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A NEW APPROACH TO CONSTRUCT AND EXTEND THE SCHUR STABLE MATRIX FAMILIES

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Abstract. In this study, Schur stability, sensitivity and continuity theorems have been mentioned. In addition, matrix families, interval matrix and extend of the intervals also have been mentioned. The $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ intervals of the matrix families have been determined so that the linear sums family $\mathcal{L}$ and convex combination family $\mathcal{C}$ are Schur stable. Samely, the $\mathcal{I}^*_{\mathcal{L}}$ and $\mathcal{I}^*_{\mathcal{C}}$ intervals have been determined and $\mathcal{L}$ and $\mathcal{C}$ are ω^*-Schur stable. Afterwards, the methods which based on continuity theorems and the algorithms which based on the methods have been given. Extended intervals have been obtained with the help of the methods and the algorithms. All definitions are supported by examples.

1. Introduction

One of the real problems of the stability analysis is to determine the stability of the matrix families. In this paper the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ have been presented to make the matrix families $\mathcal{L}$ and $\mathcal{C}$ Schur stable. Here the matrix families $\mathcal{L}$ and $\mathcal{C}$ consist of linear sum and convex combination, respectively. Also, these intervals are extended with the help of continuity theorems and the matrix families are constructed in order to provide Schur stability [13, 16]. There are many studies in the literature specifically related to linear sum and convex combination [5–8, 19, 24]. Unlike studies that control the Schur stability of interval matrices, Schur stable interval matrices are constructed in this study.

In 1892, Lyapunov studied the behavior of solutions of systems and developed the concept of stability (see, for instance, [1,9,18]). The stability problem is reduced

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to the problem of the existence of a positive definite solution of the matrix equation known as the Lyapunov equation with this concept for linear systems.

A necessary and sufficient condition for the matrix A to be discrete-asymptotic stable is that the eigenvalues of the matrix A lay in the unit disk, that is, $|\lambda_i(A)| < 1$ for all $i = 1, 2, \ldots, N$, where $\lambda_i (i = 1, 2, \ldots, N)$ are the eigenvalues of the matrix A [1, 18]. On the other hand, this is also known as spectral criterion in the literature. The spectral criterion can also be represented by the spectrum. $\sigma(A) = \{\lambda \mid \lambda = \lambda_i(A)\}$ to be spectrum, the matrix A is said to be Schur stable if it satisfies the condition $\sigma(A) \subset C_s = \{z \mid |z| < 1\}$ [27]. Let's also give the family of Schur stable matrices as follows;

$$S_N = \{A \in M_N(\mathbb{C}) \mid |\lambda_i(A)| < 1 \ \ (i = 1, 2, ..., N)\}.$$

If the locations of these eigenvalues are known approximately, stability analysis of the system can be done with help of many well-known methods. Stability analysis of many control systems is concerned with the region where the eigenvalues of the matrices are located. Gerschgorin and Rouche theorems, which are used in determining the region, can be given as an example to this situation [18, 20]. However, it is not easy to determine the eigenvalue in practice. Small changes in the inputs of the matrices lead to the big changes in the eigenvalues, i.e. the eigenvalue problem is an ill-posed problem for the non-symmetric matrices [9, 28]. We can give the example of Ostrowski to explain this situation better. $A_\omega = (a_{ij}) \in M_N(\mathbb{R})$; $a_{i,i} = 0.5, \ a_{i,i+1} = 10, \ a_{N,1} = \omega, \ i = 1, 2, ..., N-1$. It is seen that $\|A_{10^{-100}} - A_0\| = 10^{-100}$ and $\lambda_i(A_{10^{-100}}) = 1.5$ so $|\lambda_i(A_{10^{-100}}) - \lambda_i(A_0)| \leq 1$ [1]. As can be seen here, while the matrix A_0 is Schur stable, the matrix $A_{10^{-100}}$ is not Schur stable because of $\lambda_i(A_{10^{-100}}) = 1.5$. Therefore, it is more convenient to use the parameters calculated with the help of the solution of a linear algebraic equation which characterizes the stability for the determination of stability.

Thus, the stability problem is reduced to the problem of the existence of a positive definite solution of the matrix equation given as the Lyapunov equation [1, 2, 18]. According to Lyapunov's theorem, the Lyapunov matrix equation, which determines the Schur stability of the systems, is given as follow

$$A^* H A - H + I = 0. \tag{1}$$

If this system of equations has a positive definite solution

$$H = \sum_{k=0}^{\infty} (A^*)^k A^k, \ \ H = H^* > 0 \tag{2}$$

then the matrix A is said to be Schur stable [1, 9, 18, 23, 27]. The existence of $H = H^* > 0$ equivalent to having the eigenvalues of the matrix A inside the unit circle.

The parameter $\omega(A) = \|H\| \geq 1$, which determines the quality of the stability, is known as the Schur stability parameter of the matrix A [1, 9, 11]. Furthermore, ω^* is the practical Schur stability parameter of the matrix A, where $1 < \omega^* \in \mathbb{R}$

and the users choose the value ω^* in view of their problem. If $\omega(A) \leq \omega^*$ then the matrix A is ω^*-Schur stable. Otherwise, the matrix A is ω^*-Schur unstable matrix [1, 3, 26]. Let's examine the following matrices in order to see the notion of quality of stability more easily.

Let's take $A_k \in S_N$ as follow

$$A_k = \begin{pmatrix} -0.1 & 10^{k-1} - 1 \\ 0 & 0.1 \end{pmatrix}, \ k \in \mathbb{N}.$$

It is clear that, although $\sigma(A_k) = \{-0.1, 0.1\}$ for $k \in \mathbb{N}$, it can be seen from the Table 1 that the values of $\omega(A_k)$ also increase as the values of k increase.

