constants γ,Γ,ϕ,Φ such that*

$$\gamma\le g(t)\le\Gamma,\quad \forall t\in[a,x],$$

and

$$\phi\le g(t)\le\Phi,\quad \forall t\in(x,b],$$

for some $x \in (a,b)$. Then,

$$|\mathcal{R}(f,g;x)| \leq \frac{1}{2} L \left[(x-a)(\Gamma-\gamma)+(b-x)(\Phi-\phi)\right] \tag{3.1}$$

for all $a < x < b$.

Proof. Since

$$\mathcal{R}(f,g;x) = \int_a^x \left[g(t) - G(a,x)\right] df(t) + \int_x^b \left[g(t) - G(x,b)\right] df(t),$$

using the fact that for a Riemann integrable function $p : [c,d] \to \mathbb{R}$ and L-Lipschitzian function $\nu : [c,d] \to \mathbb{R}$, the inequality one has the inequality

$$\left|\int_c^d p(t)\, d\nu(t)\right| \leq L \int_c^d |p(t)|\, dt. \tag{3.2}$$

As f is L_f–Lipschitzian on $[a,b]$, by (3.2) we have

$$\begin{aligned} |\mathcal{R}(f,g;x)| &\leq \left|\int_a^x \left[g(t) - G(a,x)\right] df(t)\right| + \left|\int_x^b \left[g(t) - G(x,b)\right] df(t)\right| \\ &\leq L_f \left[\int_a^x |g(t) - G(a,x)|\, dt + \int_x^b |g(t) - G(x,b)|\, dt\right]. \end{aligned} \tag{3.3}$$

Now, using the same techniques applied in [26], we define

$$I_1(g) := \frac{1}{x-a} \int_a^x \left(g(t) - \frac{1}{x-a}\int_a^x g(s)\, ds\right)^2 dt.$$

Then, we have

$$\begin{aligned} I_1(g) : &= \frac{1}{x-a} \int_a^x \left[g^2(t) - 2g(t) \frac{1}{x-a}\int_a^x g(s)\, ds + \left(\frac{1}{x-a}\int_a^x g(s)\, ds\right)^2\right] dt \\ &= \frac{1}{x-a} \int_a^x g^2(t)\, dt - \left(\frac{1}{x-a}\int_a^x g(s)\, ds\right)^2 \end{aligned}$$

and

$$\begin{aligned} I_1(g) := &\left(\Gamma - \frac{1}{x-a}\int_a^x g(s)\, ds\right)\left(\frac{1}{x-a}\int_a^x g(s)\, ds - \gamma\right) \\ &- \frac{1}{x-a}\int_a^x (\Gamma - g(t))(g(t) - \gamma)\, dt. \end{aligned}$$

As $\gamma \leq g(t) \leq \Gamma$, for all $t \in [a,b]$, then

$$\int_a^x (\Gamma - g(t))(g(t) - \gamma)\, dt \geq 0,$$

which implies

$$\begin{aligned} I_1(g) &\leq \left(\Gamma - \frac{1}{x-a}\int_a^x g(s)\, ds\right)\left(\frac{1}{x-a}\int_a^x g(s)\, ds - \gamma\right) \\ &\leq \frac{1}{4}\left[\left(\Gamma - \frac{1}{x-a}\int_a^x g(s)\, ds\right) + \left(\frac{1}{x-a}\int_a^x g(s)\, ds - \gamma\right)\right]^2 \\ &= \frac{1}{4}(\Gamma - \gamma)^2 \end{aligned} \tag{3.4}$$

Using Cauchy–Buniakowski–Schwarz's integral inequality we have

$$I_1(g) \geq \left[\frac{1}{x-a}\int_a^x \left|g(t) - \frac{1}{x-a}\int_a^x g(s)\,ds\right| dt\right]^2$$

and thus by (3.4) we get

$$\int_a^x \left|g(t) - \frac{1}{x-a}\int_a^x g(s)\,ds\right| dt \leq \frac{1}{2}(\Gamma-\gamma)(x-a). \tag{3.5}$$

Similarly, define

$$I_2(g) := \frac{1}{b-x}\int_x^b \left(g(t) - \frac{1}{b-x}\int_x^b g(s)\,ds\right)^2 dt,$$

then one can observe that

$$\int_x^b \left|g(t) - \frac{1}{b-x}\int_x^b g(s)\,ds\right| dt \leq \frac{1}{2}(\Phi-\phi)(b-x). \tag{3.6}$$

