Adıyaman University Journal of Science

https://dergipark.org.tr/en/pub/adyujsci

DOI: 10.37094/adyujsci.493589

ADYUJSCI

9 (2) (2019)
342-351

On The Characterizations of Timelike Curves in R_2^4

M. Aykut AKGÜN

Adıyaman University, Technical Sciences Vocational High School, Adıyaman, Turkey

muslumakgun@adiyaman.edu.tr, ORCID: 0000-0002-8414-5228

Abstract

In this paper by establishing the Frenet frame $\{T, N, B_1, B_2\}$ for a timelike curve we study the different position vectors of timelike curves in Semi-Euclidean space R_2^4. We gave the position vectors of timelike curves in terms of curvature functions which lie on the three dimensional subspaces of R_2^4.

Keywords: Timelike curve, Frenet frame, Semi-Euclidean space

R_2^4 deki Timelike Eğrilerin Karakterizasyonları Üzerine

Özet

Bu çalışmada R_2^4 semi-Öklidyen uzayda bir timelike eğrinin farklı yer vektörleri $\{T, N, B_1, B_2\}$ Frenet çatısı kullanılarak çalışılmıştır. R_2^4 uzayının 3-boyutlu alt uzaylarında yatan timelike eğrilerin yer vektörleri araştırılmış ve bu yer vektörleri eğrinin eğrilik fonksiyonları türünden ifade edilmiştir.

Anahtar Kelimeler: Timelike eğri, Frenet çatısı, Semi-Öklidyen uzay

1. Introduction

The classical differential geometry of curves have been studied by several authors. Fernandez, Gimenez and Lucas introduced a reference along a null curve in an n-

* Corresponding Author

Received: 07 December 2018 Accepted: 03 December 2019

dimensional Lorentzian space with the minimum number of curvatures [1]. Çöken and Çiftçi reconstructed the Cartan frame of a null curve in Minkowski spacetime for an arbitrary parameter, and they characterized pseudo-spherical null curves and Bertrand null curves in R_1^4 [2]

İlarslan and Boyacıoğlu studied position vectors of a timelike and a null helice in R_1^3 [3]. İlarslan and Nesovic gave the necessary and sufficient conditions for null curves in E_1^4 to be osculating curves in terms of their curvature functions [4].

İlarslan studied spacelike curves with different normal vectors in Minkowski space R_1^3 [5]. İlarslan, Nesovic and Petrovic-Torgasev characterized rectifying curves in the R_1^3 [6].

Ali and Önder characterized rectifying spacelike curves with curvature functions in Minkowski spacetime [7].

Keleş, Perktaş and Kılıç studied Biharmonic Curves in LP-Sasakian Manifolds [8]. Akgün and Sivridağ gave some theorems for null Cartan curves in 4-dimensional Minkowski space [9]. Also, in [10], Akgün and Sivridağ studied the spacelike curves of Minkowski 4-space.

In that work, some basic knowledge about curves in R_2^4 is given in the second section. The original part of this paper is the third section. In the third section we gave the conditions for timelike curves to lie on subspaces of R_2^4 and gave theorems for such curves.

2. Preliminaries

Let R_2^4 denotes semi-Euclidean 4-space together with two index metric $\langle , \rangle$ of signature $(-,-,+,+)$. A vector X is called timelike if $\langle X, X\rangle < 0$, spacelike if $\langle X, X\rangle > 0$ and null (lightlike) if $\langle X, X\rangle = 0$ and $X \neq 0$, respectively. The norm of a vector $X \in R_2^4$ is denoted by $\| X \|$ and defined by $\| X \| = \sqrt{|\langle X, X\rangle|}$.

A curve α in R_2^4 is called a null curve if $\langle \alpha'(s), \alpha'(s)\rangle = 0$ and $\alpha'(s) \neq 0$, timelike curve if $\langle\alpha'(s),\alpha'(s)\rangle < 0$ and spacelike curve if $\langle\alpha'(s),\alpha'(s)\rangle > 0$, for $\forall s \in R$.

