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ON THE INTUITIONISTIC FUZZY PROJECTIVE MENELAUS AND CEVA'S CONDITIONS

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ABSTRACT. In this work, the intuitionistic fuzzy versions of Menelaus and Ceva's theorems in intuitionistic fuzzy projective plane are defined and the conditions to the intuitionistic fuzzy versions of Menelaus and Ceva 6-figures are determined.

1. INTRODUCTION

Ceva's and Menelaus theorems are two classic theorems in plane geometry. The main question of these theorems is to determine conditions under which three points are collinear and conditions under which three lines are concurrent. Ceva's theorem characterizes the concurrency of lines and Menelaus's theorem characterizes the collinearity of points. Kelly B. Funk [9] gave Menelaus and Ceva theorems in projective planes $\mathcal{P}_2(F)$ where F is the field of characteristic not equal to two. The definitions of the original Menelaus and Ceva $6-$figures are given in [3, 9].

After the introduction of Fuzzy set theory by Zadeh [15] several researches were conducted on generalizations of fuzzy theory.

A model of fuzzy projective geometries was introduced by Kuijken and Van Maldeghem [14]. This provided a link between the fuzzy versions of classical theories that are very closely related some basic results on fuzzy projective geometries are published in [1, 2, 5, 8]. Fiber geometry that is a particular kind of fuzzy geometries is introduced by Kuijken and Van Maldeghem. In these geometry, the points and lines of the base geometry mostly have multiple degrees of membership. The fibered version of Menelaus and Ceva's 6-figures was studied in [6].

Intuitionistic fuzzy set theory was firstly published by Atanassov [4]. A model of intuitionistic fuzzy projective geometry and the link between fibered and intuitionistic fuzzy projective geometry were given by Ghassan E. Arif [10].

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In the present paper, intuitionistic fuzzy projective Menelaus and Ceva's conditions in the intuitionistic fuzzy projective plane with base plane that is projective plane are given.

2. Preliminaries

We firstly recall the basic notions from the theory of projective geometry, fuzzy projective geometry and intuitionistic fuzzy projective geometry. We assume that the reader is familiar with the basic notions of fuzzy mathematics, although this is not strictly necessary as the paper is self-contained in this respect.

We denote by $\wedge$ and $\vee$, minimum and maximum operators respectively.

Definition 1. *Let $\mathcal{P} = (P, B, \sim)$ be any projective plane with point set P and line set B, i.e., P and B are two disjoint sets endowed with a symmetric relation $\sim$ (called the* incidence relation*) such that the graph $(P \cup B, \sim)$ is a bipartite graph with classes P and B, and such that two distinct points p, q in $\mathcal{P}$ are incident with exactly one line (denoted by $\langle pq \rangle$), every two distinct lines L, M are incident with exactly one point (denoted by $L \cap M$), and every line is incident with at least three points. A set S of* collinear *points is a subset of P each member of which is incident with a common line L. Dually, one defines a set of* concurrent *lines* [5].

Definition 2. (see [15])*A fuzzy set λ of a set X is a function $\lambda : X \to [0, 1]$.*

Definition 3. (see [4]) *Let X be a nonempty fixed set. An intuitionistic fuzzy set A on X is an object having the form*

$$A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$$

where the function $\lambda : X \to I$ and $\mu : X \to I$ denote the degree of membership (namely, $\lambda(x)$) and the degree of nonmembership (namely, $\mu(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \lambda(x) + \mu(x) \leq 1$ for each $x \in X$. An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ can be written in the $A = \{\langle x, \lambda, \mu\rangle : x \in X\}$, or simply $A = \langle \lambda, \mu \rangle$.

Definition 4. *(see [10]) An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ on n -dimensional projective space S is an intuitionistic fuzzy $n-$dimensional projective space on S if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$ and $\mu(p) \leq \mu(q) \vee \mu(r)$, for any three collinear points p, q, r of A we denoted $[A, S]$.*

The projective space S is called the base projective space of $[A, S]$ if $[A, S]$ is an intuitionistic fuzzy point, line, plane, ... ,we use base point, base line, base plane,... , respectively.

