Commun.Fac.Sci.Univ.Ank.Series A1
Volume 59, Number 2, Pages 35–40 (2010)
ISSN 1303–5991

UNIT DUAL QUATERNIONS AND ARCS

HESNA KABADAYI AND YUSUF YAYLI

Abstract. In this paper we obtain sine and cosine rules for dual spherical triangle on the dual unit sphere $\widetilde{S}^2$ by representing great circle arcs by dual quatrenions.

1. Introduction

As it is well known that an arc of a unit circle (subtending an angle θ at the origin) can be represented by a complex number of unit norm $\cos\theta + i\sin\theta$.

Great circle arcs on a unit sphere represented by a unit quaternion and sine and cosine rules are obtained by J. P. Ward (see [1], pg. 98-102).

A similar correspondence is possible with dual quaternions and great circle arcs on the dual unit sphere $\widetilde{S}^2$.

The sine and cosine rules for dual and real spherical trigonometry have been well known for a long time. (see [2], [3], [4], [5]).

Here in this paper we obtain sine and cosine laws by means of this correspondence between great circle arcs on dual unit sphere and dual quaternions.

2. Dual Quaternions and Arcs

Consider a unit dual quaternion $q = \cos\varphi + \hat{q}\sin\varphi$. We may associate this dual quaternion by the great circle arcs which is obtained when the diametral plane with normal $\hat{q}$ intersects the unit dual sphere.

Let A, B and C be the unit dual vectors and $q = \langle A, B\rangle + A \wedge B$ and $p = \langle B, C\rangle + B \wedge C$ be dual unit quaternions.

The quaternion product of p and q is readily checked to be

Received by the editors June 18 2010, Accepted: Nov. 29, 2010.

2000 *Mathematics Subject Classification.* 51M10, 53B99.

Key words and phrases. sine rule, cosine rule, dual spherical triangle, dual quaternions,dual spherical trigonometry.

$$\begin{aligned} pq &= \langle A,B\rangle\langle B,C\rangle - \langle A\wedge B, B\wedge C\rangle + \langle B,C\rangle A\wedge B \\ &\quad + \langle A,B\rangle B\wedge C + (B\wedge C)\wedge(A\wedge B) \\ &= \langle A,C\rangle + A\wedge C. \end{aligned}$$

Considering that p and q are unit dual quaternions, they are screw operators at the same time. Hence $q(A)=B$, $p(B)=C$ implies that $pq(A)=C$. This means that the line d_1 which corresponds A transforms into the line d_2 which corresponds C.

We write (using $\sim$ to specify the geometrical correspondence)

$$\begin{aligned} arcAB \sim q &= \langle A,B\rangle + A\wedge B \\ &= \cos\varphi + \hat{q}\sin\varphi \\ &= \cos(\theta+\varepsilon\theta^*) + \sin(\theta+\varepsilon\theta^*)\frac{A\wedge B}{\|A\wedge B\|} \end{aligned}$$

$$\begin{aligned} arcBC \sim p &= \langle B,C\rangle + B\wedge C \\ &= \cos\phi + \hat{p}\sin\phi \\ &= \cos\phi + \sin\phi\frac{B\wedge C}{\|B\wedge C\|} \end{aligned}$$

$$arcAC \sim pq = \langle A,C\rangle + A\wedge C.$$

Hence we write

$$arcAB + arcBC = arcAC$$

or

$$arcq + arcp = arcpq.$$

Theorem 2.1. *Let A_1, A_2, ...A_n be unit dual vectors. Then*

$$arcA_1A_2 + arcA_2A_3 + ... + arcA_{n-1}A_n = arcA_1A_n.$$

Proof. Denoting q_kq_{k+1} by $q_{k(k+1)}$ we have

$$\begin{aligned} arcA_1A_2 &\sim q_{12} = \langle A_1,A_2\rangle + A_1\wedge A_2 \\ arcA_2A_3 &\sim q_{23} = \langle A_2,A_3\rangle + A_2\wedge A_3 \\ &\vdots \\ arcA_{n-1}A_n &\sim q_{(n-1)n} = \langle A_{n-1},A_n\rangle + A_{n-1}\wedge A_n \\ arcA_1A_n &\sim q_{1n} = \langle A_1,A_n\rangle + A_1\wedge A_n. \end{aligned}$$