TABLE 1. The quality of Schur stability of the matrix A_k

k	1	2	3	4	5
$\omega(A_k)$	1.0101	82.0282	9803	998102	9.999e+007

Also the quality of the Schur stability increases as it approaches 1. Especially, in case of $A = 0$, when we substitute it in (2), $H = I$ and $\omega(A) = 1$ are obtained. This state is also known as the perfect state.

In [25], Hurwitz stability intervals for the matrix families were studied. The matrix families were introduced. The intervals were determined to make these families Hurwitz stable. A method and an algorithm were given to extend these intervals.

This study is an analogy of [25]. Here, Schur and ω^*-Schur stability of linear sum and convex combination families are discussed. In Section 2, $\mathcal{L}$ and $\mathcal{C}$ matrix families are introduced, $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ intervals are determined to make these families Schur stable. The illustrative examples related to the subject are given. In Section 3, $\mathcal{I}_{\mathcal{L}}^*$ and $\mathcal{I}_{\mathcal{C}}^*$ intervals are determined to make these families ω^*-Schur stable. Thereafter, the illustrative examples related to the subject are given. In Section 4, a new approach is given for the Schur stability of the matrix families. According to the approach, the methods which based on continuity theorems are given. These theorems shows the sensitivity of Schur stability and ω^*-Schur stability. The algorithms which based on the methods are given. The extended intervals $\mathcal{I}_{\mathcal{L}}^e$, $\mathcal{I}_{\mathcal{C}}^e$, $\mathcal{I}_{\mathcal{L}}^{*e}$ and $\mathcal{I}_{\mathcal{C}}^{*e}$ are obtained with the help of methods and algorithms. End of the paper, examples are given. The numerical results in the article are obtained using the computer dialogue system MVC [10].

2. SCHUR STABILITY OF THE MATRIX FAMILIES

Let's give the theorems which determining the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ for the matrix families

$$\mathcal{L} = \mathcal{L}(A_1, A_2) = \{A(r) = A_1 + rA_2 \mid A_1, \ A_2 \in M_N(\mathbb{C})\} \tag{3}$$

and

$$\mathcal{C} = \mathcal{C}\left(A_1, A_2\right) = \left\{A\left(r\right) = \left(1-r\right)A_1 + rA_2 \mid A_1,\ A_2 \in M_N\left(\mathbb{C}\right)\right\} \tag{4}$$

to be Schur stable for $A_1 \in S_N$ and $A_2 \in M_N(\mathbb{C})$. Before giving the theorems for these matrix families, let's give the continuity theorem which determines the sensitivity of the stability. We use this theorem for Schur stability. Let's remember the family of Schur stable matrices as follows;

$$S_N = \left\{A \in M_N\left(\mathbb{C}\right) \mid \omega(A) < \infty\right\}.$$

Theorem 1. *Let $A \in S_N$. If $\|B\| < \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$ then the matrix $A+B \in S_N$ and*

$$\omega(A+B) \leqslant \tfrac{\omega(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)},$$

$$|\omega(A+B) - \omega(A)| \leqslant \tfrac{(2\|A\|+\|B\|)\|B\|\omega^2(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}$$

holds [4, 14].

Theorem 2. *If $A_1 \in S_N,\ A_2 \in M_N\left(\mathbb{C}\right)$ and $r \in \mathcal{I}_{\mathcal{L}} = [\underline{r}, \overline{r}]$ then $\mathcal{L}\left(A_1, A_2\right) \subset S_N$, where $-l = u = -\frac{\|A_1\|}{\|A_2\|} + \frac{1}{\|A_2\|}\sqrt{\|A_1\|^2 + \frac{1}{\omega(A_1)}}$, $l < \underline{r} < \overline{r} < u$.*

Proof. Let us consider the given linear sum as follow

$$A\left(r\right) = A_1 + rA_2.$$

If $A_2 = 0$ then $A(r) = A_1$. We know that $A_1 \in S_N$ so $A(r) \in S_N$, too. Let's take $A_2 \neq 0$. If we substitute $A(r)$ in the Lyapunov equation, we get the equation as follow

$$\left(A_1 + rA_2\right)^* H \left(A_1 + rA_2\right) - H + I = 0$$
$$A_1^* H A_1 - H = -\left(I + rA_1^* H A_2 + rA_2^* H A_1 + r^2 A_2^* H A_2\right).$$

At that rate,

$$C = I + rA_1^* H A_2 + rA_2^* H A_1 + r^2 A_2^* H A_2 > 0$$

$C = C^* > 0$ is available. The obtained result is written as follows

$$\|C\| \leq 1 + 2\left|r\right| \|A_1\| \|H\| \|A_2\| + r^2 \|A_2\|^2 \|H\|$$

then, if the inequality is substituted in the equation which is the solution of the Lyapunov equation

$$H = \sum_{k=0}^{\infty} \left(A_1^*\right)^k C A_1^k$$
$$\|H\| \leq \|C\| \,\omega\left(A_1\right)$$
$$\|H\| \leq \Big(1 + 2\left|r\right| \|A_1\| \|H\| \|A_2\| + r^2 \|A_2\|^2 \|H\|\Big)\,\omega\left(A_1\right)$$
$$\|H\| \leq \frac{\omega\left(A_1\right)}{1 - 2\left|r\right| \|A_1\| \|A_2\| \,\omega\left(A_1\right) - r^2 \|A_2\|^2 \omega\left(A_1\right)}$$

is obtained. While $A_1 \in S_N$, the following condition must be verified for $A(r)$ to be Schur stable

$$1 - 2|r| \|A_1\| \|A_2\| \omega(A_1) - r^2 \|A_2\|^2 \omega(A_1) \geq 0.$$

Then if the inequality is arranged with according to r, the Schur stability intervals $[\underline{r}, \overline{r}]$ of the matrix $A(r)$ are obtained, where

$$\underline{r} > l = \frac{\|A_1\| \omega(A_1) - \sqrt{\|A_1\|^2 (\omega(A_1))^2 + \omega(A_1)}}{\|A_2\| \omega(A_1)}$$

and

$$\overline{r} < u = \frac{-\|A_1\| \omega(A_1) + \sqrt{\|A_1\|^2 (\omega(A_1))^2 + \omega(A_1)}}{\|A_2\| \omega(A_1)}.$$