Definition 5. *(see [10]) Consider the projective plane $\mathcal{P} = (P, B, \sim)$. Suppose $a \in P$ and $\alpha, \alpha' \in [0, 1]$. The $\mathcal{IF}$-point (a, α, α') is the following intuitionistic fuzzy set on the point set P of $\mathcal{P}$:*

$$(a, \alpha, \alpha') : P \to [0, 1] : \begin{cases} a \to \alpha, \ a \to \alpha' & \\ x \to 0 & \text{if } x \in P\backslash\{a\}. \end{cases}$$

The point a is called the base point of the $\mathcal{IF}$-point (a,α,α'). An $\mathcal{IF}$-line (L,α,α') with base line L is defined in a similar way.

The $\mathcal{IF}$-lines (L,α,α') and (M,β,β') intersect in the unique $\mathcal{IF}$- point $(L\cap M,\alpha\wedge\beta,\alpha'\vee\beta')$. The $\mathcal{IF}$-points (a,α,α') and (b,β,β') span the unique $\mathcal{IF}$-line $(\langle a,b\rangle,\alpha\wedge\beta,\alpha'\vee\beta')$.

Definition 6. *(see [10]) Suppose $\mathcal{P}$ is a projective plane $\mathcal{P}=(P,B,\sim)$. The intuitionistic fuzzy set $Z=\langle\lambda,\mu\rangle$ on $P\cup B$ is an intuitionistic fuzzy projective plane on $\mathcal{P}$ denoted by $\mathcal{IFP}$ if*

1) $\lambda(L)\geq\lambda(p)\wedge\lambda(q)$ *and* $\mu(L)\leq\mu(p)\vee\mu(q)$, $\forall p,q$: $\langle p,q\rangle=L$
2) $\lambda(p)\geq\lambda(L)\wedge\lambda(M)$ *and* $\mu(p)\leq\mu(L)\vee\mu(M)$, $\forall L,M$: $L\cap M=p$.

Theorem 7. *(see [7]) Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$ that is Desarguesian. Choose three $\mathcal{IF}$-points (a_i,α_i,α_i'), $i\in\{1,2,3\}$ with noncollinear base points, and three other points (b_i,β_i,β_i'), $i\in\{1,2,3\}$ with noncollinear base points, such that the f-lines $(\langle a_i,b_i\rangle,\alpha_i\wedge\beta_i,\alpha_i'\vee\beta_i')$, for $i\in\{1,2,3\}$, meet in an IF-point (p,γ,η) of $\mathcal{IFP}$, with $a_i\neq b_i\neq p\neq a_i$. Then the three $\mathcal{IF}$-points $(c_{\{i,j\}},\gamma_{\{i,j\}},\gamma'_{\{i,j\}})$ obtained by intersecting $(\langle a_i,a_j\rangle,\alpha_i\wedge\alpha_j,\alpha_i'\vee\alpha_j')$ and $(\langle b_i,b_j\rangle,\beta_i\wedge\beta_j,\beta_i'\vee\beta_j')$, for $i\neq j$ and $i,j\in\{1,2,3\}$), are collinear.*

Theorem 8. *(see [7]) Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with Pappian base plane $\mathcal{P}$. Choose two different lines L_1 and L_2 in $\mathcal{P}$. Choose two triples of $\mathcal{IF}-$points (a_i,α_i,α_i') and (b_i,β_i,β_i') with a_i on L_1 and b_i on L_2, $i=1,2,3$ and such that no three of the base points a_1, a_2, b_1, b_2 are collinear. Then the three intersection $\mathcal{IF}-$points $(c_1,\gamma_1,\gamma_1')=(a_2b_3\cap a_3b_2,\alpha_2\wedge\alpha_3\wedge\beta_2\wedge\beta_3,\alpha_2'\vee\alpha_3'\vee\beta_2'\vee\beta_3')$, $(c_2,\gamma_2,\gamma_2')=(a_1b_3\cap a_3b_1,\alpha_1\wedge\alpha_3\wedge\beta_1\wedge\beta_3,\alpha_1'\vee\alpha_3'\vee\beta_1'\vee\beta_3')$ and $(c_3,\gamma_3,\gamma_3')=(a_1b_2\cap a_2b_1,\ \alpha_1\wedge\alpha_2\wedge\beta_1\wedge\beta_2,\ \alpha_1'\vee\alpha_2'\vee\beta_1'\vee\beta_2')$ are collinear.*