Therefore, from (3.3) we have

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\leq L_f\left[\int_a^x |g(t)-G(a,x)|\,dt + \int_x^b |g(t)-G(x,b)|\,dt\right]\\ &\leq \frac{1}{2}L_f\left[(x-a)(\Gamma-\gamma)+(b-x)(\Phi-\phi)\right],\end{aligned}$$

which gives the inequality (3.1). □

Remark 3.1. In Theorem 3.1, if $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a,b]$, then we have

$$|\mathcal{R}(f,g;x)| \leq \frac{1}{2}L_f(b-a)(\Gamma-\gamma). \tag{3.7}$$

for all $x \in (a,b)$.

Corollary 3.1. *In Theorem 3.1, choose $g(t) = t$, $t \in [a,b]$, then we have the inequality:*

$$\left|\frac{1}{2}\left[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] - \int_a^b f(t)\,dt\right| \leq \frac{1}{2}L_f(b-a)^2 \tag{3.8}$$

for all $x \in (a,b)$. Moreover, if we choose $x = \frac{a+b}{2}$, then we get

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \int_a^b f(t)\,dt\right| \leq \frac{1}{2}L_f(b-a)^2. \tag{3.9}$$

Theorem 3.2. *Let $f,g : [a,b] \to \mathbb{R}$ be such that f is L_f–Lipschitzian on $[a,b]$ and g is of r-H_g–Hölder type on $[a,b]$, where $r \in (0,1]$ and $H_g > 0$ are given. Then,*

$$|\mathcal{R}(f,g;x)| \leq \frac{2L_fH_g}{(r+1)(r+2)}\left[(x-a)^{r+1} + (b-x)^{r+1}\right], \tag{3.10}$$

for all $x \in (a,b)$.

Proof. Since f is L_f–Lipschitzian on $[a,b]$, then (3.3) holds; that is,

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\leq \left|\int_a^x [g(t)-G(a,x)]\,df(t)\right| + \left|\int_x^b [g(t)-G(x,b)]\,df(t)\right| \\ &\leq L_f\left[\int_a^x |g(t)-G(a,x)|\,dt + \int_x^b |g(t)-G(x,b)|\,dt\right].\end{aligned}$$

Also, since g is of r-H_g–Hölder type on $[a,b]$, then we have

$$\begin{aligned}|g(t)-G(a,x)| &= \left|g(t)-\frac{1}{x-a}\int_a^x g(s)\,ds\right| \\ &\leq \frac{1}{x-a}\int_a^x |g(t)-g(s)|\,ds \\ &\leq \frac{H_g}{x-a}\int_a^x |t-s|^r\,ds = \frac{H_g}{x-a}\cdot\frac{(t-a)^{r+1}+(x-t)^{r+1}}{r+1},\end{aligned}$$

and

$$\begin{aligned}|g(t)-G(x,b)| &= \left|g(t)-\frac{1}{b-x}\int_x^b g(s)\,ds\right| \\ &\leq \frac{1}{b-x}\int_x^b |g(t)-g(s)|\,ds \\ &\leq \frac{H_g}{b-x}\int_x^b |t-s|^r\,ds = \frac{H_g}{b-x}\cdot\frac{(t-x)^{r+1}+(b-t)^{r+1}}{r+1},\end{aligned}$$

which gives by (3.3), we have

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\leq \frac{L_fH_g}{x-a}\cdot\int_a^x \frac{(t-a)^{r+1}+(x-t)^{r+1}}{r+1}dt \\ &\quad + \frac{L_fH_g}{b-x}\cdot\int_x^b \frac{(t-x)^{r+1}+(b-t)^{r+1}}{r+1}dt \\ &= \frac{2L_fH_g}{(r+1)(r+2)}\left[(x-a)^{r+1}+(b-x)^{r+1}\right],\end{aligned}$$

and thus the proof is completed. □

Corollary 3.2. *In Theorem 3.2, if g is L_g–Lipschitzian on $[a,b]$, then we have*

$$|\mathcal{R}(f,g;x)| \leq \frac{1}{3}L_fL_g\left[(x-a)^2+(b-x)^2\right], \tag{3.11}$$

for all $x\in(a,b)$. Moreover, if we choose $x=\frac{a+b}{2}$, then

$$\left|\mathcal{R}\left(f,g;\frac{a+b}{2}\right)\right| \leq \frac{1}{6}L_fL_g(b-a)^2. \tag{3.12}$$

