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by

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A note on Multiple Group Method of Factor Analysis

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ABSTRACT

This article extends and generalizes Guttman's works on multiple group method of factor analysis when data matrix is Gramian**, where data matrix can be considered as covariance matrix or correlation matrix.

Key words: multiple group method of factor analysis, matrix algebra, generalized inverse.

INTRODUCTION

In his previous works on multiple group method of factor analysis, Guttman developed a basic theorem of the method and gave computing procedure [3], [4]. His proof of the theorem based on the supermatrix, where data matrix is a part, and of the condition of non- singularity.

This paper discuss a different proof of the theorem which is analogous to the seperation of the quadratic forms and ranks of the related matrices. There is also the discussion on a general proof of the theorem without using the condition of non-singularity.

1. A New Proof Of The Theorem

Theorem: 1)

Let S be a Gramian matrix of order nxn and of rank $r > O$. Let A be of order mxn $(m \leq r)$ and such that ASA' is non-singular. Then the residual matrix

(1.1) $S_{res} = S - SA' (ASA')^{-1} AS$

is of rank $(r - m)$ and is Gramian.

Proof: If S is a Gramian matrix, then there exists a matrix E of order nxr and of rank $r > O$ such that

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** A Gramian matrix is a symmetric matrix in which all principal minors of all orders are nonnegative.

(1.2) $S = EE'$

[3]. Combining equations (1.1) and (1.2) we get the following equations:

(1.3) $S_{res} = E(I_r - E'A' \ (AEE'A')^{-1}AE)E'$,

(1.4) $S_{res} = EG_{res}E'$.

where

(1.5) $G_{res} = I_r - E'A' \ (AEE'A')^{-1}AE$.

and I_r is identity matrix of order rxr. If we define the reproduced matrix

(1.6) $S_{rep} = S - S_{res}$

then by using equations (1.2) and (1.3) we get the followings:

(1.7) $S_{rep} = E(E'A' \ (AEE'A')^{-1}AE)E'$.

(1.8) $S_{rep} = EG_{rep}E'$

where

(1.9) $G_{rep} = E'A' \ (AEE'A')^{-1}AE'$

Then the following equations could be written:

(1.10) $S = S_{rep} + S_{res}$

(1.11) $I_r = G_{rep} \ G_{res}$

(1.12) $EIE' = EG_{rep} \ E' + EG_{res}E'$

It can be shown that I_r, G_{rep} and G_{res} are symmetric matrices of order rxr and hold the following properties:

(1.13) G_{rep} and G_{res} are each idempotent

(1.14) $I_r = G_{rep} + G_{res}$ is idempotent

(1.15) $G_{rep} \ G_{res} = G_{res} \ G_{rep} = \Phi_r$

where Φ_r is a null matrix of order rxr. If any two of the equations (1.13), (1.14) and (1.15) hold, then the rank of $(G_{rep} + G_{res})$ equals the sum of the ranks of the G_{rep} and G_{res} [1]. Therefore one could write

(1.16) $r(G_{rep} + G_{res}) = r(I_r) = r(Grep) + r(G_{res})$

where r(G) denotes the rank of matrix G. Since G_{rep} is symmetric and idempotent then one could write

(1.17) $r(G_{rep}) = tr\,(G_{rep}) = trI_m = m.$

Substituting (1.17) in (1.16) one could obtain

(1.18) $r(G_{res}) = r - m.$

Since G_{res} is symmetric and idempotent then there exits an orthogonal matrix P of order rxr, such that

$$(1.19)\qquad P'G_{res}P = \begin{pmatrix} I_{r-m} & \Phi \\ \Phi & \Phi \end{pmatrix}$$

Considering equation (1.4) and (1.19) one could write the followings:

$$r(S_{res}) = r(EG_{res}E') = r(EPP'G_{res}PP'E')$$

$$(1.20)\qquad r(S_{res}) = r\left[EP\begin{pmatrix} I_{r-m} & \Phi \\ \Phi & \Phi \end{pmatrix}P'\,E'\right]$$

Partitioning P into P_1 of order $rx(_{r-m})$ and P_2 of order rxm, then inserting in equation (1.20) one could get:

$$r\left[E\,(P_1, P_2)\begin{pmatrix} I_{r-m} & \Phi \\ \Phi & \Phi \end{pmatrix}\begin{pmatrix} I_{r-m} & \Phi \\ \Phi & \Phi \end{pmatrix}\begin{pmatrix} P'_1 \\ P'_2 \end{pmatrix}E'\right]$$

$$(1.21)\qquad = r(EP_1P'_1E') = r(EP_1) = r - m$$

Combining equation (1.20) and (1.21), the result $r(S_{res}) = r - m$ follows.

To prove that S_{res} is Gramian one could substituted symmetry and idempotency of G_{res} in equation (1.4)

(1.22) $S_{res} = EG_{res}G'_{res}E' = BB'$

where $B = EG_{res}$ of order nxr and of rank (r—m). Furthermore it can be shown that S_{rep} is also Gramian.