Let α be a timelike curve in R_2^4 with the Frenet frame $\{T,N,B_1,B_2\}$ and let N be a spacelike vector, B_1 and B_2 be null vectors. In this case there exists a unique Frenet frame $\{T,N,B_1,B_2\}$ for the timelike curve $\alpha(s)$ with Frenet equations [11]

$$\begin{aligned} \nabla_T T &= k_1 N, \\ \nabla_T N &= -k_1 T - k_2 B_1, \\ \nabla_T B_1 &= -k_2 N - k_3 B_2, \\ \nabla_T B_2 &= k_3 B_1, \end{aligned} \tag{1}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying

$$\langle B_1,B_2\rangle = \langle N,N\rangle = 1, \langle B_1,B_1\rangle = \langle B_2,B_2\rangle = 0, \langle T,T\rangle = -1. \tag{2}$$

3. On The Timelike Curves in R_2^4

In this section we will give the conditions under which the timelike curves lie subspaces of R_2^4.

Let α be a timelike curve in R_2^4 with the Frenet frame $\{T,N,B_1,B_2\}$. Then, the 3-dimensional subspaces of R_2^4 are spanned by $\{T,N,B_1\}$, $\{T,N,B_2\}$, $\{T,B_1,B_2\}$ and $\{N,B_1,B_2\}$.

Case 1. Let the timelike curve α lies on the space Span$\{T,N,B_1\}$. We can write

$$\alpha(s) = \lambda(s)T + \mu(s)N + \gamma(s)B_1, \tag{3}$$

for differentiable functions λ, μ and γ of the parameter s. If we Differentiate (3.1) we have

$$\begin{aligned} \alpha'(s) = (\lambda'(s) - \mu(s)k_1(s))T + (\lambda(s)k_1(s) + \mu'(s) - \gamma(s)k_2(s))N \\ +(-\mu(s)k_2(s) + \gamma'(s))B_1 + \gamma(s)k_3(s)B_2. \end{aligned} \tag{4}$$

From (4) we have

$$\begin{cases} \lambda'(s) - \mu(s)k_1(s) = 1, \\ \lambda(s)k_1(s) + \mu'(s) - \gamma(s)k_2(s) = 0, \\ -\mu(s)k_2(s) + \gamma'(s) = 0, \\ \gamma(s)k_3(s) = 0. \end{cases} \tag{5}$$

If $\gamma(s) = 0$ we see that $\mu(s)k_2(s) = 0$. From this result if $\mu(s) = 0$ we find $\lambda(s) = s + c$. So we find

$$\alpha(s) = (s + c)T. \tag{6}$$

If $k_2(s) = 0$, we find the equations

$$\lambda'(s) - \mu(s)k_1(s) = 1, \tag{7}$$

$$\lambda(s)k_1(s) + \mu'(s) = 0.$$

From (7) we find the differential equation

$$\frac{d}{ds}\left(\frac{1}{k_1(s)}\frac{d\mu(s)}{ds}\right) + \mu(s)k_1(s) = -1. \tag{8}$$

By using exchange variable $t = \int_0^s k_1(s)ds$ in (3.6) we have

$$\frac{d^2\mu(s)}{ds^2} + \mu(s) = -1. \tag{9}$$

The general solution of (9) is

$$\mu(s) = c_1 cost + c_2 sint - 1, \tag{10}$$

where $c_1, c_2 \in R$. Replacing variable $t = \int_0^s k_1(s)ds$ in (10) we obtain

$$\mu(s) = c_1 cos \int_0^s k_1(s)ds + c_2 sin \int_0^s k_1(s)ds - 1. \tag{11}$$

From the equation (7) we have

$$\lambda(s) = -c_1 sin \int_0^s k_1(s)ds + c_2 sin \int_0^s k_1(s)ds. \tag{12}$$

So we have

$$\alpha(s) = \left(-c_1 sin \int_0^s k_1(s)ds + c_2 sin \int_0^s k_1(s)ds\right)T \tag{13}$$

$$+\left(c_1 cos \int_0^s k_1(s)ds + c_2 sin \int_0^s k_1(s)ds - 1\right)N.$$

If $k_3(s) = 0$, then we have

$$\begin{cases} \lambda'(s) - \mu(s)k_1(s) = 1, \\ \lambda(s)k_1(s) + \mu'(s) - \gamma(s)k_2(s) = 0, \\ -\mu(s)k_2(s) + \gamma'(s) = 0. \end{cases} \tag{14}$$