Noting that $q_{12}(A_1)=A_2$ we have

$$\begin{aligned} \left(q_{(n-1)n}...q_{23}q_{12}\right)(A_1) &= \left(q_{(n-1)n}...q_{23}\right)(A_2) \\ &= q_{(n-1)n}(A_{n-1}) \\ &= A_n. \end{aligned}$$

Hence we get

$$\begin{aligned} q_{(n-1)n} \cdots q_{23} q_{12} &= \langle A_1, A_n \rangle + A_1 \wedge A_n \\ &= q_{1n.} \end{aligned}$$

Thus

$$arc\left(q_{(n-1)n} \cdots q_{23} q_{12}\right) = arcq_{1n.}$$

Therefore

$$arcA_1A_2 + arcA_2A_3 + ... + arcA_{n-1}A_n = arcA_1A_n$$

or

$$arcq_{(n-1)n} + ... + arcq_{23} + arcq_{12} = arcq_{1n}.$$

Note that when dual quaternions are taken as real quaternions this result reduces the case in [1]. □

3. The Sine and Cosine Laws for a Dual Spherical Triangle

We consider two different points A and B on the dual unit sphere given by dual unit vectors $\hat{x} = x + \varepsilon x^*$, $\hat{y} = y + \varepsilon y^*$ respectively. We introduce the set of all dual vectors given by $\hat{c_\lambda} = c_\lambda + \varepsilon c_\lambda^* = (1 - \hat{\lambda})\hat{x} + \hat{\lambda}\,\hat{y}$, where $\hat{\lambda} = \lambda + \varepsilon\lambda^*$ and $0 \leq \lambda \leq 1$. We put $\hat{c_\lambda} = \left|\hat{c_\lambda}\right| \hat{e_\lambda}$; then $\hat{e_\lambda}$ is a point C_λ on the dual unit sphere. The set of all points C_λ with $0 \leq \lambda \leq 1$ is called the dual great -circle- arc $arcAB$. We will say that C_λ runs along $arcAB$ from A to B if λ increases from 0 to 1. With the $arcAB$ we will always mean this arc in the sense from A to B.

Let A, B and C be three points on the dual unit sphere $\tilde{S}^2$ given by the linearly independent dual unit vectors $\hat{x} = x + \varepsilon x^*$, $\hat{y} = y + \varepsilon y^*$ and $\hat{z} = z + \varepsilon z^*$ respectively. We will always suppose that the notations is such that $\det(x, y, z) > 0$. These points together with the dual great -circle- arcs $arcAB$, $arcBC$, $arcCA$ form a dual spherical triangle ΔABC. (see [2])

Having defined a dual spherical triangle there is naturally defined six dual angles $a^o = a + \varepsilon a^*$, $b^o = b + \varepsilon b^*$, $c^o = c + \varepsilon c^*$ called arc angles and $A^o = u + \varepsilon u^*$, $B^o = v + \varepsilon v^*$, $C^o = w + \varepsilon w^*$ called vertex angles (see figure 3.1).

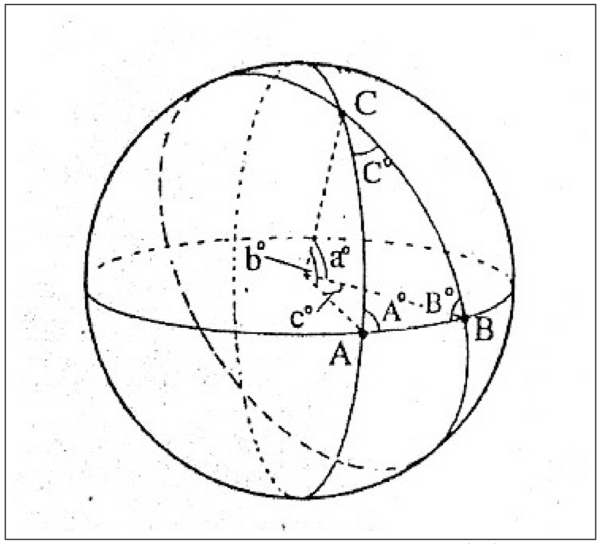

Figure 3.1 Dual spherical triangle ΔABC

We can represent arcs dual quaternionically. If $q = \cos a^o + \hat{q} \sin a^o$, $p = \cos c^o + \hat{p} \sin c^o$ then