□

Theorem 3. *If $A_1 \in S_N, \ A_2 \in M_N(\mathbb{C})$ and $r \in \mathcal{I}_{\mathcal{C}} = [\underline{r}, \overline{r}]$ then $\mathcal{C}(A_1, A_2) \subset S_N$, where $-l = u = -\frac{\|A_1\|}{\|A_2 - A_1\|} + \frac{1}{\|A_2 - A_1\|} \sqrt{\|A_1\|^2 + \frac{1}{\omega(A_1)}}$, $l < \underline{r} < \overline{r} < u$.*

Proof. If we write $A_2 - A_1$ instead of A_2 in Theorem 2, proof is clear from Theorem 2. □

Here, the equation expressed as a convex combination is shown with $A(r) = (1-r)A_1 + rA_2$ and the values r are examined in such a way that the convex sums of two matrices are Schur stable without the condition $r \in (0, 1)$.

Let's examine the values r of the matrix families $\mathcal{L}(A_1, A_2)$ and $\mathcal{C}(A_1, A_2)$, which provide the Schur stability, by using the Schur stability of A_1. During this review, the articles of Duman and Aydın were taken into consideration [14, 15].

It is possible to write the convex combination as a special case of the linear sum. In other words, we can express the convex combination given as $A(r) = (1-r)A_1 + rA_2$ as a linear sum as $A(r) = A_1 + r(A_2 - A_1)$. In order to the matrix $A(r)$ to be Schur stable, let's determine the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ using the Schur stability of the matrix A_1.

Example 1. *For $\alpha \in (-1, 1)$, $A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let's examine the interval $\mathcal{I}_{\mathcal{L}}$ which leaves the matrix families $\mathcal{L}(A_1, A_2)$ Schur stable.*

According to the Theorem 2, from $\|A_1\| = |\alpha|$, $\|A_2\| = 1$, $\omega(A_1) = \frac{1}{1-\alpha^2}$, we obtained as follows,

$$l = |\alpha| - 1 \ , \ u = -|\alpha| + 1.$$

$\omega(A(r)) = \frac{1}{1-(\alpha+r)^2}$ is known, then

$$\begin{cases} \alpha < 0 \ , \ \lim_{r \to |\alpha| - 1} \left(\frac{1}{1-(\alpha+r)^2} \right) = \infty \\ \\ \alpha > 0 \ , \ \lim_{r \to -|\alpha| + 1} \left(\frac{1}{1-(\alpha+r)^2} \right) = \infty \end{cases}$$

is obtained.

Example 2. *Let's examine the following matrices*

$$A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For these matrices we obtained $\|A_1\| = 0.5$, $\|A_2 - A_1\| = 0.5$, $\omega(A_1) = 1.33333$. *So we know that* A_1 *is Schur stable.*

Table 2. The effectiveness of the interval $\mathcal{I}_{\mathcal{C}}$

r	-0.9999	-0.99	-0.9	$\dots$	0.9	0.99	0.9999
$\omega(A(r))$	10000.3	100.251	10.2564	$\dots$	10.2564	100.251	10000.3

According to the Theorem 3, we obtained $l = -1$, $u = 1$. *As can be seen in the Table 2, the condition numbers change according to the values* r *selected from the intervals* $\mathcal{I}_{\mathcal{C}}$. *Also the quality of the stability decrease as the value* r *approaches* -1 *or* 1.

Remark 1. *In particular, if taken* $A_1 = A_2 = 0$, *we get the matrix family* $\mathcal{L}(0,0) = \{0\} \subset S_N$. *Lets take* $A_2 \neq 0$, *the matrix family* $\mathcal{L}(0, A_2)$, $r \in \mathcal{I}_{\mathcal{L}}$ *specified here, which is obtained in the form of* $-l = u = \frac{1}{\|A_2\|}$ *for* $\|A_1\| = 0$ *and* $\omega(A_1) = 1$. *If we call this interval obtained for the* r *value "perfect interval", we can say that the result obtained here is the "perfect state".*

3. ω^*-Schur Stability of the Matrix Families

Let ω^* be the practical Schur stability parameter, where $1 < \omega^* \in \mathbb{R}$ and the users choose the value ω^* in view of their problem. If $\omega(A) \leq \omega^*$ then the matrix A is ω^*-Schur stable matrix. Otherwise, the matrix A is ω^*-Schur unstable matrix [1,3,26].

Although there are theorems known as continuity theorems in the literature that determine the sensitivity of the problem, these theorems show under which conditions the given problems maintain the same property [1,9,11,14,15,17]. Let's give the continuity theorem which determines the sensitivity of the ω^*-Schur stability.

Theorem 4. *Let* A *be a* ω^*-*Schur stable matrix* ($\omega(A) \leq \omega^*$). *If the matrix* B *satisfies* $\|B\| \leq \sqrt{\|A\|^2 + \frac{\omega^* - \omega(A)}{\omega^* \omega(A)}} - \|A\|$, *then* $A + B$ *is* ω^*-*Schur stable [14].*

Let's define the family of ω^*-Schur stable matrices as follows;

$$S_N^* = \{A \in S_N \mid \omega(A) \leq \omega^*\}.$$

Now, considering Theorem 4, let's give the following two theorems.