Corollary 3.3. *In Theorem 3.2, choose $g(t) = t$, $t \in [a,b]$, then we have the inequality:*

$$\left|\frac{1}{2}\left[f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right]-\int_a^b f(t)\,dt\right| \leq \frac{2L_f}{(r+1)(r+2)}\left[(x-a)^{r+1}+(b-x)^{r+1}\right] \tag{3.13}$$

for all $a < x < b$. Moreover, if we choose $x = \frac{a+b}{2}$, then we get

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\int_a^b f(t)\,dt\right| \leq \frac{1}{6}L_f(b-a)^2. \tag{3.14}$$

Theorem 3.3. *Let $f,g : [a,b] \to \mathbb{R}$ be such that f is L_f–Lipschitzian on $[a,b]$ and g is of bounded variation on $[a,b]$. Then,*

$$|\mathcal{R}(f,g;x)| \leq \frac{3}{4}L_f\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(g), \tag{3.15}$$

for all $x \in (a,b)$.

Proof. Since f is L_f–Lipschitzian on $[a,b]$, then (3.3) holds; that is,

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\leq \left|\int_a^x [g(t)-G(a,x)]\,df(t)\right|+\left|\int_x^b [g(t)-G(x,b)]\,df(t)\right| \\ &\leq L_f\left[\int_a^x |g(t)-G(a,x)|\,dt+\int_x^b |g(t)-G(x,b)|\,dt\right].\end{aligned}$$

Using the Ostrowski integral inequality for the bounded variation function g we have

$$\begin{aligned}\int_a^x |g(t)-G(a,x)|\,dt &= \int_a^x \left|g(t)-\frac{1}{x-a}\int_a^x g(s)\,ds\right|dt \\ &\leq \int_a^x \left[\frac{1}{2}+\left|\frac{t-\frac{a+x}{2}}{x-a}\right|\right]dt\bigvee_a^x(g) \\ &\leq \frac{3}{4}(x-a)\bigvee_a^x(g),\end{aligned}$$

similarly, we observe

$$\int_x^b |g(t)-G(x,b)|\,dt \leq \frac{3}{4}(b-x)\bigvee_x^b(g),$$

which gives by (3.3), we have

$$\begin{aligned}|\mathcal{R}(f,g;x)| &\leq \frac{3}{4}L_f\left[(x-a)\bigvee_a^x(g)+(b-x)\bigvee_x^b(g)\right] \\ &\leq \frac{3}{4}L_f\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(g),\end{aligned}$$

for all $x \in (a,b)$, and thus the proof is completed. □

Remark 3.2. Let f be a monotonic nondecreasing in the theorems above. By applying the same techniques used in the corresponding proofs of each theorem, we may obtain several inequalities for monotonic non-decreasing integrator f using the fact that for a monotonic non-decreasing function $\nu : [a,b] \to \mathbb{R}$ and continuous function $p : [a,b] \to \mathbb{R}$, one has the inequality

$$\left| \int_a^b p(t)\, d\nu(t) \right| \le \int_a^b |p(t)|\, d\nu(t).$$

We leave the details to the interested reader.

4. Applications to A Three-point Quadrature rule

Consider $I_n : a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$, be a division of the interval $[a,b]$, $L_i := g(x_{i+1}) - g(x_i)$, $(i = 0, 1, \ldots n-1)$ and $\nu(L) := \max\{L_i | i = 0, 1, \ldots n-1\}$. Consider the following Three–point quadrature rule as

$$S(f,g,I_n,\xi) = \sum_{i=0}^{n} [G(x_i,\xi_i) - g(x_i)]\, f(x_i) + [G(\xi_i, x_{i+1}) - G(x_i,\xi_i)]\, f(\xi_i) + [g(x_{i+1}) - G(\xi_i, x_{i+1})]\, f(x_{i+1}) \tag{4.1}$$

for all $\xi_i \in (x_i, x_{i+1})$, where $G(\alpha,\beta) := \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\, dt$.