Is S is a positive definite matrix of order nxn as usually the case in applications, then obviously the residual matrix in equation (1.1) is of rank n—m and is also Gramian.

2. Generalization Of The Theorem.

Another theorem will be stated and proved here, which can be named as a generalized version of Guttman's basic theorem.

Theorem 2: Let S be a Gramian matrix of order nxn and of rank $r > O$. Let A be a matrix of order mxn, and of rank t, ($t < m < n$). Then the residual matrix

(2.1) $S_{res} = S - SA' (ASA')^{+} AS$

is of rank r—t and is Gramian. Where $(ASA')^{+}$ denotes the generalized inverse of ASA' [2].

This theorem enlarges the applicability of multiple group method of factor analysis to the case of singular matrix (ASA'), which is sometimes encountered by the researchers in practical work.

Proof: From equations (1.2) and (2.1) one could write

(2.2) $S_{res} = E(I_r - E'A' (AEE'A')^{+} AE)E' = EG_{res}E'$.

where $G_{res} = I_r - E'A' (AEE'A')^{+} AE$ as in (1.5).

Taking the defination in below

(2.3) $S_{rep} = S - S_{res} = E (E'A' (AEE'A')^{+} AE)E' = EG_{rep}E'$

where $G_{rep} = E'A' (AEE'A')^{+} AE$, the following equations could be written:

(2.4) $G_{res} = I_r - C'(CC')^{+} C$

(2.5) $G_{rep} = C'(CC')^{+} C$

(2.6) $I_r = G_{rep} + G_{res}$

where $C = AE$, of rank t.

Now we can show that, I_r, G_{rep}, G_{res} are idempotent, G_{rep} and G_{res} are orthogonal to each other and are symmetric.

Since generalized inverse X^{+} of a matrix X, holds the following properties [2]:

(2.7) $X X^{+} X = X$

(2.8) $X^{+} X X^{+} = X^{+}$

(2.9) $(X X^{+})' = X X^{+}$

(2.10) $(X^{+} X)' = X^{+} X$

we can substitute (2.8) into (2.5) and write

$$(G_{rep})^2 = C' (C C')^{+} C.C' (CC')^{+} C$$

(2.11) $(G_{rep})^2 = C'.(C\,C')^+.C = G_{rep}$

From equation (2.6) and (2.11) one could get

(2.12) $(G_{res})^2 = (I_r - G_{rep})^2 = I_r - G_{rep} = G_{res}$

It follows from equations (2.6) and (2.11) that

(2.13) $G_{res}.G_{rep} = G_{rep}.G_{res} = \Phi_r$.

From equations (2.7) through (2.10) the following could be written:

(2.14) $(X)'^+ = (X^+)'$

[2]. Substituting (2.14) into (2.4) and (2.5) one could see that G_{res} and G_{rep} are symmetric matrices.

From equations (2.6) and (2.13) the following will be hold:

(2.15) $r(I_r) = r(G_{rep}) + r(G_{res})$

Since C is of order mxr and of rank txr and $I_r - C'(CC')^+C$ is idempotent, then

(2.16) $r(I_r - C'(CC')^+C = tr(I_r - C'\,(CC')^+C)$

$= r(G_{res}) = r - t.$

[2]. Substitution (2.16) into (2.2) one could write

(2.17) $r(S_{res}) = r - t.$

Now it can be shown, as we did in theorem 1, that S_{res} is Gramian. Taking equation (2.2), (2.12) and propetry of symmetry of G_{res} we could write the following:

(2.18) $S_{res} = EG_{res}\,G'_{res}\,E' = DD'$

where $D = EG_{res}$.

Equation (2.18) shows that S_{res} is Gramian. It can be shown that S_{rep} is also Gramian.

For the non-singular case, we had the following additional property:

(2.19) $AS_{res} = S_{res}A' = \Phi_r$

Let us show that it is also true for the singular case. The following equations were given by Rao [5].

(2.20) $CC'(CC')^{+}C=C$

(2.21) $C'(CC')^{+}CC'=C'$

Substituting equations (1.3), (2.20) and (2.21) into equation (2.19) one could write

$$AS_{res}=AE\,(I_r-E'A'(AEE'A')^{+}AE)\,E'$$

(2.22) $AS_{res}=(C-CC')(CC')^{+}C)\,E'=\Phi_r$

and

$$S_{res}A'=E\,(I_r-E'A'(AEE'A')^{+}AE)\,E'A'$$

(2.23) $S_{res}A'=E(C'-C'(CC')^{+}CC')=\Phi_r$

where $C=AE$.

Therefore, instead of matrix A, a new hypothesis matrix should be used to reapply the theorem 2 to S_{res} which is Gramian. Since S_{res} is substitued instead of Gramian matrix S in theorem 2, process continues until exhausting the final S_{res}.

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ÖZET

L. Guttman, veri matrisi Gramian bir matris olduğu zaman "faktör analiz'in çoklu gruplandırma yöntemi" diye bir yöntem önermiş ve üzerinde çalışmıştır. Çalışmamızda Guttman'ın teoremi hem değişik bir yolla ispatlanmış hem de genelleştirilmiştir.

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