Theorem 5. *If* $A_1 \in S_N^*$, $A_2 \in M_N(\mathbb{C})$ *and* $r \in \mathcal{I}_{\mathcal{L}}^* = [\underline{r}, \overline{r}]$ *then* $\mathcal{L}(A_1, A_2) \subset S_N^*$, *where* $-l^* = u^* = -\frac{\|A_1\|}{\|A_2\|} + \frac{1}{\|A_2\|}\sqrt{\|A_1\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_1)}}$, $l^* \leq \underline{r} < \overline{r} \leq u^*$.

Proof. If $A_2 = 0$ then $A(r) = A_1$. We know that $A_1 \in S_N^*$ so $A(r) \in S_N^*$ too. Lets take $A_2 \neq 0$. For $r \in \mathcal{I}_{\mathcal{L}}^*$ we can write $l^* \leq r \leq u^*$. Then we get following inequality,

$$r^2 \left\|A_2\right\|^2 \omega\left(A_1\right) \omega^* + 2\left|r\right| \left\|A_1\right\| \left\|A_2\right\| \omega\left(A_1\right) \omega^* - \omega^* + \omega\left(A_1\right) \leq 0.$$

If we arrange above inequality

$$\frac{\omega\left(A_1\right)}{1 - r^2 \left\|A_2\right\|^2 \omega\left(A_1\right) - 2\left|r\right| \left\|A_1\right\| \left\|A_2\right\| \omega\left(A_1\right)} \leq \omega^*$$

holds. Since $\omega\left(A_1 + rA_2\right) \leq \frac{\omega(A_1)}{1 - r^2\|A_2\|^2\omega(A_1) - 2|r|\|A_1\|\|A_2\|\omega(A_1)}$ is valid from the Theorem 2, the inequality $\omega\left(A_1 + rA_2\right) \leq \omega^*$ is found. □

Theorem 6. *If* $A_1 \in S_N^*,\ A_2 \in M_N\left(\mathbb{C}\right)$ *and* $r \in \mathcal{I}_{\mathcal{C}}^* = [\underline{r}, \overline{r}]$ *then* $\mathcal{C}\left(A_1, A_2\right) \subset S_N^*$, *where* $-l^* = u^* = -\frac{\|A_1\|}{\|A_2 - A_1\|} + \frac{1}{\|A_2 - A_1\|}\sqrt{\left\|A_1\right\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_1)}}$, $l^* \leq \underline{r} < \overline{r} \leq u^*$.

Proof. It is obvious from the previous proof. □

Now let's give the following illustrative example on this subject.

Example 3. *For* $\alpha \in \left(-\sqrt{1 - \frac{1}{\omega^*}}, \sqrt{1 - \frac{1}{\omega^*}}\right) \subset (-1, 1)$, $A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \in S_N^*$ *and* $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. *Let's examine the interval* $\mathcal{I}_{\mathcal{L}}^*$ *which leaves the matrix family* $\mathcal{L}\left(A_1, A_2\right)$ *is* ω^*-*Schur stable.*

According to the Theorem 5, we obtained,

$$l^* = |\alpha| - \sqrt{1 - \frac{1}{\omega^*}}\ ,\ u^* = -\left|\alpha\right| + \sqrt{1 - \frac{1}{\omega^*}}$$

from $\|A_1\| = |\alpha|$, $\|A_2\| = 1$, $\omega\left(A_1\right) = \frac{1}{1-\alpha^2}$. $\omega\left(A\left(r\right)\right) = \frac{1}{1-(\alpha+r)^2}$ *is known, then*

$$\begin{cases} \alpha < 0\ ,\ \lim_{r \to |\alpha| - \sqrt{1 - \frac{1}{\omega^*}}} \left(\frac{1}{1-(\alpha+r)^2}\right) = \omega^* \\ \\ \alpha > 0\ ,\ \lim_{r \to -|\alpha| + \sqrt{1 - \frac{1}{\omega^*}}} \left(\frac{1}{1-(\alpha+r)^2}\right) = \omega^* \end{cases}$$

is obtained.

Example 4.

$$A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\ ,\ A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For these matrices we obtained $\omega\left(A_1\right) = 1.33333$, $\|A_1\| = 0.5$, $\|A_2 - A_1\| = 0.5$. *So we know that* $A_1 \in S_N^*$. *If we choose* $\omega^* = 10$ *then we get* $-l = u = 0.897367$. *Let's examine the interval* $r \in \mathcal{I}_{\mathcal{C}}^*$ *which leaves the matrix family* $\mathcal{C}\left(A_1, A_2\right)$ *is* $10-$*Schur stable. According to the Theorem 6, as can be seen in the Table 3, sharp intervals are obtained for the specified* $\omega^* = 10$ *parameter. It is seen that* $\omega^* < \omega(A(r))$ *for the* r *value selected outside these intervals.*

TABLE 3. Sharpness of the interval $\mathcal{I}_{\mathcal{C}}^{*}$ of 10−Schur stability

r	-0.897368	l	-0.897366	...	0.897366	u	0.897368
$\omega\left(A(r)\right)$	10.0003	10	9.99937	...	9.99937	10	10.0003

Remark 2. *From the above example, when values of Schur stability parameter $\omega\left(A\left(r\right)\right)$ are checked for the r values, it can be seen clearly that Theorem 2, Theorem 3, Theorem 5 and Theorem 6 gave sharp bounds.*