From (14) we find the differential equation

$$\mu'' + \mu(s)(k_1^2(s) - k_2^2(s)) = -k_1(s). \tag{15}$$

Here $k_1(s)$ and $k_2(s)$ are nonzero constants. From (15) we find

$$\mu(s) = c_1 cos\sqrt{(k_1^2(s) - k_2^2(s))s} + c_2 sin\sqrt{(k_1^2(s) - k_2^2(s))s} + \frac{k_1(s)}{k_1^2(s)-k_2^2(s)}. \tag{16}$$

If we use the equation $\lambda'(s) + \mu(s)k_1(s) = 1$ we have

$$\lambda(s) = \left(\frac{k_1(s)}{k_1^2(s)-k_2^2(s)} + 1\right)s \tag{17}$$

$$-\frac{k_1(s)}{\sqrt{(k_1^2(s) - k_2^2(s))s}}\left(c_2 cos\sqrt{(k_1^2(s) - k_2^2(s))s} + c_1 sin\sqrt{(k_1^2(s) - k_2^2(s))s}\right).$$

From the equation $\mu(s)k_2(s) + \gamma'(s) = 0$ we find

$$\gamma(s) = \frac{k_1(s)}{k_1^2(s)-k_2^2(s)}s \tag{18}$$

$$-\frac{k_2(s)}{\sqrt{(k_1^2(s) - k_2^2(s))s}}\left(c_2 cos\sqrt{(k_1^2(s) - k_2^2(s))s} - c_1 sin\sqrt{(k_1^2(s) - k_2^2(s))s}\right).$$

So we can give the following theorem:

Theorem 1. *Let α be a timelike curve in R_2^4. If the curve α lies on the subspace spanned by $\{T, N, B_1\}$, then it is one of the following forms*

$$(i)\ \ \alpha(s) = (s+c)T$$

or

$$(ii)\ \ \alpha(s) = \left(-c_1 sin \int_0^s k_1(s)ds + c_2 sin \int_0^s k_1(s)ds\right)T$$

$$+\left(c_1 cos \int_0^s k_1(s)ds + c_2 sin \int_0^s k_1(s)ds - 1\right)N$$

where $k_2(s) = 0$ or

$$(iii)\ \ \alpha(s) = \Bigg[\left(\frac{k_1(s)}{k_1^2(s)-k_2^2(s)} + 1\right)s$$

$$-\frac{k_1(s)}{\sqrt{(k_1^2(s)-k_2^2(s))s}}\left(c_2 cos\sqrt{(k_1^2(s)-k_2^2(s))s} + c_1 sin\sqrt{(k_1^2(s)-k_2^2(s))s}\right)\Bigg]T$$

$$+\left[c_1 cos\sqrt{(k_1^2(s)-k_2^2(s))s} + c_2 sin\sqrt{(k_1^2(s)-k_2^2(s))s} - \frac{k_1(s)}{k_1^2(s)-k_2^2(s)}\right]N$$

$$+\Bigg[\frac{k_1(s)}{k_1^2(s)-k_2^2(s)}s$$

$$-\frac{k_2(s)}{\sqrt{(k_1^2(s)-k_2^2(s))s}}\left(c_2 cos\sqrt{(k_1^2(s)-k_2^2(s))s} - c_1 sin\sqrt{(k_1^2(s)-k_2^2(s))s}\right)\Bigg]B_1,$$

where $k_1(s)$ and $k_2(s)$ are nonzero constant.

Case 2. Let the timelike curve α lies on the space Span $\{T, N, B_2\}$. In that case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)N + \gamma(s)B_2. \tag{19}$$

Differentiating (19) we have

$$\alpha'(s) = (\lambda'(s) - \mu(s)k_1(s))T + (\lambda(s)k_1(s) + \mu'(s))N \tag{20}$$

$$+(-\mu(s)k_2(s) + \gamma(s)k_3(s))B_1 + \gamma'(s)B_2.$$

From (20) we have the following equations:

$$\begin{cases} \lambda'(s) - \mu(s)k_1(s) = 1, \\ \lambda(s)k_1(s) + \mu'(s) = 0, \\ \gamma'(s) = 0, \\ -\mu(s)k_2(s) + \gamma(s)k_3(s) = 0. \end{cases} \tag{21}$$

