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Time Fractional Diffusion Equation with Periodic Boundary Conditions

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Abstract

The aim of this research is to establish the analytic solution of time fractional diffusion equations with periodic boundary conditions in one dimension by implementing well-known separation of variables method. First, the eigenvalues of the obtained Sturm-Liouville problem are determined by investigating all cases. The corresponding eigenfunctions are obtained in the second step. Utilizing eigenvalues and eigenfunctions, the Fourier series of the solution is constructed in terms of Mittag-Leffler function and the coefficients are computed by taking L^2 inner product and initial condition into account at the final step.

Keywords: *Caputo fractional derivative; Mittag-Leffler function; Periodic boundary conditions; Spectral method; Time-fractional diffusion equation*
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1. Introduction

Mathematical modeling plays undeniably powerful role for the quantitative and qualitative analysis of systems which describes the physical and scientific processes. Moreover mathematical models with fractional differential equations allow us to figure out the features of quantitative and qualitative behavior of complicated systems with memory and hereditary properties in diverse areas of science and engineering much more better since fractional derivatives are non-local operators. The suitable type of the fractional derivative in mathematical modelling is chosen by analyzing the experimental data of the phenomena with memory. Therefore recently modelling with fractional differential equations gain interest from many scientist in various range of fields such as mathematics, biology, engineering and so on. Because of advantages of Caputo fractional derivative such that the derivative of constant function is zero unlike the other fractional derivatives and the initial conditions can be taken in classical sense, it is one of the widely used in various branches of sciences. Consequently fractional order mathematical models in Caputo sense are one of the most preferable to do research on the behaviour of the processes with memory and hereditary properties. Since finding the analytic solution of fractional differential equations is not possible for many times, solving them numerically gains considerable attention. In literature increasing number of studies about theory and diverse applications of fractional differential equations, can be found supporting this conclusion [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. The solutions of fractional PDEs and ordinary differential equations (ODEs) are determined in terms of Mittag-Leffler function.

2. Preliminary Results

In this section, we recall fundamental definitions and well known results about fractional derivative in Caputo sense.

Definition 2.1. *The q^{th} order fractional derivative of $u(t)$ in Caputo sense is defined as*

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} u^{(n)}(s) ds, t \in [t_0, t_0+T],$$

where $u^{(n)}(t) = \frac{d^n u}{dt^n}, n-1 < q < n$. Note that Caputo fractional derivative is equal to integer order derivative when the order of the derivative is integer.

Definition 2.2. *The q^{th} order Caputo fractional derivative for $0 < q < 1$ is defined in the following form:*

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds, \quad t \in [t_0, t_0+T].$$

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The Mittag–Leffler function with two-parameters which is taken into account in eigenvalue problem, is given by

$$E_{\alpha,\beta}\left(\lambda\left(t-t_0\right)^{\alpha}\right)=\sum_{k=0}^{\infty}\frac{\left(\lambda\left(t-t_0\right)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\beta\right)},\alpha,\beta>0,$$

including constant λ. Especially, for $t_0=0,\alpha=\beta=q$ we have

$$E_{q,q}\left(\lambda t^{q}\right)=\sum_{k=0}^{\infty}\frac{\left(\lambda t^{q}\right)^{k}}{\Gamma\left(qk+q\right)},q>0.$$

Mittag–Leffler function coincides with exponential function i.e., $E_{1,1}\left(\lambda t\right)=e^{\lambda t}$ for $q=1$. For details see [9], [10].
The main goal of this study is to establish the analytic solution of following time fractional differential equations with periodic boundary and initial condition.

$$D_t^{\alpha}u\left(x,t\right)=\gamma^{2}u_{xx}\left(x,t\right), \tag{2.1}$$

$$\begin{cases} u\left(-l,t\right)=u\left(l,t\right), \\ u_x\left(-l,t\right)=u_x\left(l,t\right), \end{cases} \tag{2.2}$$

$$u\left(x,0\right)=f\left(x\right), \tag{2.3}$$

where $0<\alpha<1,-l\leq x\leq l,0\leq t\leq T,\gamma\in\mathbb{R}$.

3. Main Results

By means of separation of variables method, The generalized solution of above problem is constructed in analytical form. Thus a solution of problem (2.1)-(2.3) have the following form:

$$u\left(x,t;\alpha\right)=X\left(x\right)T\left(t;\alpha\right), \tag{3.1}$$

where $-l\leq x\leq l,0\leq t\leq T$.
Plugging (3.1) into (2.1) and arranging it, we have

$$\frac{D_t^{\alpha}\left(T\left(t;\alpha\right)\right)}{T\left(t;\alpha\right)}=\gamma^{2}\frac{X''\left(x\right)}{X\left(x\right)}=-\lambda^{2}. \tag{3.2}$$

