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Research Article

A review of radial kernel methods for the resolution of Fredholm integral equations of the second kind

Dedicated to Professor Paolo Emilio Ricci, on occasion of his 80th birthday, with respect and friendship.

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ABSTRACT. The paper presents an overview of the existing literature concerning radial kernel meshfree methods for the numerical treatment of second-kind Fredholm integral equations. More in detail, it briefly recalls radial basis function (RBF) interpolation and cubature to properly describe numerical methods for two-dimensional linear Fredholm equations. The RBF approach allows us to consider the case when the involved functions are not known analytically, but only as vectors of scattered data samples. The described methods do not require any underlying mesh and hence are also independent on the geometry of the domain.

Keywords: Radial basis functions, Fredholm integral equations, meshfree methods, scattered data.

2020 Mathematics Subject Classification: 65D12, 45B05, 65R20.

1. INTRODUCTION

A huge variety of problems arising in mathematical physics, engineering and mechanics can be described by Fredholm integral equations. Some examples are the rendering equation [31, 32] that generalizes a great variety of known rendering algorithms for computer graphics, and Love's equation [35] that arises in the electrostatic problem of a circular plate condenser in an unbounded perfect fluid. Furthermore, solving a related Fredholm equation is the key in broadband dielectric spectroscopy [45] to study molecular dynamics in complex systems such as glass-forming liquids and liquid crystalline materials, the image deblurring problem [30], the diffraction theory [41] and the study of microtearing modes [18] as an explanation for the anomalous electron thermal transport in tokamak experiments. Many numerical procedures to approximate the solution of Fredholm integral equations, such as projection and iterated projection methods, Nyström methods [14, 3, 29, 36, 38], discrete Galerkin methods [4] and Monte Carlo methods [19, 17] are widely available in the current literature. However, most of these methods are based on piecewise approximating polynomials or rely on zeros of orthogonal polynomials. Hence, to guarantee accurate results, they require that all the involved functions should be known in their analytical form or, at least, sampled at the zeros of orthogonal polynomials. Another possible approach is to consider the case where the involved functions are only known as samples over grids of equally spaced points. This research topic has been approached during the last twenty years and led to the development of some numerical methods based on the generalized Bernstein operator [39, 40] and the constrained mock-Chebyshev operator [16], respectively. Nonetheless, it is a common occurrence that in real-life applications the available data are scattered and, consequently, all the previous accurate methods are not

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applicable. In this context, meshless methods, in particular those based on radial basis function (RBF) interpolation [6, 47], are an adequate tool to tackle this problem. RBF interpolation has been extensively studied also to develop methods for the numerical treatment of partial differential equations (PDEs) [20, 21, 8] or cubature formulae to approximate integrals starting from scattered data [9, 12, 46]. It is worth noting that a huge variety of RBFs are scaled in terms of a shape parameter and the task of finding the optimal one to increase the accuracy of the interpolation is still a key topic in the literature [10, 23, 27, 37]. Other approaches can involve the use of variably scaled kernels [5, 43] or the construction of stable bases like RBF-QR [24, 26], Hilbert-Schmidt SVD [13, 22], WSVD [7, 15], and greedy algorithms for the convergence of RBF approximants [48]. More recently, meshless methods based on machine learning have emerged as a novel approach for the numerical treatment of integral equations, offering data-driven techniques that complement traditional strategies [28, 49].

This paper aims to collect recent results about the employment of RBF interpolation in the context of the numerical treatment of Fredholm integral equations of the second kind, focusing on the bivariate case in particular. To our knowledge, the literature about this topic is fragmentary and not so wide. So we decided to gather all the known information about it, intending to work on it in the future. We consider the following two-dimensional Fredholm integral equation of the second kind

$$f(x,y) - \mu \int_\Omega k(x,y,s,t) f(s,t) ds dt = g(x,y), \quad (x,y) \in \Omega, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^2$ is a bi-dimensional bounded domain, f is the unknown solution, g is the right-hand side term, k is the kernel and μ is a non-zero real parameter.