In the following, we establish an upper bound for the error approximation of the Riemann–Stieltjes integral $\int_a^b f(t)\, dg(t)$ by its Riemann sum $S(f,g,I_n,\xi)$. As a sample we consider (2.1).

Theorem 4.1. *Under the assumptions of Theorem 2.1, we have*

$$\int_a^b f(t)\, dg(t) = S(f,g,I_n,\xi) + R(f,g,I_n,\xi)$$

where, $S(f,g,I_n,\xi)$ is given in (4.1) and the remainder $R(f,g,I_n,\xi)$ satisfies the bound

$$\begin{aligned} |R(f,g,I_n,\xi)| &\le \left[\frac{1}{2}\nu(L) + \max_{0,n-1} \left| g(\xi_i) - \frac{g(x_i) + g(x_{i+1})}{2} \right| \right] \cdot \bigvee_a^b (f) \\ &\le \nu(L) \cdot \bigvee_a^b (f). \end{aligned} \tag{4.2}$$

Proof. Fix $\xi_i \in (x_i, x_{i+1})$. Applying Theorem 2.1 on the intervals $[x_i, x_{i+1}]$, we may state that

$$\begin{aligned} &\Big| [G(x_i,\xi_i) - g(x_i)]\, f(x_i) + [G(\xi_i,x_{i+1}) - G(x_i,\xi_i)]\, f(\xi_i) \\ &\qquad + [g(x_{i+1}) - G(\xi_i,x_{i+1})]\, f(x_{i+1}) - \int_{x_i}^{x_{i+1}} f(t)\, dg(t) \Big| \\ &\qquad\qquad \le \left[\frac{g(x_{i+1}) - g(x_i)}{2} + \left| g(\xi_i) - \frac{g(x_i) + g(x_{i+1})}{2} \right| \right] \cdot \bigvee_{x_i}^{x_{i+1}} (f), \end{aligned}$$

for all $i \in \{0,1,2,\cdots,n-1\}$. Summing the above inequality over i from 0 to $n-1$, we deduce

$$
\begin{aligned}
&|R(f,g,I_n,\xi)| \\
&= \sum_{i=0}^{n-1} \Big\{ \Big| [G(x_i,\xi_i) - g(x_i)] f(x_i) + [G(\xi_i,x_{i+1}) - G(x_i,\xi_i)] f(\xi_i) \\
&\qquad + [g(x_{i+1}) - G(\xi_i,x_{i+1})] f(x_{i+1}) - \int_{x_i}^{x_{i+1}} f(t)\,dg(t) \Big| \Big\} \\
&\le \sum_{i=0}^{n-1} \left[\frac{g(x_{i+1}) - g(x_i)}{2} + \left| g(\xi_i) - \frac{g(x_i)+g(x_{i+1})}{2} \right| \right] \cdot \bigvee_{x_i}^{x_{i+1}} (f) \\
&\le \max_{0,n-1} \left[\frac{g(x_{i+1}) - g(x_i)}{2} + \left| g(\xi_i) - \frac{g(x_i)+g(x_{i+1})}{2} \right| \right] \cdot \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) \\
&\le [g(b) - g(a)] \cdot \bigvee_a^b (f),
\end{aligned}
$$

since

$$
\begin{aligned}
&\max_{0,n-1} \left[\frac{g(x_{i+1}) - g(x_i)}{2} + \left| g(\xi_i) - \frac{g(x_i)+g(x_{i+1})}{2} \right| \right] \\
&\qquad\qquad \le \frac{1}{2}\nu(L) + \max_{0,n-1} \left| g(\xi_i) - \frac{g(x_i)+g(x_{i+1})}{2} \right|
\end{aligned}
$$

and

$$
\sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) = \bigvee_a^b (f).
$$

For the second inequality, we observe that

$$
\max_{0,n-1} \left| g(\xi_i) - \frac{g(x_i)+g(x_{i+1})}{2} \right| \le \frac{1}{2} \max_{0,n-1} L_i = \frac{1}{2}\nu(L)
$$

which completes the proof. □

Remark 4.1. Several error estimations for the quadrature $S(f,g,I_n,\xi)$ (4.1) by using the results in section 2, we shall omit the details.

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