$qp = \cos c^o \cos a^o - \left\langle \hat{q}, \hat{p} \right\rangle \sin c^o \sin a^o + \hat{q} \sin a^o \cos c^o + \hat{p} \cos a^o \sin c^o + \hat{q} \wedge \hat{p} \sin c^o \sin a^o.$

On the other hand $arcAB \sim p$, $arcBC \sim q$, $arcAC \sim qp$ and writing $arcAC \sim \cos b^o + \hat{m} \sin b^o$ we get, by equating scalar and vector parts:

$$\cos c^o \cos a^o - \left\langle \hat{q}, \hat{p} \right\rangle \sin c^o \sin a^o = \cos b^o \tag{3.1}$$

$$\hat{q} \sin a^o \cos c^o + \hat{p} \cos a^o \sin c^o + \hat{q} \wedge \hat{p} \sin c^o \sin a^o = \hat{m} \sin b^o. \tag{3.2}$$

Note that $\hat{p}$, $\hat{q}$, $\hat{m}$ are unit dual vectors in the direction of $A \wedge B$, $B \wedge C$ and $A \wedge C$ respectively i.e.

$$\hat{p} = \frac{A \wedge B}{\|A \wedge B\|}, \ \hat{q} = \frac{B \wedge C}{\|B \wedge C\|}, \ \hat{m} = \frac{A \wedge C}{\|A \wedge C\|}.$$

We define

$$\langle B, C \rangle = \cos a^o, \ B \wedge C = \hat{q} \sin a^o.$$

Similar definitions are given for dual angle b^o and c^o.

If $a^o = a + \varepsilon a^*$, we have consequently $\sin a > 0$. This implies (see [2]) $|\sin a^o| = \sin a^o$. Similarly $|\sin b^o| = \sin b^o$, $|\sin c^o| = \sin c^o$. It is moreover readily seen that A, B, C are dual unit vectors having the same sense as $\hat{p} \wedge \hat{m}$, $\hat{p} \wedge \hat{q}$ and $\hat{m} \wedge \hat{q}$ respectively. The angle A^o of ΔABC is defined as the dual angle given by

$$\left\langle \hat{p}, \hat{m} \right\rangle = -\cos A^o, \ \hat{p} \wedge \hat{m} = A \sin A^o.$$

Similar definitions for the angles B^o and C^o are given, i.e.

$$\left\langle \hat{p}, \hat{q} \right\rangle = -\cos B^o,\ \hat{p} \wedge \hat{q} = B \sin B^o \text{ and } \left\langle \hat{m}, \hat{q} \right\rangle = -\cos C^o,\ \hat{m} \wedge \hat{q} = C \sin C^o.$$

Now (3.1) implies the law of cosine in dual spherical trigonometry as follows:

Theorem 3.1. *Let ΔABC be a dual spherical triangle on the dual unit sphere $\widetilde{S}^2$. Then*

$$\cos c^o \cos a^o + \cos B^o \sin c^o \sin a^o = \cos b^o \tag{3.3}$$

and

$$\cos a^o = \cos b^o \cos c^o + \cos A^o \sin b^o \sin c^o \tag{3.4}$$

$$\cos c^o = \cos a^o \cos b^o + \cos C^o \sin a^o \sin b^o. \tag{3.5}$$

Corollary 1. *The real and dual parts of the formula* (3.3) , (3.4) , (3.5) *are given by*

$$\cos u = \frac{\cos a - \cos c \cos b}{\sin c \sin b},\ \ \sin u = \frac{-\sin a}{u^* \sin b \sin c}\left(b^* \cos w + c^* \cos v - a^*\right)$$

$$\cos v = \frac{\cos b - \cos c \cos a}{\sin c \sin a},\ \ \sin v = \frac{-\sin b}{v^* \sin a \sin c}\left(a^* \cos w + c^* \cos u - b^*\right)$$

$$\cos w = \frac{\cos c - \cos a \cos b}{\sin a \sin b},\ \ \sin w = \frac{-\sin c}{w^* \sin a \sin b}\left(b^* \cos u + a^* \cos v - c^*\right).$$