The paper is structured as follows. Section 2 recalls some basic definitions and properties about radial basis functions, while Section 3 is devoted to a RBF cubature rule for scattered data. In Section 4, the available methods for the numerical treatment of Fredholm integral equations are described. Finally, Section 5 concludes the paper.

2. Radial basis functions

Suppose that we are given a compact domain $\Omega \subset \mathbb{R}^2$, a set $X = \{P_1, \dots, P_N\} \subset \Omega$, of scattered data $P_i = (x_i, y_i)$, $i = 1, \dots, N$, and the corresponding data (or function) value set $F = \{f_1, \dots, f_N\} \subset \mathbb{R}$ that is obtained by possibly sampling any (unknown) function $f : \Omega \to \mathbb{R}$.

Let $\kappa : \Omega \times \Omega \to \mathbb{R}$ be a given radial kernel. Starting from κ, we may define a RBF $\phi : \mathbb{R}_+ \to \mathbb{R}$ that is strictly conditionally positive definite (SCPD) of order m by setting

$$\kappa(P, P_i) = \phi_i(P) = \phi(\|P - P_i\|_2) = \phi(r), \tag{2.2}$$

we can find a unique interpolating function $\psi : \Omega \to \mathbb{R}$ of the form

$$\begin{aligned} \psi(P) &= \sum_{i=1}^{N} c_i \kappa(P, P_i) + \sum_{i=N+1}^{N+M} c_i \pi_{i-N}(P) \\ &= \sum_{i=1}^{N} c_i \phi_i(P) + \sum_{i=N+1}^{N+M} c_i \pi_{i-N}(P), \end{aligned} \tag{2.3}$$

where $\{\pi_k\}_{k=1}^{M}$ generate a basis for the $M = \binom{m+1}{m-1}$-dimensional linear space $\mathbb{P}^2_{m-1}$ of bivariate real valued polynomials of total degree less than or equal to $m-1$, and $\|\cdot\|_2$ is the Euclidean norm. The coefficients $c_1, \dots, c_{N+M}$ are determined by enforcing the interpolation conditions

$$\psi(P_i) = f_i, \qquad i = 1, \dots, N.$$

Since these conditions lead to a system of N linear equations in the $N+M$ unknowns c_i, one usually adds the M additional conditions

$$\sum_{i=1}^{N} c_i \pi_k(P_i) = 0, \qquad k = 1, \dots, M,$$

to ensure a unique solution. Moreover, from theory it is known that a SCPD function of order 0 is strictly positive definite (SPD), and so in this case the polynomial part in (2.3) is omitted. Solving the interpolation problem for a SCPD function ϕ of order m leads to a symmetric linear system

$$\mathcal{A}\mathbf{c} = \mathbf{b}, \tag{2.4}$$

where

$$\mathcal{A} = \begin{bmatrix} A & Q \\ Q^T & O \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}.$$

The interpolation matrix $\mathcal{A}$ in (2.4) has entries

$$A_{ij} = \phi(\|P_i - P_j\|_2), \qquad Q_{ik} = \pi_k(P_i), \qquad i,j = 1, \dots, N, \ k = 1, \dots, M,$$

and O is a $M \times M$ zero matrix. Moreover, $\mathbf{c} = [c_1, \dots, c_{N+M}]^T$, $\mathbf{f} = [f_1, \dots, f_N]^T$ and $\mathbf{0}$ is a zero vector of length M. Note that for a SPD RBF ϕ the matrix reduces to $\mathcal{A} = A$, and hence the polynomial part vanishes. Recalling that every SCPD kernel has an associated normalized PD kernel [44], from now on we confine our treatise to the case of SPD kernels.

In the literature, many kernels are dependent on a shape parameter $\epsilon > 0$ such that, by recalling (2.2), it is

$$\kappa_\epsilon(P, P_i) = \phi_\epsilon(\|P - P_i\|_2) = \phi_\epsilon(r) = \phi(\epsilon r).$$

In what follows, to simplify the notation, we refer to $\phi_\epsilon(\|P - P_i\|_2)$ as $\phi_{\epsilon,i}(P)$, $\forall\, P, P_i \in \Omega$, omitting the subscript ϵ when clear from context.