From (21) we see that $\gamma(s) = c$ and $\mu(s) = c\frac{k_3(s)}{k_2(s)}$. If we use these equations in $\lambda(s)k_1(s) + \mu'(s) = 0$, we obtain

$$\lambda(s) = c\frac{k_3(s)k'_2(s) - k'_3(s)k_2(s)}{k_1(s)k_2^2(s)}, \tag{22}$$

So we have

$$\alpha(s) = \left(c\frac{k_3(s)k'_2(s) - k'_3(s)k_2(s)}{k_1(s)k_2^2(s)}\right)T + \left(c\frac{k_3(s)}{k_2(s)}\right)N + cB_2. \tag{23}$$

So we can give the following theorem.

Theorem 2. *Let α be a timelike curve in R_2^4 . If it lies on the subspace spanned by $\{T, N, B_2\}$, then it is in the form*

$$\alpha(s) = \left(c\frac{k_3(s)k'_2(s) - k'_3(s)k_2(s)}{k_1(s)k_2^2(s)}\right)T + \left(c\frac{k_3(s)}{k_2(s)}\right)N + cB_2,$$

where $c \in R$.

Case 3. If we suppose that the timelike curve α lies on the space Span$\{T, B_1, B_2\}$ we can write

$$\alpha(s) = \lambda(s)T + \mu(s)B_1 + \gamma(s)B_2, \tag{24}$$

Differentiating (24) we have

$$\alpha'(s) = \lambda'(s)T + (\lambda(s)k_1(s) - \mu(s)k_2(s))N + (\mu'(s) + \gamma(s)k_3(s))B_1 \tag{25}$$

$$+(\gamma'(s) - \mu(s)k_3(s))B_2.$$

From (25) we have the following equations:

$$\begin{cases}\lambda'(s) = 1,\\ \lambda(s)k_1(s) - \mu(s)k_2(s) = 0,\\ \mu'(s) + \gamma(s)k_3(s) = 0,\\ \gamma'(s) - \mu(s)k_3(s) = 0.\end{cases} \tag{26}$$

From (26) we can write $\lambda(s) = s + c$ and $\mu(s) = \frac{k_1(s)}{k_2(s)}(s + c)$. If we take account the equation $\mu'(s) + \gamma(s)k_3(s) = 0$, we obtain

$$\gamma(s) = \frac{k_1(s)}{k_2(s)k_3(s)} - \frac{k'_1(s)k_2(s) - k_1(s)k'_2(s)}{k_3(s)k_1^2(s)}(s + c). \tag{27}$$

So we have

$$\alpha(s) = (s + c)T + \left(\frac{k_1(s)}{k_2(s)}(-s + c)\right)B_1 + \left(\frac{k_1(s)}{k_2(s)k_3(s)} - \frac{k'_1(s)k_2(s) - k_1(s)k'_2(s)}{k_3(s)k_1^2(s)}(s + c)\right)B_2.$$

Theorem 3. *Let α be a timelike curvein R_2^4 . If it lies on the subspace spanned by $\{T, B_1, B_2\}$, then it is in the form*

$$\alpha(s) = (s + c)T + \left(\frac{k_1(s)}{k_2(s)}(-s + c)\right)B_1 + \left(\frac{k_1(s)}{k_2(s)k_3(s)} - \frac{k'_1(s)k_2(s) - k_1(s)k'_2(s)}{k_3(s)k_1^2(s)}(s + c)\right)B_2.$$

Here c is a constant.

Case 4. Let the timelike curve α lies on the space Span$\{N, B_1, B_2\}$. In that case we have

$$\alpha(s) = \lambda(s)N + \mu(s)B_1 + \gamma(s)B_2. \tag{28}$$

Differentiating (28) we find that

$$\alpha'(s) = \lambda(s)k_1(s)T + (\lambda'(s) - \mu(s)k_2(s))N \tag{29}$$

$$+(\lambda(s)k_2(s) + \mu'(s) - \gamma(s)k_3(s))B_1 + (\gamma'(s) + \mu(s)k_3(s))B_2.$$