4. Obtaining the Extended Intervals

The intervals $\mathcal{I}_{\mathcal{L}}, \mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{L}}^{*}$ and $\mathcal{I}_{\mathcal{C}}^{*}$ are given in the Section 3. Although these intervals are found, actually it has been realized that big intervals which preserve Schur stability or $\omega^{*}-$ Schur stability of the matrix families $\mathcal{L}$ and $\mathcal{C}$ can be found. For this reason, the intervals are extended with certain rule in this section. Here, the extended intervals for the matrix families which preserve the Schur stability or $\omega^{*}-$ Schur stability are given. In addition, the extended intervals also allow us to introduce the Schur stable interval matrices or and $\omega^{*}-$ Schur stable interval matrices. To extend the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$, the methods which based on continuity theorems are given and the algorithms which based on the methods are given. Similarly, to extend the intervals $\mathcal{I}_{\mathcal{L}}^{*}$ and $\mathcal{I}_{\mathcal{C}}^{*}$, the methods and the algorithms are given. So it can be obtained bigger intervals which preserve the Schur stability or $\omega^{*}-$ Schur stability of the matrix families $\mathcal{L}$ and $\mathcal{C}$. In this process, the stepsize is determined from the continuity theorems which are Theorem 2, Theorem 3, Theorem 5 and Theorem 6. The extended intervals $\mathcal{I}_{\mathcal{L}}^{e}, \mathcal{I}_{\mathcal{C}}^{e}, \mathcal{I}_{\mathcal{L}}^{*e}$ and $\mathcal{I}_{\mathcal{C}}^{*e}$ are obtained at the end of processing. Let's give the methods and the algorithms as below.

4.1. A method and an algorithm to find the extended interval $\mathcal{I}_{\mathcal{L}}^{e}$.

4.1.1. *A method.* Keeping the Schur stability of the matrix family $\mathcal{L}\left(A_1, B\right)$, a method is given to extend the intervals with the Schur stable matrix A_1 and the matrix B. $\mathcal{I}_{\mathcal{L}} = [\underline{r}, \overline{r}]$ has been chosen with Theorem 2. For $r \in \mathcal{I}_{\mathcal{L}}$, the matrices $A\left(r\right) = A_r = A_1 + rB$ are Schur stable.

i) Defining the stepsize

The stepsize parameter r is used to extend the interval $\mathcal{I}_{\mathcal{L}}$. So, generalizing form the Theorem 2, it is chosen as $r_k \lessapprox -\frac{\|A_k\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_k\|^2 + \frac{1}{\omega(A_k)}}$.

ii) Determining the initial value

From the Theorem 2, the first value of the parameter r_1 is taken as $r_1 \lessapprox u$.

iii) Calculating the upper bound u^e

To extend the upper bound of the intervals $\mathcal{I}_{\mathcal{L}}$, the following steps are done,

$$A_k = A_{k-1} + r_{k-1}B \ , \ r_1 \lessapprox u \ , \ k \geq 2, \tag{5}$$

$$r_k \lessapprox -\frac{\|A_k\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_k\|^2 + \frac{1}{\omega(A_k)}}, \tag{6}$$

$$u_k = u_{k-1} + r_k \ , \ u_1 = r_1. \tag{7}$$

The new matrix A_k in the equality (5) is obtained as Schur stable. r_k in equality (6) is calculated with Theorem 2. u_k in the equality (7) is the upper bound of the extended interval obtained in step k. At the end of this process, the upper bound u^e of the extended interval $\mathcal{I}_{\mathcal{L}}^e$ is obtained.

iv) Calculating the lower bound l^e

Similar to the above application, to extend the lower bound of the intervals $\mathcal{I}_{\mathcal{L}}$, the matrix A_k is taken as $A_k = A_{k-1} - r_{k-1}B$ in the equality (5) and the equality (7) is replaced by the recurrence relation $l_k = l_{k-1} - r_k \ , l_1 = -r_1$. l_k is the lower bound of the extended interval obtained in step k. The result obtained with the new equations, the lower bound l^e of the extended interval $\mathcal{I}_{\mathcal{L}}^e$ is obtained.

Remark 3. *If the method is applied consecutively to get the upper bound, the stepsize r_k is become smaller and the parameter ω continues to grow by increasing. A similar situation is also observed for the lower bound. Because of these reasons, the working with very small numbers is non-practical.*

4.1.2. *An algorithm.* As given in the Remark 3, to stop the calculation, the stopping criterion is given as follow.

Stopping parameter r^*

After a certain step, the new stepsize becomes too small. Calculations with such values are not practical due to some reasons (i.e. floating point arithmetic)(see. [12, 16]). r^* is called the practical parameter for the stepsize which chosen by user small enough [21, 22]. With this criterion, less processing is needed and the given method run smoothly.

Let's give the algorithm to extend the upper bound of the intervals $\mathcal{I}_{\mathcal{L}}$.

Algorithm 1.1 (for the upper bound u^e)

(1) Input; $A \in S_N,\ B,\ r^*,\ \gamma \lessapprox 1$.
(2) Calculate $\omega(A),\ \|A\|\ ,\ \|B\|$
$\beta = -\frac{\|A\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A\|^2 + \frac{1}{\omega(A)}},\ r_1 = \gamma.\beta.$
(3) Take $k = 1,\ A_1 := A,\ u_1 := r_1$.
(4) If $r_1 < r^*$ then write "The interval cannot be extended based on the available data." and go 7. step.
(5) Calculate;
$A_{k+1} = A_k + r_k B\ ,\ \|A_{k+1}\|\ ,\ \omega(A_{k+1}),$
$\beta_{k+1} = -\frac{\|A_{k+1}\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_{k+1}\|^2 + \frac{1}{\omega(A_{k+1})}},$
$r_{k+1} = \gamma.\beta_{k+1}.$
(6) If $r_{k+1} \geq r^*$ then calculate $u_{k+1} = u_k + r_{k+1}$, take $k := k + 1$ and go 5. step.

(7) Write as $M := k$ and the upper bound of interval $u^e = u_M$.

To extend the lower bound of the intervals $\mathcal{I}_{\mathcal{L}}$, steps (5)-(7) in Algorithm 1.1 are taken as follow.

Algorithm 1.2 (for the lower bound l^e)

(5́) Calculate;

$A_{k+1} = A_k - r_k B$, $\|A_{k+1}\|$, $\omega(A_{k+1})$,

$r_{k+1} = \gamma.\beta_{k+1}$.