RBF	$\phi_\epsilon(r)$	SCPD order
Gaussian C^∞ (GA)	$e^{-\epsilon^2 r^2}$	0
Inverse quadratic C^∞ (IQ)	$(1+\epsilon^2 r^2)^{-1}$	0
Inverse Multiquadric C^∞ (IMQ)	$(1+\epsilon^2 r^2)^{-1/2}$	0
Multiquadric C^∞ (MQ)	$(1+\epsilon^2 r^2)^{1/2}$	1
Wendland C^2 (W2)	$\max(1-\epsilon r, 0)^4(4\epsilon r + 1)$	0
Wendland C^4 (W4)	$\max(1-\epsilon r, 0)^6(35\epsilon^2 r^2 + 18\epsilon r + 3)$	0
Matérn C^0 (M0)	$e^{-\epsilon r}$	0
Matérn C^2 (M2)	$e^{-\epsilon r}(\epsilon r + 1)$	0

TABLE 1. Some examples of well-known RBFs depending on a shape parameter.

3. A CUBATURE RULE ON SCATTERED DATA

A key role in the numerical treatment of integral equations of the type (1.1) is played by the cubature rule used to approximate the integral operator

$$(\mathcal{I}f)(x,y) = \int_\Omega k(x,y,s,t) f(s,t) ds dt. \tag{3.5}$$

A cubature rule for integrals of the type

$$\int_\Omega h(P)dP,$$

where $h(P)$ is an integrable function, can be obtained by replacing the function h with its RBF interpolant (2.3), i.e.

$$\int_\Omega h(P)dP \approx \int_\Omega \psi(P)dP = \sum_{i=1}^{N} c_i \int_\Omega \phi_i(P)dP + \sum_{i=N+1}^{N+M} c_i \int_\Omega \pi_{i-N}(P)dP. \tag{3.6}$$

Denoting by

$$\mathbf{I} = \left[\int_\Omega \phi_1(P)dP, \ldots, \int_\Omega \phi_N(P)dP, \int_\Omega \pi_1(P)dP, \ldots, \int_\Omega \pi_M(P)dP\right]^T,$$

the moment vector and by $< \cdot, \cdot >$ the scalar product in $\mathbb{R}^2$, we have

$$\int_\Omega h(P)dP \approx \int_\Omega \psi(P)dP =< \mathbf{c}, \mathbf{I} >=< \mathcal{A}^{-1}\mathbf{b}, \mathbf{I} >=< \mathbf{f}, \mathbf{w} >= \sum_{i=1}^{N} w_i f_i. \tag{3.7}$$

In fact, by symmetry of the matrix $\mathcal{A}$ and noting that $\mathcal{A}\mathbf{w} = \mathbf{I}$, the cubature formula (3.6) can be easily rewritten in the usual form of a weighted sum of the sampled values.

By employing the cubature rule (3.7), the integral operator (3.5) is approximated as follows:

$$(\mathcal{I}f)(x, y) \approx \sum_{i=1}^{N} w_i k(x, y, x_i, y_i) f(x_i, y_i). \tag{3.8}$$

The accuracy of the rule can be improved by choosing the optimal RBF shape parameter. For a comprehensive study of the influence of the shape parameter scaling on the accuracy of the RBF interpolation the reader can refer to [34] and the references therein.

A technique to estimate the optimal shape parameter is the *leave one out cross validation* (LOOCV) method. The strategy underlying this algorithm is to minimize a cost function that collects the errors for a sequence of partial fits to the data. More in detail, the data are split into two sets:

- a *training set* of $N - 1$ data to obtain a "partial" interpolation;
- a *validation set* with the remaining data to compute the error.

After iterating this procedure N times (one for each given data), the cost function uses the so-obtained vector of error estimates to determine the optimal shape parameter. More details about the RBF moments computation and the LOOCV method can be found in [12, 11, 42].