Since $\hat{q} \wedge \hat{p} = -B \sin B^o$ *and since* $\left\langle B, \hat{p} \right\rangle = 0$ *and* $\left\langle B, \hat{q} \right\rangle = 0$, *from (3.2) we get*

$$\sin B^o \sin c^o \sin a^o = -\left\langle B, \hat{m} \right\rangle \sin b^o.$$

Hence

$$\frac{\sin B^o}{\sin b^o} = \frac{-\left\langle B, \hat{m} \right\rangle}{\sin c^o \sin a^o} = \frac{-\langle B, A \wedge C \rangle}{\sin a^o \sin b^o \sin c^o} = \frac{\langle A, B \wedge C \rangle}{\sin a^o \sin b^o \sin c^o}.$$

Thus, since the right hand side is unchanged on cyclic interchange we obtain:

Theorem 3.2. *Let ΔABC be a dual spherical triangle on the dual unit sphere $\widetilde{S}^2$ then*

$$\frac{\sin A^o}{\sin a^o} = \frac{\sin B^o}{\sin b^o} = \frac{\sin C^o}{\sin c^o}. \tag{3.6}$$

Corollary 2. *The real and dual part of the Formula (3.6) is given by*

$$\frac{\sin u}{\sin a} = \frac{\sin v}{\sin b} = \frac{\sin w}{\sin c}$$

and

$$u^* \frac{\cos u}{\sin a} - a^* \cot a \frac{\sin u}{\sin a} = v^* \frac{\cos v}{\sin b} - b^* \cot b \frac{\sin v}{\sin b} = w^* \frac{\cos w}{\sin c} - c^* \cot c \frac{\sin w}{\sin c}.$$

Note also that sine law is obtained from (3.2) by taking the scalar product of both sides with B. One other possibility is taking the vector product of this equation with B implies

$$0 = B \wedge \hat{q} \sin a^o \cos c^o + B \wedge \hat{p} \cos a^o \sin c^o - B \wedge \hat{m} \sin b^o. \tag{3.7}$$

Note that

$$B \wedge \hat{q} = \frac{B \wedge (B \wedge C)}{\sin a^o} = \frac{[\langle B, C\rangle B - C]}{\sin a^o}$$

$$B \wedge \hat{p} = \frac{B \wedge (A \wedge B)}{\sin c^o} = \frac{[A - \langle B, A\rangle B]}{\sin c^o}$$

and

$$B \wedge \hat{m} = \frac{B \wedge (A \wedge C)}{\sin b^o} = \frac{[\langle B, C\rangle A - \langle B, A\rangle C]}{\sin b^o}.$$

Using $\langle B, A\rangle = \cos c^o$, $\langle B, C\rangle = \cos a^o$ *and* $\langle A, C\rangle = \cos b^o$ *implies the identity in* (3.7).

Remark 3.3. The results above coincide with the ones for real spherical triangles when the vectors are real.

ÖZET: Bu çalışmada, birim dual kuaterniyonlara dual yaylar karşılık getirilmiş, birim dual kuaterniyonları kullanarak dual küresel üçgenler için bilinen kosinüs ve sinüs bağıntılarını elde edilmiştir.

References

[1] J. P. Ward, Quaternions and Cayley Numbers, Kluwer Academic Publisher, 1997.
[2] G. R. Veldkamp, On the use of dual numbers, vectors and matrices in instantenaous, spatial kinematics, Mechanism and Machine Theory, 1976 vol. 11, pp. 141-156.
[3] A. F. Beardon, The Geometry of Discrete Groups, Springer-Verlag, New York, Berlin 1983.
[4] H. H. Uğurlu and H. Gündoğan, The cosine hyperbolic and sine hyperbolic rules for dual hyperbolic spherical trigonometry, Mathematical and Computational Applications, Vol. 5, No. 3, 185-190, 2000.
[5] M. Kazaz, H.H. Uğurlu, A. Özdemir, The cosine rule II for a spherical triangle on the dual unit sphere $\widetilde{S}^2$, Math. Comput. Appl. 10 (2005), no.3, 313-320.

Current address: Ankara University, Faculty of Sciences, Dept. of Mathematics, Ankara, TURKEY
E-mail address: kabadayi@science.ankara.edu.tr, yayli@science.ankara.edu.tr
URL: http://communications.science.ankara.edu.tr