Definition 9. *(see [11]) Let $\mathcal{P}$ be a projective plane. A $6-$figure in $\mathcal{P}$ is a sequence of six distinct points $(a_1a_2a_3\ ,\ b_1b_2b_3)$ such that $a_1a_2a_3$ constitutes a non-degenerate triangle with $b_1\in\langle a_2,a_3\rangle$, $b_2\in\langle a_1,a_3\rangle$, $b_3\in\langle a_1,a_2\rangle$. The points a_1,a_2,a_3,b_1,b_2,b_3 are called vertices of this $6-$figures. Such a configuration is said to be a Menelaus $6-$figure or a Ceva $6-$figure if b_1,b_2 and b_3 are collinear or if $\langle a_1,b_1\rangle$, $\langle a_2,b_2\rangle$, $\langle a_3,b_3\rangle$ are concurrent, respectively.*

Definition 10. *(see [6]) Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three $f-$points (a_i,α_i), $i\in\{1,2,3\}$ in $\mathcal{FP}$ with non collinear base points and the other three $f-$points (b_k,β_k), $k\in\{1,2,3\}$ with $(b_k,\beta_k)\in(\langle a_i,a_j\rangle,\alpha_i\wedge\alpha_j)$ for $i\neq j\neq k$, $\{i,j,k\}=\{1,2,3\}$. If the $f-$points (b_k,β_k) are $f-$collinear, the configuration that consists of these six $f-$points is called an $f-$Menelaus $6-$figure. It is called $f-$Menelaus line spanned with $f-$points (b_k,β_k) for $k=\{1,2,3\}$.*

Theorem 11. *(see [6]) Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three $f-$points (a_i,α_i), $i\in\{1,2,3\}$ in $\mathcal{FP}$ with non collinear base points and the other three f-points (b_k,β_k), $k\in\{1,2,3\}$ with $(b_k,\beta_k)\in(\langle a_i,a_j\rangle,\alpha_i\wedge\alpha_j)$ for*

$i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. The configuration that consists of these six $f-$points is Menelaus $6-$figure if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.

Corollary 12. *(see [6]) Let $\mathcal{FP}$ be a fibered projective plane with base plane $\mathcal{P}$. Choose three $f-$points (a_i, α_i), $i \in \{1, 2, 3\}$ in $\mathcal{FP}$ with non collinear base points and the other three f-points (b_k, β_k), $k \in \{1, 2, 3\}$ with $(b_k, \beta_k) \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. The configuration that consists of these six $f-$points is Ceva $6-$figure iff $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$.*

We view Menalaus and Ceva's theorem in projective plane and extend them $\mathcal{IFP}$, intuitionistic fuzzy projective plane.

Theorem 13. *Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$. Let a_1, a_2, a_3 be three non-collinear points in $\mathcal{P}$ and be*

$$(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$$

be three points of $\mathcal{IFP}$. Suppose that the point b_3 on $\langle a_1, a_2 \rangle$ is obtained by intersecting $\langle a_1, a_2 \rangle$ with the join of two chosen points b_1 and b_2 where b_1 on $\langle a_2, a_3 \rangle$ and b_2 on $\langle a_1, a_3 \rangle$. Then the point (b_3, β_3, β_3') obtained by intersecting $(\langle a_1, a_2 \rangle, \alpha_1 \wedge \alpha_2, \alpha_1' \vee \alpha_2')$ with the join of the two points (b_1, β_1, β_1') and (b_2, β_2, β_2'), where $(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$ and $(b_2, \beta_2, \beta_2'), (a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$ are collinear, is independent of the chosen points (b_1, β_1, β_1') and (b_2, β_2, β_2').