(6́) If $r_{k+1} \geq r^*$ then calculate $l_{k+1} = l_k - r_{k+1}$ ($l_1 := -r_1$), take $k := k+1$ and go 5́. step.

(7́) Write as $M := k$ and the lower bound of interval $l^e = l_M$.

Finally, the found values u^e and l^e are combined and these values constitute of the Schur stability interval $\mathcal{I}_{\mathcal{L}}^e = [l^e, u^e]$ of the matrix family $\mathcal{L}(A_1, B)$. Here, the interval $\mathcal{I}_{\mathcal{L}}^e$ preserves the Schur stability of the given matrix family.

Theorem 7 (Generalization of the Theorem 2)**.** *If $A_1 \in S_N$, $B \in M_N(\mathbb{C})$ and $r \in \mathcal{I}_{\mathcal{L}}^e = [l^e, u^e]$ then $\mathcal{L}(A_1, B) \subset S_N$, where u^e and l^e are defined as in Algorithm 1.1 and Algorithm 1.2, respectively.*

Proof. It is clear from the Theorem 2, Algorithm 1.1 and Algorithm 1.2. □

4.2. A method and an algorithm to find the extended interval $\mathcal{I}_{\mathcal{L}}^{*e}$.

4.2.1. *A method.* Keeping the ω^*-Schur stability of the matrix family $\mathcal{L}(A_1, B)$, a method is given to extend the intervals with the ω^*-Schur stable matrix A_1 and the matrix B. $\mathcal{I}_{\mathcal{L}}^* = [\underline{r}, \overline{r}]$ has been chosen with Theorem 5. For $r \in \mathcal{I}_{\mathcal{L}}^*$, the matrices $A(r) = A_r = A_1 + rB$ are ω^*-Schur stable. The stepsize chosen as $r_k = -\frac{\|A_k\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_k\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_k)}}$, the initial value taken as u^*. To extend the upper bound of the intervals $\mathcal{I}_{\mathcal{L}}^*$, the following steps are done,

$$A_k = A_{k-1} + r_{k-1}B \ , \ r_1 = u^* \ , \ k \geq 2 \tag{8}$$

$$r_k = -\frac{\|A_k\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_k\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_k)}}, \tag{9}$$

$$u_k = u_{k-1} + r_k \ , \ u_1 = r_1. \tag{10}$$

On the other hand, to extend the lower bound of the intervals $\mathcal{I}_{\mathcal{L}}^*$, the matrix A_k is taken as $A_k = A_{k-1} - r_{k-1}B$ in the equality (8) and the equality (10) is replaced by the recurrence relation $l_k = l_{k-1} - r_k$, $l_1 = -r_1$. At the end of this process, the upper bound u^{*e} and lower bound l^{*e} of the extended interval $\mathcal{I}_{\mathcal{L}}^{*e}$.

Remark 4. *Let's take A_1 and B.*

$$A_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix} \ , \ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

From the Theorem 5, it is known that $u = 0.748683$ for $\omega^* = 10$. If the method is applied consecutively to get the upper bound, the stepsize is become smaller and the *parameter ω approaches to 10 as in Table 4. A similar situation is also observed for the lower bound. Because of these reasons, the working with very small numbers is non-practical. For this reason, as in Remark 3 for the Algorithm 1.1 and Algorithm 1.2, the stopping parameter r^* is needed for the algorithm to stop.*

TABLE 4. The values r and $\omega(A_k)$ corresponding to the number of steps k

k	*1*	*50*	*100*	*200*	*300*	*380*
r	*0.748683*	*0.00817741*	*0.00310605*	*0.00079299*	*0.000245453*	*9.98825e-005*
$\omega(A_k)$	*1.66462*	*7.23792*	*8.61111*	*9.57978*	*9.86377*	*9.94385*

4.2.2. *An algorithm.* As given in the Remark 4, to stop the calculation, the stopping criterion r^* is used as follow.

Let's give the algorithm to extend the upper bound of the intervals $\mathcal{I}_{\mathcal{L}}^*$.

Algorithm 2.1 (for the upper bound u^{*e})

(1) Input; $A \in S_N\ ,\ B,\ \omega^*,\ r^*$.
(2) Calculate $\omega(A)$.
(3) If $\omega(A) > \omega^*$ then "The matrix A is not ω^*-Schur stable" and finish the algorithm.
(4) Calculate $\|A\|,\ \|B\|,\ u^* = -\frac{\|A\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A)}}$.
(5) Take $k = 1,\ A_1 := A,\ r_1 := u^*,\ u_1 := r_1$.
(6) Calculate;
$$A_{k+1} = A_k + r_k B\ ,\ \|A_{k+1}\|\ ,\ \omega(A_{k+1}),$$
$$r_{k+1} = -\frac{\|A_{k+1}\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_{k+1}\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_{k+1})}}.$$
(7) If $r_{k+1} \geq r^*$ then calculate $u_{k+1} = u_k + r_{k+1}$, take $k := k+1$ and go 6. step.
(8) Write as $M := k$ and the upper bound of interval $u^{*e} = u_M$.

To extend the lower bound of the intervals $\mathcal{I}_{\mathcal{L}}^*$, steps (6)-(8) in Algorithm 2.1 are taken as follow.

Algorithm 2.2 (for the lower bound l^{*e})

(6́) Calculate;
$$A_{k+1} = A_k - r_k B\ ,\ \|A_{k+1}\|\ ,\ \omega(A_{k+1}),$$
$$r_{k+1} = -\frac{\|A_{k+1}\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_{k+1}\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_{k+1})}}.$$
(7́) If $r_{k+1} \geq r^*$ then calculate $l_{k+1} = l_k - r_{k+1}$ ($l_1 := -r_1$), take $k := k+1$ and go 6́. step.
(8́) Write as $M := k$ and the lower bound of interval $l^{*e} = l_M$.