4. Methods for Fredholm integral equations

4.1. **Method 1.** The discrete collocation method introduced in [2] assumes that the unknown solution of the Fredholm integral equation (1.1) is approximated by

$$\widetilde{f}(x, y) = \sum_{j=1}^{N} c_j \phi_j(x, y),$$

and hence consider the equation

$$R(x, y) = \widetilde{f}(x, y) - \mu \int_\Omega k(x, y, s, t)\widetilde{f}(s, t)dsdt - g(x, y),$$

where $R(x,y)$ is the residual of the RBF approximation. Then, collocating at the points $(x_i,y_i) \in X \subset \Omega$ and requiring that

$$R(x_i,y_i) = 0, \quad \forall\, i = 1,\dots,N,$$

the following system is obtained

$$\sum_{j=1}^{N} \left[\phi_j(x_i,y_i) - \mu \int_\Omega k(x_i,y_i,s,t)\phi_j(s,t)dsdt \right] c_j = g(x_i,y_i), \quad i = 1,\dots,N. \tag{4.9}$$

To approximate the integrals in (4.9), the authors in [2] introduce a composite Gauss-Legendre cubature rule in the case the domain Ω is of the type

$$\Omega = \{(x,y) \in \mathbb{R}^2 \ : \ a \le x \le b, \ \alpha(x) \le y \le \beta(x)\}, \tag{4.10}$$

with $a,b \in \mathbb{R}$ and α, β continuous functions of x. They also consider the case when Ω is union of domains of the type (4.10).

The rule is achieved recalling that for any function $h(x,y)$ continuous on Ω of the type (4.10), we have

$$\int_\Omega h(x,y)dxdy = \int_a^b \int_{\alpha(x)}^{\beta(x)} h(x,y)dxdy = \int_a^b H(x)dx,$$

with

$$H(x) = \int_{\alpha(x)}^{\beta(x)} h(x,y)dy.$$

Hence, applying a composite m_N-point Gauss-Legendre quadrature rule first to the integral $H(x)$ and then to the integral $\int_a^b H(x)dx$ (both conveniently shifted into the interval $[-1,1]$), we get

$$\int_\Omega k(x,y,s,t)\phi_j(s,t)dsdt \approx \mathcal{G}_{m_N}^M(x,y),$$

$$\mathcal{G}_{m_N}^M(x,y) := \frac{1}{2M} \sum_{h=1}^{M} \sum_{k=1}^{m_N} w_k \frac{\Delta t(\theta_k^h)}{2} \sum_{\ell=1}^{M} \sum_{\iota=1}^{m_N} w_\iota k(x,y,\theta_k^h,\eta_\iota^\ell)\phi_j(\theta_k^h,\eta_\iota^\ell),$$

where

$$\begin{aligned} \Delta t(s) &= \frac{\beta(s)-\alpha(s)}{M}, \\ \theta_k^h &= \frac{1}{M}\left[\frac{z_k+2h-1}{2}\right], \quad h = 1,\dots,M, \quad k = 1,\dots,m_N, \\ \eta_\iota^\ell &= \frac{1}{M}\left[\frac{z_\iota+2\ell-1}{2}\right], \quad \ell = 1,\dots,M, \quad \iota = 1,\dots,m_N, \end{aligned} \tag{4.11}$$

M is the number of subintervals employed and $\{z_k,w_k\}_{k=1}^{m_N}$ are the zeros and weights of the m_N-point Gauss-Legendre rule, respectively.

Finally, the linear system is

$$\sum_{j=1}^{N} \left[\phi_j(x_i,y_i) - \mu \mathcal{G}_{m_N}^M(x_i,y_i) \right] c_j = g(x_i,y_i), \quad i = 1,\dots,N. \tag{4.12}$$

4.2. **Method 2.** The spectral meshless radial point interpolation (SMRPI) method has been proposed as a combination of meshless methods and spectral collocation techniques. More specifically, the point interpolation method with the help of RBFs is used to construct shape functions Φ_i so that

$$\Phi_i(P_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 1, \dots, n. \tag{4.13}$$

Moreover, the shape functions satisfy the following partition of the unity property

$$\sum_{i=1}^{n} \Phi_i(P) = 1. \tag{4.14}$$

In [25], the authors assume to work only with SPD RBFs. Hence, the function f at a point of interest $P \in \Omega$ is expressed in the form

$$f(P) \approx \psi(P) = \sum_{i=1}^{n} c_i \phi_i(P). \tag{4.15}$$

The idea is that here the coefficients c_i are determined taking into consideration a support domain for the point of interest that includes $n < N$ nodes of Ω. More precisely, the coefficients are determined by solving a linear system of the type (2.4), obtained by requiring that equation (4.15) is satisfied at the n nodes surrounding the point of interest P in the support domain. Note that the support domain $D_P \subset \Omega$ is usually a disk of radius r_P and centred at the considered point of interest P.