Proof. In $\mathcal{IFP}$, since the three points $(b_2, \beta_2, \beta_2'), (a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$ and the three points

$$(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$$

are collinear,

$$\alpha_1 \wedge \alpha_3 = \alpha_1 \wedge \beta_2 = \alpha_3 \wedge \beta_2$$
$$\alpha_1' \vee \alpha_3' = \alpha_1' \vee \beta_2' = \alpha_3' \vee \beta_2'$$

and

$$\alpha_2 \wedge \alpha_3 = \alpha_2 \wedge \beta_1 = \alpha_3 \wedge \beta_1$$
$$\alpha_2' \vee \alpha_3' = \alpha_2' \vee \beta_1' = \alpha_3' \vee \beta_1'.$$

One can easily calculate that $\beta_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\beta_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3^{2\prime}$. It is seen that the point (b_3, β_3, β_3') is independent of the choice of the points (b_1, β_1, β_1') and (b_2, β_2, β_2'). □

Theorem 14. *Let an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$ be $\mathcal{IFP}$. Let three points in this plane no three base points of which are collinear be $(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$. If the point (b_3, β_3, β_3') be obtain by intersecting of the lines $(\langle a_1, a_2 \rangle, \alpha_1 \wedge \alpha_2, \alpha_1' \vee \alpha_2')$ and $(\langle b_1, b_2 \rangle, \beta_1 \wedge \beta_2, \beta_1' \vee \beta_2')$, where $(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$ and $(b_2, \beta_2, \beta_2'), (a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$ are collinear, then the configuration that consists of the six points*

$$(a_i, \alpha_i, \alpha_i'), (b_i, \beta_i, \beta_i'), \quad i \in \{1, 2, 3\}$$

is an intuitionistic fuzzy Menelaus $6-$figure.

Proof. Since the three points (b_i, β_i, β_i'), $i \in \{1,2,3\}$ are collinear, from Definition 9 the configuration that consists of the six points $(a_i, \alpha_i, \alpha_i'), (b_i, \beta_i, \beta_i')$, $i \in \{1,2,3\}$ is an intuitionistic fuzzy Menelaus 6−figure. □

Theorem 15. *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and with $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$. If b_1, b_2 and b_3 in $\mathcal{P}$ are collinear, then the three points (b_j, β_j, β_j'), $j \in \{1,2,3\}$ are collinear if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.*

Proof. A configuration is picked such that three points

$$(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$$

and

$$(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k') \text{ for } i \neq j \neq k, \quad \{i,j,k\} = \{1,2,3\}.$$

Suppose the three points (b_j, β_j, β_j'), $j \in \{1,2,3\}$ be collinear. Since three points (b_j, β_j, β_j') are collinear and the three points $(a_i, \alpha_i, \alpha_i')$, $(a_j, \alpha_j, \alpha_j')$ and (b_k, β_k, β_k'), for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$ are collinear $\beta_i \wedge \beta_j = \beta_i \wedge \beta_k$, $\beta_i' \vee \beta_j' = \beta_i' \vee \beta_k'$ and $\alpha_i \wedge \alpha_j = \alpha_i \wedge \beta_k = \alpha_j \wedge \beta_k$, $\alpha_i' \vee \alpha_j' = \alpha_i' \vee \beta_k' = \alpha_j' \vee \beta_k'$. Then it is seen that $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.

Conversely, if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^3$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$ are satisfied, $\beta_1 \wedge \beta_2 = \beta_1 \wedge \beta_3 = \beta_2 \wedge \beta_3$ and $\beta_1' \vee \beta_2' = \beta_1' \vee \beta_3' = \beta_2' \vee \beta_3'$. Then three points (b_i, β_i, β_i'), $i \in \{1,2,3\}$ are collinear. □

Corollary 16. ***The intuitionistic fuzzy projective Menelaus condition ($\mathcal{IFPMC}$):*** *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2')$ and $(a_3, \alpha_3, \alpha_3')$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$ with $(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$. The configuration that consists of these six points is Menelaus 6−figure if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.*

Definition 17. *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$ with*

$$(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j') \text{ for } i \neq j \neq k, \{i,j,k\} = \{1,2,3\}.$$

If the lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$, $i = 1,2,3$ are concurrent, the configuration that consists of these six points is called an intuitionistic fuzzy Ceva 6−figure. The intersection point of the lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$, $i = 1,2,3$ is called intuitionistic fuzzy Ceva point.