Equation (3.2) produce a fractional equation with respect to time and an ordinary differential equation with respect to space. The first ordinary differential equation is obtained by taking the equation on the right hand side of Eq. (3.2). Hence with boundary conditions (2.2), we have the following problem:

$$X''\left(x\right)+\lambda^{2}X\left(x\right)=0, \tag{3.3}$$

$$\begin{cases} X\left(-l\right)=X\left(l\right), \\ X'\left(-l\right)=X'\left(l\right). \end{cases} \tag{3.4}$$

The solution of eigenvalue problem (3.3)-(3.4) is accomplished by making use of the exponential function of the following form:

$$X\left(x\right)=e^{rx}.$$

Hence the characteristic equation is computed in the following form:

$$r^{2}+\lambda^{2}=0. \tag{3.5}$$

Case 1. If $\lambda=0$, the characteristic equation have two coincident roots $r_1=r_2$, leading to the general solution of the eigenvalue problem (3.3)-(3.4) having the following form:

$$X\left(x\right)=k_1x+k_2,$$

$$X'\left(x\right)=k_1.$$

The first boundary condition yields

$$X\left(-l\right)=-k_1l+k_2=k_1l+k_2=X\left(l\right)\Rightarrow k_1=0,$$

which leads to the following solution

$$X\left(x\right)=k_2.$$

Similarly second boundary condition leads to

$X'(-l) = 0 = X'(l).$

The representation of the solution is established as

$X_0(x) = k_2.$

Case 2. If $\lambda > 0$, the Eq. (3.5) have two distinct real roots $r_1,, r_2$ yielding the general solution of the problem (3.3)-(3.4) in the following form:

$X(x) = c_1 e^{r_1 x} + c_1 e^{r_2 x}.$

By making use of the first boundary condition, we have

$X(-l) = c_1 e^{r_1(-l)} + c_2 e^{r_2(-l)} = c_1 e^{r_1 l} + c_2 e^{r_2 l} = X(l).$

$$c_1 \left(e^{r_1(-l)} - c_1 e^{r_1 l}\right) + c_2 \left(e^{r_2(-l)} - e^{r_2 l}\right) = 0. \tag{3.6}$$

Since $\left(e^{r_1(-l)} - c_1 e^{r_1 l}\right)$ and $\left(e^{r_2(-l)} - e^{r_2 l}\right)$ are linearly independent the equation (3.6) is satisfied if and only if $c_1 = 0 = c_2$ which implies that $X(x) = 0$ which implies that there is not any solution for $\lambda > 0$.
Case 3. If $\lambda < 0$, the characteristic equation have two complex roots yielding the general solution of the problem (3.3)-(3.4) in the following form:

$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$

By making use of the first boundary condition we have

$X(-l) = c_1 \cos(\lambda l) - c_2 \sin(\lambda l) = c_1 \cos(\lambda l) + c_2 \sin(\lambda l) = X(l),$

which implies that

$2c_2 \sin(\lambda l) = 0 \Rightarrow c_2 = 0.$

Hence the solution becomes

$X(x) = c_1 \cos(\lambda x),$

$X'(x) = -c_1 \lambda \sin(\lambda x).$

Similarly last boundary condition leads to

$X'(-l) = c_1 \lambda \sin(\lambda l) = -c_1 \lambda \sin(\lambda l) = X'(l),$

$\Rightarrow 2c_1 \lambda \sin(\lambda l) = 0,$

which implies that

$\sin(\lambda l) = 0,$

which yields the following eigenvalues

$\lambda_n = \frac{w_n}{l}, \lambda_1 < \lambda_2 < \lambda_3 < \ldots,$

where $w_n = n\pi$ satisfy the equation $sin(w_n) = 0$. As a result the solution is obtained as follows:

$X_n(x) = c_1 \cos\left(w_n \left(\frac{x}{l}\right)\right), n = 1, 2, 3, \ldots$

The second equation in (3.2) for eigenvalue λ_n yields the ordinary differential equation below:

$\frac{D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} = -\gamma^2 \lambda_n^2,$

which yields the following solution

$T_n(t;\alpha) = E_{\alpha,1}\left(-\gamma^2 \lambda_n^2 t^{\alpha}\right) = E_{\alpha,1}\left(-\gamma^2 \frac{w_n^2}{l^2} t^{\alpha}\right), n = 0, 1, 2, 3, \ldots$

The solution for every eigenvalue λ_n is constructed as

$u_n(x,t;\alpha) = X_n(x) T_n(t;\alpha) = E_{\alpha,1}\left(-\gamma^2 \frac{w_n^2}{l^2} t^{\alpha}\right) \cos\left(w_n \left(\frac{x}{l}\right)\right), n = 0, 1, 2, 3, \ldots,$

which leads to the following general solution

$$u(x,t;\alpha) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(w_n\left(\frac{x}{l}\right)\right) E_{\alpha,1}\left(-\gamma^2 \frac{w_n^2}{l^2} t^\alpha\right).$$

Note that it satisfies boundary condition and fractional differential equation.
The coefficients of general solution are established by taking the following initial condition into account:

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(w_n\left(\frac{x}{l}\right)\right).$$

The coefficients A_n for $n = 0,1,2,3,\ldots$ determined by the help of inner product defined on $L^2[-l,l]$:

$$A_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx, \tag{3.7}$$

$$A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(w_n\left(\frac{x}{l}\right)\right). \tag{3.8}$$

4. Illustrative Example

In this section, we first consider the following initial periodic boundary value problem:

$$u_t(x,t) = u_{xx}(x,t),$$
$$\begin{cases} u(-1,t) = u(1,t), \\ u_x(-1t,t) = u_x(1,t), \end{cases}$$
$$u(x,0) = \cos(\pi x)$$

which has the solution in the following form:

$$u(x,t) = \cos(\pi x)e^{-\pi^2 t}, \tag{4.1}$$

where $-1 \le x \le 1, 0 \le t \le T$. Now let the following problem called fractional heat-like problem be taken into consideration:

$$D_t^\alpha u(x,t) = u_{xx}(x,t), \tag{4.2}$$

$$\begin{cases} u(-1,t) = u(1,t), \\ u_x(-1t,t) = u_x(1,t), \end{cases} \tag{4.3}$$

$$u(x,0) = \cos(\pi x), \tag{4.4}$$

where $0 < \alpha < 1, -1 \le x \le 1,\ 0 \le t \le T$.
The separation of the variables method yields the following equations:

$$\frac{D_t^\alpha(T(t;\alpha))}{T(t;\alpha)} = \frac{X''(x)}{X(x)} = -\lambda^2. \tag{4.5}$$

Equation (4.5) produce a fractional equation with respect to time and an ordinary differential equation with respect to space. The first ordinary differential equation is obtained by taking the equation on the right hand side of Eq. (4.5). Hence with boundary conditions (4.3), we have the following problem:

$$X''(x) + \lambda^2 X(x) = 0, \tag{4.6}$$

$$\begin{cases} X(-l) = X(l), \\ X'(-l) = X'(l). \end{cases} \tag{4.7}$$

The representation of the solution for the eigenvalue problem (4.6)-(4.7) is obtained as

$$X_n(x) = \cos(w_n x), n = 1,2,3,\ldots$$

The second eqution in (4.5) for every eigenvalue λ_n yields the following equation:

$$\frac{D_t^\alpha(T(t;\alpha))}{T(t;\alpha)} = -\lambda^2,$$

which has the following solution

$$T_n(t;\alpha) = E_{\alpha,1}\left(-w_n^2 t^\alpha\right), n = 0,1,2,3,\ldots$$

For each eigenvalue λ_n, we obtain the following solution:

$$u_n(x,t;\alpha) = E_{\alpha,1}\left(-w_n^2 t^\alpha\right)\cos(w_n x), n = 0,1,2,3,\ldots$$

and hence we have the following sum:

$$u(x,t;\alpha) = A_0 + \sum_{n=1}^{\infty} A_n \cos(w_n x)\, E_{\alpha,1}\left(-w_n^2 t^\alpha\right). \tag{4.8}$$

Note that the general solution (4.8) satisfy both boundary conditions (4.3) and the fractional equation (4.2). By making use of the inner product defined on $L^2[-l,l]$, we determine the coefficients A_n in such a way that the general solution (4.8) satisfies the initial condition (4.4). Plugging $t = 0$ in to the general solution (4.8) and making equal to the initial condition (4.4) we have

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(w_n x)\ .$$

The coefficients A_n for $n = 0,1,2,3,\ldots$ are determined by the help of the inner product as follows:

$$A_0 = \frac{1}{2}\int_{-1}^{1} \cos(\pi x)dx = \left(\frac{1}{2\pi}\sin(\pi x)\right)_{x=-1}^{x=1} = 0,$$

$$A_n = \int_{-1}^{1} \cos(\pi x)\cos(n\pi x)dx.$$

For $n \neq 1, A_n = 0$. $n = 1$ we get

$$A_1 = \int_{-1}^{1} \cos^2(\pi x)dx = \int_{-1}^{1}\left(\frac{1}{2} + \frac{\cos(2\pi x)}{2}\right)dx = \left(\frac{x}{2} + \frac{\sin(2\pi x)}{4\pi}\right)\Big|_{x=-1}^{x=1} = 1.$$

Thus

$$u(x,t;\alpha) = \cos(\pi x)E_{\alpha,1}\left(-w_1^2 t^\alpha\right). \tag{4.9}$$

It is important to note that plugging $\alpha = 1$ in to the solution (4.9) gives the solution (4.1) which confirm the accuracy of the method we apply.

5. Conclusion

In this research, the analytic solution of time fractional differential equation with periodic boundary conditions in one dimension is constructed. By making use of separation of variables the solution is formed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including fractional derivative in Caputo sense. In the future work, construction of solutions for space-time fractional differential equations with various boundary conditions will be investigated by the method implemented in this research and modifications of this method.

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