Consequently, we have the following linear system

$$A_n \mathbf{c}_n = \mathbf{f}_n,$$

with $\mathbf{c}_n = [c_1, \dots, c_n]^T$ and $\mathbf{f}_n = [f_1, \dots, f_n]^T$ and

$$A_n = \begin{pmatrix} \phi_1(P_1) & \phi_1(P_2) & \dots & \phi_1(P_n) \\ \phi_2(P_1) & \phi_2(P_2) & \dots & \phi_2(P_n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_n(P_1) & \phi_n(P_2) & \dots & \phi_n(P_n) \end{pmatrix}.$$

Since A_n is a symmetric positive definite matrix, it follows that

$$\mathbf{c}_n = A_n^{-1} \mathbf{f}_n. \tag{4.16}$$

Setting $\mathbf{r}(P) = [\phi_1(P), \dots, \phi_n(P)]^T$, equation (4.15) can be rewritten as

$$f(P) \approx \mathbf{r}^T(P)\mathbf{c}.$$

By (4.16), we have

$$\psi(P) = \mathbf{r}^T(P) A_n^{-1} \mathbf{f}_n = \mathbf{\Phi}^T(P) \mathbf{f}_n = \sum_{i=1}^{n} f_i \Phi_i(P), \tag{4.17}$$

being $\mathbf{\Phi}^T(P) = \mathbf{r}^T(P) A_n^{-1} = [\Phi_1(P), \dots, \Phi_n(P)]$. The n functions in the vector $\mathbf{\Phi}(P)$ are called the radial point interpolation method (RPIM) *shape functions* corresponding to the nodal displacements and satisfy the properties (4.13) and (4.14).

Remark 4.1. *We point out that the number n of nodes included in the support domain D_P clearly depends on the point of interest P. When necessary, to underline this dependence we write it as n_P. Moreover, we can rewrite the RPIM approximation of f as follows*

$$\psi(P) = \sum_{j=1}^{N} f_j \Phi_j(P),$$

noting that for any

$$P_j \in D_P^c := \{P_j \ : \ P_j \notin D_P\}$$

we have $\Phi_j(P) = 0$.

The SMRPI method is constructed by replacing the unknown solution $f(x,y)$ in (1.1) with its RPIM approximation

$$\psi(x,y) = \sum_{j=1}^{N} f_j \Phi_j(x,y). \tag{4.18}$$

This leads to

$$\sum_{j=1}^{N} \Phi_j(x,y) f_j - \mu \sum_{j=1}^{N} f_j \int_\Omega k(x,y,s,t)\Phi_j(s,t)dsdt = g(x,y).$$

By Remark 4.1, the above integral can be rewritten as follows

$$\sum_{j=1}^{N} \Phi_j(x,y) f_j - \mu \sum_{j=1}^{N} f_j \int_{D_{P_j}} k(x,y,s,t)\Phi_j(s,t)dsdt = g(x,y), \tag{4.19}$$

where D_{P_j} is the support domain of the point $P_j = (x_j, y_j)$. Collocating equation (4.19) at the points of $X \subset \Omega$, we obtain the following system of order N

$$\sum_{j=1}^{N} \left[\Phi_j(x_i,y_i) - \mu G(x_i,y_i)\right] f_j = g(x_i,y_i), \quad i = 1,\dots,N.$$