Theorem 18. *Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$. Let a_1, a_2, a_3 be three non-collinear points in $\mathcal{P}$ and be*

$$(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2') \text{ and } (a_3, \alpha_3, \alpha_3')$$

be three points of $\mathcal{IFP}$. Let points $b_1$and b_2 be chosen such that b_1 on $\langle a_2, a_3\rangle$ and b_2 on $\langle a_1, a_3\rangle$. Suppose that the point b_3 on $\langle a_1, a_2\rangle$ is obtained by intersecting $\langle a_1, a_2\rangle$ with the join $(\langle a_1, b_1\rangle \cap \langle a_2, b_2\rangle)$ and a_3. Then the point (b_3, β_3, β_3') obtained by intersecting $(\langle a_1, a_2\rangle, \alpha_1 \wedge \alpha_2, \alpha_1' \vee \alpha_2')$ with the join of the two points $(\langle a_1, b_1\rangle, \alpha_1 \wedge \beta_1, \alpha_1' \vee \beta_1') \cap (\langle a_2, b_2\rangle, \alpha_2 \wedge \beta_2, \alpha_2' \vee \beta_2')$ and $(a_3, \alpha_3, \alpha_3')$, where $(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$ and $(b_2, \beta_2, \beta_2'), (a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$ are collinear, and independent of the chosen points (b_1, β_1, β_1') and (b_2, β_2, β_2').

Proof. Since three points$(a_1, \alpha_1, \alpha_1'), (a_3, \alpha_3, \alpha_3')$,$(b_2, \beta_2, \beta_2')$ and three points

$$(b_1, \beta_1, \beta_1'), (a_2, \alpha_2, \alpha_2'), (a_3, \alpha_3, \alpha_3')$$

are collinear,

$$\begin{aligned} \alpha_1 \wedge \alpha_3 = \alpha_1 \wedge \beta_2 = \alpha_3 \wedge \beta_2 \text{ and } \alpha_2 \wedge \alpha_3 = \alpha_2 \wedge \beta_1 = \alpha_3 \wedge \beta_1 \\ \alpha_1' \vee \alpha_3' = \alpha_1' \vee \beta_2' = \alpha_3' \vee \beta_2' \text{ and } \alpha_2' \vee \alpha_3' = \alpha_2' \vee \beta_1' = \alpha_3' \vee \beta_1'. \end{aligned}$$

One calculates that $\beta_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\beta_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3^{2\prime}$ hence the point (b_3, β_3, β_3') is independent of the chosen of the points (b_1, β_1, β_1') and (b_2, β_2, β_2'). □

Theorem 19. *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1, 2, 3\}$ in $\mathcal{IFP}$ with non collinear base points and with $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k\rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. If the lines $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle$ and $\langle a_3, b_3\rangle$ in $\mathcal{P}$ are concurrent, then three lines $(\langle a_i, b_i\rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$, for $i \in \{1, 2, 3\}$ are concurrent if and only if $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.*

Proof. A configuration is chosen such that three points$(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2')$ and $(a_3, \alpha_3, \alpha_3')$ and $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k\rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$. Suppose three lines $(\langle a_i, b_i\rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$, for $i \in \{1, 2, 3\}$ are concurrent. Then three membership degree pairs in concurrent point $\alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$ and $\alpha_i' \vee \alpha_j' \vee \beta_i' \vee \beta_j'$, $i \neq j$, $\{i, j\} = \{1, 2, 3\}$ are equal. Since three points $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k\rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$, $\alpha_i \wedge \alpha_j = \alpha_i \wedge \beta_k = \alpha_j \wedge \beta_k$ and $\alpha_i' \vee \alpha_j' = \alpha_i' \vee \beta_k' = \alpha_j' \vee \beta_k'$ for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$ are valid. One can easily get $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.

Conversely, by using points (b_j, β_j, β_j'), $(a_i, \alpha_i, \alpha_i')$ and $(a_i, \alpha_k, \alpha_k')$ are collinear for $i \neq j \neq k$, $\{i, j, k\} = \{1, 2, 3\}$ in $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$ it is shown that three pair of values $\alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$ and $\alpha_i' \vee \alpha_j' \vee \beta_i' \vee \beta_j'$, $i \neq j$, $\{i, j\} = \{1, 2, 3\}$ are equal. □

Corollary 20. ***(The intuitionistic fuzzy projective Ceva condition($\mathcal{IFPCC}$))*** *Let $\mathcal{FP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$ with $(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$. The configuration that consists of these six points is Ceva 6−figure iff $\alpha_1^2 \wedge \alpha_2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2^2 \wedge \alpha_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3^2$ and $\alpha_1'^2 \vee \alpha_2' \vee \alpha_3' = \alpha_1' \vee \alpha_2'^2 \vee \alpha_3' = \alpha_1' \vee \alpha_2' \vee \alpha_3'^2$.*

The following theorem show that intuitionistic fuzzy Ceva 6−figures can be obtained as a corollary of intuitionistic fuzzy Menelaus 6−figures.