Finally, the found values u^{*e} and l^{*e} are combined and these values constitute of the ω^*-Schur stability interval $\mathcal{I}_{\mathcal{L}}^{*e} = [l^{*e}, u^{*e}]$ of the matrix family $\mathcal{L}(A_1, B)$. Here, the interval $\mathcal{I}_{\mathcal{L}}^{*e}$ preserves the ω^*-Schur stability of the given matrix family.

Theorem 8 (Generalization of the Theorem 5)**.** *If $A_1 \in S_N^*$, $B \in M_N(\mathbb{C})$ and $r \in \mathcal{I}_{\mathcal{L}}^{*e} = [l^{*e}, u^{*e}]$ then $\mathcal{L}(A_1, B) \subset S_N^*$, where u^{*e} and l^{*e} are defined as in Algorithm 2.1 and Algorithm 2.2, respectively.*

Proof. It is clear from the Theorem 5, Algorithm 2.1 and Algorithm 2.2. □

Example 5. *Let us consider the matrices A_1 and B as follow,*

$$A_1^1 = \begin{pmatrix} -0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \; A_1^2 = \begin{pmatrix} 0.2 & 1 \\ 0 & 0.1 \end{pmatrix},$$
$$B_1 = E_{11} + E_{22}, \; B_2 = E_{12}, \; B_3 = E_{11} + E_{12} + E_{22}$$

Here E_{ij} is a real matrix which the element in position (i, j) equals 1 and all other elements are 0.

Let's examine the Table 5 (Table 6). The matrices A_1, B and the parameters r^* and ω^* are the input elements, where r^* and ω^* selected by the users. l (l^*) are the lower bounds and u (u^*) are the upper bounds of the interval $\mathcal{I}_{\mathcal{L}}$ ($\mathcal{I}_{\mathcal{L}}^*$) which is calculated with the help of Theorem 2 (Theorem 5). l^e (l^{*e}) are the lower bounds and u^e (u^{*e}) are the upper bounds of the interval $\mathcal{I}_{\mathcal{L}}^e$ ($\mathcal{I}_{\mathcal{L}}^{*e}$) which is the extended interval obtained by the Algorithm 1.1 (Algorithm 2.1) and Algorithm 1.2 (Algorithm 2.2). M indicates how many steps the algorithms stopped.

Table 5. The computed values for the data A_1, B, r^*

A_1	B	γ	r^*	$\overline{r} = \gamma.u$	u^e	M
A_1^1	B_1	0.9	0.01 0.001	0.81	0.891 0.8991	2 3
		0.95	0.01 0.001	0.855	0.89775 0.899888	2 3
A_1^2	B_1	0.9	0.1 0.01	0.185918	0.320172 0.638475	2 12
		0.95	0.1 0.01	0.196247	0.334315 0.635644	2 11
A_1^2	B_2	0.9	0.1 0.01	0.185918	0.429824 2.35532	3 88
		0.95	0.1 0.01	0.196247	0.448457 2.42436	3 90
A_1^2	B_3	0.9	0.01 0.001	0.114904	0.486169 0.667356	14 77
		0.95	0.01 0.001	0.121287	0.494272 0.66961	14 75

(a) The computed values $\overline{r}$ and u^e

A_1	B	γ	r^*	$\underline{r} = -\gamma.u$	l^e	M
A_1^1	B_1	0.9	0.01 0.001	-0.81	-0.891 -0.8991	2 3
		0.95	0.01 0.001	-0.855	-0.89775 -0.899888	2 3
A_1^2	B_1	0.9	0.1 0.01	-0.185918	-0.663416 -0.932985	4 13
		0.95	0.1 0.01	-0.196247	-0.6847 -0.940808	4 13
A_1^2	B_2	0.9	0.1 0.01	-0.185918	-2.40175 -4.36254	8 94
		0.95	0.1 0.01	-0.196247	-2.45135 -4.42464	8 95
A_1^2	B_3	0.9	0.01 0.001	-0.114904	-1.08201 -1.09759	9 14
		0.95	0.01 0.001	-0.121287	-1.08676 -1.0975	9 13

(b) The computed values $\underline{r}$ and l^e

For example, according to Table 5a (Algorithm 1.1) and Table 5b (Algorithm 1.2), the initial value is obtained as $u = -l = 0.206575$ for the matrices A_1^2, B_2.

- *For $\gamma = 0.9$,*
  - *The extended upper bound is obtained as $u^e = 0.320172$ in 2 steps for $r^* = 0.1$ and $u^e = 0.638475$ in 12 steps for $r^* = 0.01$.*
  - *The extended lower bound is obtained as $l^e = -0.663416$ in 4 steps for $r^* = 0.1$ and $l^e = -0.932985$ in 13 steps for $r^* = 0.01$.*

The extended interval $\mathcal{I}_{\mathcal{L}}^{e} = [l^e, u^e] = [l_{13}, u_{12}] = [-0.932985, 0.638475]$ *is obtained from the Table 5 for the matrices* A_1^2, B_2 *and the parameter* $r^* = 0.01$ *and* $\gamma = 0.9$.

- *For* $\gamma = 0.95$,
  - *The extended upper bound is obtained as* $u^e = 0.334315$ *in* 2 *steps for* $r^* = 0.1$ *and* $u^e = 0.635644$ *in* 11 *steps for* $r^* = 0.01$.
  - *The extended lower bound is obtained as* $l^e = -0.6847$ *in* 4 *steps for* $r^* = 0.1$ *and* $l^e = -0.940808$ *in* 13 *steps for* $r^* = 0.01$.