Approximating the integral by a suitable Gaussian cubature rule

$$G_j(x_i,y_i) \approx \int_{D_{P_j}} k(x_i,y_i,s,t)\Phi_j(s,t)dsdt, \quad i = 1,\dots,N, \tag{4.20}$$

and recalling the Kronecker delta property (4.13), we finally obtain the linear system

$$\sum_{j=1}^{N} \left[\delta_{ij} - \mu G_j(x_i,y_i)\right] f_j = g(x_i,y_i), \quad i = 1,\dots,N, \tag{4.21}$$

that can also be written in the following matrix form

$$(\mathbf{I} - \mu\mathbf{G})\mathbf{f} = \mathbf{g},$$

where $\mathbf{I}$ is the identity matrix of order N, the square matrix $\mathbf{G}$ has entries $G_j(x_i,y_i)$, $i,j = 1,\dots,N$, $\mathbf{g} = [g_1,\dots,g_N]^T$ with $g_i = g(x_i,y_i)$, $i = 1,\dots,N$, and $\mathbf{f} = [f_1,\dots,f_N]^T$ with $f_i = fx_i,y_i)$, $i = 1,\dots,N$.

4.3. **Method 3.** The method introduced in [1] is based on hybrid kernels. A *hybrid kernel* is a combination of radial kernels, usually one infinitely smooth depending on a shape parameter along with one of finite smoothness that is shape-parameter-free, in order to combine their benefits and reduce the respective drawbacks. It is known that infinitely smooth kernels generally have a high order of convergence while their stability usually decreases by decreasing the shape parameter and increasing the number of points. Conversely, kernels of finite smoothness often provide lower accuracy and convergence order but better stability in the computations.

RBF	$\phi(r)$
Thin Plate Spline (TPS)	$r^2 \log r$
Cubic (CU)	r^3
Polyharmonic (P7)	r^7

TABLE 2. Some examples of shape-parameter-free RBFs.

A family of hybrid radial kernels (HRKs) is constructed as follows:

$$\Psi_{\epsilon,i}(P) = \phi_{\epsilon,i}(P) + \rho\phi_i(P), \qquad i = 1, \dots, N, \tag{4.22}$$

where $\phi_{\epsilon,i}$ is an infinitely smooth radial kernel with shape parameter ϵ, ϕ_i is a piecewise smooth shape-parameter-free radial kernel and $\rho \in \mathbb{R}$ is the HRK weight parameter. The shape parameter ϵ and the weight parameter ρ control the accuracy and the stability of the HRK interpolation of the function f that is given by

$$\widetilde{\psi}(P) = \sum_{i=1}^{N} \tilde{c}_i \Psi_{\epsilon,i}(P) + \sum_{i=N+1}^{N+M} \tilde{c}_i \pi_{i-N}(P). \tag{4.23}$$

The coefficients $\tilde{c}_1, \dots, \tilde{c}_{N+M}$ are also in this case determined by enforcing the interpolation conditions

$$\widetilde{\psi}(P_i) = f_i, \qquad i = 1, \dots, N,$$

and the M additional conditions

$$\sum_{i=1}^{N} \tilde{c}_i \pi_k(P_i) = 0, \qquad k = 1, \dots, M,$$

with the polynomial part in (4.23) omitted if dealing with SPD hybrid radial kernels. Consequently, we obtain the following symmetric linear system

$$\mathcal{H}\tilde{\mathbf{c}} = \mathbf{b}, \tag{4.24}$$

where

$$\mathcal{H} = \begin{bmatrix} H & Q \\ Q^T & O \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}.$$

with

$$H_{ij} = \Psi_{\epsilon,i}(P_j), \qquad Q_{ik} = \pi_k(P_i), \qquad i, j = 1, \dots, N, \ k = 1, \dots, M,$$

$\tilde{\mathbf{c}} = [\tilde{c}_1, \dots, \tilde{c}_{N+M}]^T$ the HRK coefficient vector and $\mathbf{f} = [f_1, \dots, f_N]^T$ the scattered data vector. We recall that for a SPD HRK $\Psi_{\epsilon,i}$ the matrix in (4.24) reduces to $\mathcal{H} = H$.