Theorem 21. *Let $\mathcal{IFP}$ be an intuitionistic fuzzy projective plane with base plane $\mathcal{P}$. Choose three points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$ with $(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$, three lines $\langle a_i, b_i \rangle$ are concurrent in $\mathcal{P}$. If the configuration that consists of these six points is intuitionistic fuzzy Menelaus 6−figure, it is intuitionistic fuzzy Ceva 6−figure.*

Proof. Let the configuration chosen such that three points $(a_1, \alpha_1, \alpha_1'), (a_2, \alpha_2, \alpha_2')$ and $(a_3, \alpha_3, \alpha_3')$ and $(b_j, \beta_j, \beta_j') \in (\langle a_i, a_k \rangle, \alpha_i \wedge \alpha_k, \alpha_i' \vee \alpha_k')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$ be intuitionistic fuzzy Menelaus 6−figure. Three membership degree pairs in intersection point of three lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$ are $\alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$ and $\alpha_i' \vee \alpha_j' \vee \beta_i' \vee \beta_j'$, $i \neq j$, $\{i,j\} = \{1,2,3\}$. It is easily seen that these are equal. So three lines $(\langle a_i, b_i \rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$ for $i \in \{1,2,3\}$ are concurrent. □

The reverse of this theorem isn't true in $\mathcal{IFP}$.

Fano projective plane, denoted by $PG(2,2)$, consists seven points and seven lines. Fano projective plane is only example that is both Menelaus 6−figure and Ceva 6−figure. Even if the base plane $\mathcal{P}$ of $\mathcal{IFP}$ is Fano plane, the reverse of the process is not always valid in $\mathcal{IFP}$.

Theorem 22. *Let $\wedge$ and $\vee$be a triangular norm and conorm, respectively. Let $\mathcal{IFP}$ be any nontrivial intuitionistic fuzzy projective plane with base plane $\mathcal{P}$ that is Fano plane. Let three points be $(a_i, \alpha_i, \alpha_i')$, $i \in \{1,2,3\}$ in $\mathcal{IFP}$ with non collinear base points and the other three points (b_k, β_k, β_k'), $k \in \{1,2,3\}$ with $(b_k, \beta_k, \beta_k') \in (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ for $i \neq j \neq k$, $\{i,j,k\} = \{1,2,3\}$, three lines $\langle a_i, b_i \rangle$ are concurrent in $\mathcal{P}$. If the configuration that consists of these six points is intuitionistic fuzzy Ceva 6−figure, it can not be intuitionistic fuzzy Menalaus 6−figure.*

Proof. The configuration picked such that points

$$(a_1, 0.5, 0.5), (a_2, 0.5, 0.5), (a_3, 0, 5, 0.5) \text{ and } (b_1, 0.6, 0.4), (b_2, 0.7, 0.3), (b_3, 0.8, 0.2)$$

is Ceva 6−figure in $\mathcal{IFP}$. But, using the minimum and maximum operators for $\wedge$ and $\vee$, it is easily seen that the points $(b_1, 0.6, 0.4), (b_2, 0.7, 0.3)$ and $(b_3, 0.8, 0.2)$ are not collinear. □

Conclusion 23. *In this study, the intuitionistic fuzzy versions of Menelaus and Ceva 6-figures in intuitionistic fuzzy projective plane are given. So, the obtained conditions and results for the intuitionistic fuzzy versions of Menelaus and Ceva will contribute to the intuitionistic fuzzy projective geometry. While the fibered and fuzzy versions of some classical results in projective planes by using t-norm are given, the intuitionistic fuzzy versions of these theorems include both t-norm and conorm. It seen that the triangular norms and conorms have important role in the intuitionistic fuzzy versions of theorems related to theory.*

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