From (29) we have the following equations:

$$\begin{cases}\lambda(s)k_1(s) = 1,\\ \lambda'(s) - \mu(s)k_2(s) = 0,\\ -\lambda(s)k_2(s) + \mu'(s) + \gamma(s)k_3(s) = 0,\\ \gamma'(s) - \mu(s)k_3(s) = 0.\end{cases} \tag{30}$$

From (30) we can write $\lambda(s)=\frac{1}{k_1(s)}$ and $\mu(s)=-\frac{k'_1(s)}{k_2(s)k_1^2(s)}$. If we take account the equation $\lambda'(s)-\mu(s)k_2(s)=0$, we obtain

$$\gamma(s)=\frac{k_2(s)}{k_1(s)k_3(s)}+\frac{k''_1(s)k_1(s)k_2(s)-k'_1(s)(2k'_1(s)k_2(s)+k_1(s)k'_2(s))}{k_1^3(s)k_2^2(s)k_3(s)}.$$

So we have

$$\alpha(s)=\left(\frac{1}{k_1(s)}\right)N+\left(-\frac{k'_1(s)}{k_2(s)k_1^2(s)}\right)B_1 \tag{31}$$

$$+\left(\frac{k_2(s)}{k_1(s)k_3(s)}+\frac{k''_1(s)k_1(s)k_2(s)-k'_1(s)(2k'_1(s)k_2(s)+k_1(s)k'_2(s))}{k_1^3(s)k_2^2(s)k_3(s)}\right)B_2.$$

So we can give the theorem.

Theorem 4. *Let α be a timelike curve in R_2^4. If it lies on the subspace spanned by $\{N,B_1,B_2\}$, then it is in the form*

$$\alpha(s)=\left(\frac{1}{k_1(s)}\right)N+\left(-\frac{k'_1(s)}{k_2(s)k_1^2(s)}\right)B_1$$
$$+\left(\frac{k_2(s)}{k_1(s)k_3(s)}+\frac{k''_1(s)k_1(s)k_2(s)-k'_1(s)(2k'_1(s)k_2(s)+k_1(s)k'_2(s))}{k_1^3(s)k_2^2(s)k_3(s)}\right)B_2.$$

References

[1] Fernandez, A., Gimenez, A., Lucas, P., *Null helices in Lorentzian space forms*, International Journal of Modern Physics A, 16, 4845-4863, 2001.

[2] Çöken, A.C., Çiftçi, Ü., *On the Cartan curvatures of a null curve in Minkowski spacetime*, Geometriae Dedicate, 114 71-78, 2005.

[3] İlarslan, K., Boyacıoğlu, Ö., *Position vectors of a timelike and a null helix in Minkowski 3-space*, Chaos, Solitions and Fractals, 38, 1383-1389, 2008.

[4] İlarslan, K., Nesovic, E., *Some characterizations of null osculating curves in the Minkowski space-time*, Proceedings of the Estonian Academy of Sciences, 61(I), 1-8, 2012.

[5] İlarslan, K., *Spacelike normal curves in Minkowski E_1^3*, Turkish Journal of Mathematics, 29, 53-63, 2005.

[6] İlarslan, K., Nesovic, E., Petrovic-Torgasev, M., *Some characterizations of*

rectifying curves in Minkowski 3-space, Novi Sad Journal of Mathematics, 33(2), 23-32, 2003.

[7] Ali, A.T., Önder, M., *Some characterizations of rectifying spacelike curves in the Minkowski space-time*, Global Journal of Science Frontier Research Mathematics, 12(1), 2249-4626, 2012.

[8] Keleş, S., Yüksel Perktaş S., Kılıç, E., *Biharmonic curves in lp-sasakian manifolds*, Bulletin of the Malasyian Mathematical Society, 33(2), 325-344, 2010.

[9] Akgün, M.A., Sivridağ, A.İ., *On the null Cartan curves of* R_1^4, Global Journal of Mathematics, 1(1), 41-50, 2015.

[10] Akgün, M.A., Sivridağ, A.İ., *On the spacelike curves of* R_1^4, British Journal of Mathematics and Computer Science, 15(6), 1-10, 2016.

[11] İlarslan, K., Nesovic, E., *Spacelike and timelike normal curves*, Publications de l'Institut Mathematique, 85(99), 111-118, 2009.