The HRK method is constructed by replacing the unknown solution $f(x, y)$ in (1.1) with its HRK approximation

$$\widetilde{\psi}(x, y) = \sum_{i=1}^{N} \tilde{c}_i \Psi_{\epsilon,i}(x, y)$$

in the case of SPD hybrid kernels obtaining

$$\sum_{j=1}^{N}\left[\Psi_{\epsilon,j}(x,y)-\mu\int_{\Omega}k(x,y,s,t)\Psi_{\epsilon,j}(s,t)dsdt\right]\tilde{c}_j=g(x,y).$$

By collocating the previous equation at the points $(x_i,y_i)\in X\subset\Omega$, we get the following system:

$$\sum_{j=1}^{N}\left[\Psi_{\epsilon,j}(x_i,y_i)-\mu\int_{\Omega}k(x_i,y_i,s,t)\Psi_{\epsilon,j}(s,t)dsdt\right]\tilde{c}_j=g(x_i,y_i),\quad i=1,\ldots,N. \tag{4.25}$$

To approximate the integrals in (4.25), the authors in [1] apply the composite Gauss-Legendre cubature rule that we have recalled in Subsection 4.1

$$\int_{\Omega}k(x,y,s,t)\Psi_{\epsilon,j}(s,t)dsdt\approx\widetilde{\mathcal{G}}^{M}_{m_N}(x,y),$$

$$\widetilde{\mathcal{G}}^{M}_{m_N}(x,y):=\frac{1}{2M}\sum_{h=1}^{M}\sum_{k=1}^{m_N}w_k\frac{\Delta t(\theta^h_k)}{2}\sum_{\ell=1}^{M}\sum_{\iota=1}^{m_N}w_\iota k(x,y,\theta^h_k,\eta^\ell_\iota)\Psi_{\epsilon,j}(\theta^h_k,\eta^\ell_\iota),$$

with $\Delta t(s)$, θ^h_k and η^ℓ_ι defined as in (4.11). Hence, the following linear system is finally obtained

$$\sum_{j=1}^{N}\left[\Psi_{\epsilon,j}(x_i,y_i)-\mu\widetilde{\mathcal{G}}^{M}_{m_N}(x_i,y_i)\right]\tilde{c}_j=g(x_i,y_i),\quad i=1,\ldots,N.$$

Regarding the search for the optimal parameter values of the hybrid kernels, the authors in [1] present a global particle swarm optimization (PSO) method. PSO is a robust algorithm introduced by Kennedy and Eberhart [33] that uses swarm intelligence for solving optimization problems. A swarm in the PSO algorithm comprises a group of particles navigating the problem's search-space, with each particle assigned a fitness value that represents a potential solution to the optimization problem at hand. Key advantages of PSO include its straightforward implementation, information sharing, mutual cooperation among particles, memory utilization, fast convergence, and low computational complexity. However, the conventional PSO algorithm is susceptible to premature convergence and can easily become trapped in local optima. To address these limitations, the authors in [1] introduced some modifications to the standard PSO algorithm to enhance its performance and the reader can refer to their article for further details about it.

4.4. **Advantages and drawbacks.** The three methods presented in this review paper all rely on RBFs and, consequently, face similar challenges, particularly in selecting an optimal shape parameter. At first glance, the hybrid kernel method might seem preferable, as it combines the strengths of two different RBFs: one dependent on a shape parameter and another that is shape-parameter-free. However, this introduces a new challenge: optimizing both the shape parameter and the weight parameter, ρ, which balances the contributions of the two RBFs. While algorithms such as LOOCV and PSO can optimize both parameters, the computational effort increases because these algorithms must now handle two parameters instead of one. In contrast, the SMRPI method is more computationally expensive due to the construction of the shape functions Φ_i, as described in (4.17), whereas the discrete collocation method is the simplest to implement. Regarding the cubature rule used to approximate the integral in (1.1), all three methods rely on Gaussian-type rules. Despite these differences, it is important to note that all three methods are reliable and exhibit comparable performance as far as we know.

5. Conclusions and future research

In this paper, we reviewed three methods for the numerical treatment of Fredholm integral equations of the second kind using RBFs. The described methods are meshless, independent of the geometry of the domain, and reduce the problem to solving a linear system of algebraic equations. To our knowledge, this research topic has not yet been fully explored. Our future goal is to further engage in this field by introducing Nyström-type methods, which, as far as we know, have not yet been considered. Additionally, we plan to carry out a comparison with the methods discussed in this review paper, providing multiple numerical examples. This work could help strengthen the connections between RBF interpolation and the theory of integral equations.

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