The extended interval $\mathcal{I}_{\mathcal{L}}^{e} = [l^e, u^e] = [l_{13}, u_{11}] = [-0.940808, 0.635644]$ *is obtained from the Table 5 for the matrices* A_1^2, B_2 *and the parameter* $r^* = 0.01$ *and* $\gamma = 0.95$.

TABLE 6. The computed values for the data A_1, B, ω^*, r^*

A_1	B	ω^*	r^*	u^*	u^{*e}	M
A_1^1	B_1	10	0.01	0.848683	0.848683	-
		100	0.01	0.894987	0.894987	-
A_1^2	B_1	10	0.1	0.165268	0.281339	1
			0.01		0.476099	7
		100	0.1	0.202507	0.341518	1
			0.01		0.615616	9
A_1^2	B_2	10	0.05	0.165268	0.627035	6
			0.005		1.51639	69
		100	0.1	0.202507	0.457834	2
			0.01		2.3151	82
A_1^2	B_3	10	0.01	0.102141	0.344321	8
			0.001		0.403404	24
		100	0.01	0.125156	0.479542	12
			0.001		0.612721	54

(a) The computed values u^* and u^{*e}

A_1	B	ω^*	r^*	l^*	l^{*e}	M
A_1^1	B_1	10	0.01	-0.848683	-0.848683	-
		100	0.01	-0.894987	-0.894987	-
A_1^2	B_1	10	0.1	-0.165268	-0.604153	3
			0.01		-0.769007	8
		100	0.1	-0.202507	-0.694796	3
			0.01		-0.921211	11
A_1^2	B_2	10	0.05	-0.165268	-2.60586	11
			0.005		-3.51925	75
		100	0.1	-0.202507	-2.48489	7
			0.01		-4.31782	87
A_1^2	B_3	10	0.01	-0.102141	-1.03721	8
			0.001		-1.04426	10
		100	0.01	-0.125156	-1.07752	7
			0.001		-1.08955	10

(b) The computed values l^* and l^{*e}

On the other hand, according to Table 6a (Algorithm 2.1) and Table 6b (Algorithm 2.2), if the parameter ω^* *is chosen as* 10, *the initial value is obtained as* $u^* = -l^* = 0.165268$ *for the matrices* A_1^2, B_2. *If the stopping parameter* r^* *is chosen as* $r^* = 0.05$ $(r^* = 0.005)$,

- *the extended upper bound is obtained as* $u^{*e} = 0.627035$ $(u^{*e} = 1.51639)$ *in 6 (69) steps.*
- *the extended lower bound is obtained as* $l^{*e} = -2.60586$ $(l^{*e} = -3.51925)$ *in 11 (75) steps.*

The extended interval $\mathcal{I}_{\mathcal{L}}^{*e} = [l^{*e}, u^{*e}] = [-2.60586, 0.627035]$ *is obtained from the Table 6 for the matrices* A_1^2, B_2 *and the parameters* $\omega^* = 10$, $r^* = 0.05$.

According to the Table 5 and the Table 6, let's give the following;

- *The interval* $\mathcal{I}_{\mathcal{L}}^{e}$ *is bigger than the interval* $\mathcal{I}_{\mathcal{L}}^{*e}$ *with same condition.*
- *The number of steps increases while the stopping parameter decreases.*
- *If the matrices* A_1 *and* B *are taken diagonal, the extended intervals are obtained by the theorems.*
- *If the parameter* ω^* *is chosen bigger, the extended interval* $\mathcal{I}_{\mathcal{L}}^{*e}$ *is obtained bigger in the same conditions.*

4.3. **Methods and algorithms to find the extended interval $\mathcal{I}_{\mathcal{C}}^{e}$ and $\mathcal{I}_{\mathcal{C}}^{*e}$.** The methods and the algorithms can be given to extend the intervals $\mathcal{I}_{\mathcal{C}}$ and $\mathcal{I}_{\mathcal{C}}^{*}$ as similar to the methods and algorithms to extend the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{L}}^{*}$ in Section 4. So, in this paper, the methods and the algorithms to find the intervals $\mathcal{I}_{\mathcal{C}}^{e}$ and $\mathcal{I}_{\mathcal{C}}^{*e}$ won't be given to avoid repeat.

5. Conclusion

In this study, the matrix families $\mathcal{L}$ and $\mathcal{C}$ based on linear sum and convex combination were constructed, respectively. This construction is a new approach that preserves the Schur stability of the matrix families. The intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ that make these matrix families Schur stable were determined in the Theorem 2 and Theorem 3 and supported by the illustrative examples. Here it is seen that the sharp results are obtained from the Theorem 2 and Theorem 3, especially in the Example 1 and Example 2, for the matrix families $\mathcal{L}$ and $\mathcal{C}$. Similarly, the intervals $\mathcal{I}_{\mathcal{L}}^{*}$ and $\mathcal{I}_{\mathcal{C}}^{*}$ that provide $\omega^{*}-$Schur stability of the matrix families $\mathcal{L}$ and $\mathcal{C}$ are determined in the Theorem 5 and Theorem 6 and supported with the numerical examples. It is seen that the Theorem 5 and Theorem 6 give sharp results in the Example 3 and Example 4. At the end, the methods and the algorithms are given to extended the intervals $\mathcal{I}_{\mathcal{L}}, \mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{L}}^{*}$ and $\mathcal{I}_{\mathcal{C}}^{*}$. Here, the methods are based on continuity theorems and the algorithms based on the methods. With the help of these theorems, the obtained intervals are extended and the results are presented with the numerical example.

On the other hand, unlike other studies in the literature, this study shows the importance of continuity theorems which guarantee Schur stability. With the help of these theorems, the matrix families are extended in such a way that their Schur stability is preserved. Also, in many studies, the matrices A_1 and B were taken as Schur stable but in this study there is no need for the matrix B to be Schur stable or ω^{*}-Schur stable.

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