Improved New Conditions for Qualitative Behaviors of Time-varying Delay Differential Systems

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ABSTRACT

This study is devoted to a few qualitative behaviors of solutions of continuous-time systems of DDEs. Here, certain the continuous-time delay unperturbed and perturbed systems of DDEs are considered, respectively. The qualitative behaviors of solutions of the considered equations are studied based on the L-K-F approach. Firstly, an L-K-F is defined, and then by aid of this L-K-F, three new theorems, Theorems 1-3, which have improved and adequate conditions for asymptotic stability (AS), integrability and bounded solutions, are proved. Two specific examples are also offered, which illustrate the efficiency of the L-K-F approach. The new results of this study are also more general than that obtained in the past literature.

Keywords: System, Delay differential equations, Asymptotic stability, Integrability, Boundedness, Lyapunov-Krasovskii functional approach

Değişken Gecikmeli Diferansiyel Sistemlerin Niteliksel Davranışları için Geliştirilmiş Yeni Şartlar

ÖZ

Bu makalede, birinci mertebeden gecikmeli diferansiyel denklemlerin sürekli zaman sistemlerinin çözümlerinin bazı niteliksel analizleri ele alınmaktadır. Burada belli formda sürekli zaman gecikmeli pertürbe ve pertürbe olmayan diferansiyel denklemler sistemleri sırasıyla göz önüne alınmaktadır. Bu sistemlerin çözümlerinin asimptotik kararlılık, integrallenebilirlik ve sınırlılık davranışları Lyapunov-Krasovskii fonksiyonel metodu yardımıyla incelenmektedir. Elden edilen sonuçların uygulanabilirliğini göstermek için iki örnek verilmiştir. Verilen yeni sonuçlar, geçmiş literatürde elde edilmiş sonuçlardan daha genel niteliktedir.

Anahtar Kelimeler: Sistem, Gecikmeli diferansiyel denklemler, Asimptotik kararlılık, Integrallenebilirlik, Sınırlılık, Lyapunov-Krasovskii fonksiyonel yaklaşımı

INTRODUCTION

As it is known, in many applied science fields, including engineering, medicine, economics, artificial neural networks, control theory, mechanics, electricity and more, differential equations correspond to the mathematical models denoting of various problems in that areas. A significant type of these differential equations is known as delay differential equations (DDEs). Stability problems of solutions of that kind equations in the Lyapunov sense have a significance place in applications. However, in generally solving DDEs explicitly is a very hard task, indeed sometimes it is impossible to find explicit solutions except numerically. Across that kind of difficulties, the Lyapunov-Krasovskii functional (L-K-F) method allows to get information regarding the stability and a few other behaviors of solutions without any prior knowledge of solutions depending upon DDEs under study.

Thereby, based on the data from the present literature, it can be seen that functional differential equations (FDEs), in particular, DDEs of first and second order, have widely applications across a range of fields of sciences, medicine, engineering and so on. In these researches fields and some related ones, most of real problems can be modeled mathematically as FDEs, in particular cases, as first and second order DDEs, see, for examples, (Arino et al. [1], Azbelev et al. [2], Burton [3], Hale and Verduyn Lunel [9], Kolmanovskii and Myshkis ([10],[11]], Kolmanovskii and Nosov [12], Krasovskii [13], Kuang [14], Lakshmikantham et al. [15], Slyn'ko and Tunç [20], Smith [21]). Indeed, to reach analytical solutions of nonlinear DDEs explicitly is a hard task. However, from the past up to now, there have been done intensively works regarding the research of the behaviors of solutions of DDEs without finding solutions. The L-K-F approach is one of very effective methods to discuss the qualitative behaviors of solutions of FDEs without needing to find their explicitly solutions.

In the recent literature, the AS of the below systems of DDEs and their modified forms were investigated by many researchers (in particular, see, Chuan-Ke et al.([4], [5]), Ge et al. [7], Lee et al. [16], Li et al. ([17], [18]). Seuret and Gouaisbaut [19], Tian and Ren [20] and Wang et al. [35]) considered DDEs as follows:

$$\dot{\omega}(t) = A\omega(t) + B\omega(t - h(t)) \quad (1)$$

and

$$\dot{\omega}(t) = A\omega(t) + A_d\omega(t - d(t)) \quad (2)$$

depending upon the initial function

$$\omega(t) = \phi(t),\ t \in [-h_2, 0],$$

where $\omega(t)$ is the system state, and $h(t)$ and $d(t)$ are the time-varying retardations, and they also have lower and upper constant bounds. The proper reason to study these DDEs intensively may be their effective roles in applications of engineering.

In Chuan-Ke et al. ([4], [5]), Ge et al. ([6], [7]), Lee et al. [16], Li et al. ([17], [18]), Seuret and Gouaisbaut [19], Tian and Ren [20], Wang et al. [35], Wu and He [36], and Zevin [37], suitable L-K-Fs were defined to discuss the AS of the systems of DDEs (1), (2) and their modified forms. Based on the defined L-K-Fs, some interesting results that have sufficient conditions on the AS of the systems of DDEs (1), (2) and their modified forms were proved. In that works, it was also determined various different upper bounds for the constant and variable delays related to the considered FDEs. Indeed, additionally, in the papers (Slny'ko and Tunç, [20], Tunç ([23],[24],[25]), [26], [27], Tunç and Tunç ([28],[29],[30],[31]), it can also be found some interesting and recent results on the various qualitative features of different classes of various FDEs. This paper's inspiration came from recent publications of Chuan-Ke et al. ([4], [5]), Ge et al. [7], Lee et al. [16], Li et al. [17], Li et al. [18], Seuret and Gouaisbaut [19], Tian and Ren [22], Wang et al. [35] and that in the references of this study. In this study, we will think about the subsequent system of DDEs consisting of the time-varying retardation:

$$\dot{x}(t) = -A(t)F(x(t)) - G(X(t)) + CH(x(t - d(t))) + Q(t, x(t), x(t - d(t))), \quad (3)$$

where $x \in \mathbb{R}^n, t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, $d \in C^1(\mathbb{R}^+, (0, \infty))$ is the time-varying retardation, and $A \in C(\mathbb{R}^+, \mathbb{R}^{n\times n})$, $C \in \mathbb{R}^{n\times n}$, F, G, $H \in C(\mathbb{R}^n, \mathbb{R}^n)$, $F(0) = G(0) = H(0) = 0$ and $Q \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$. It is also supposed that

$$0 \le d_1 \le d(t) \le d_2,\ d_{12} = d_2 - d_1,$$
$$0 \le d'(t) \le d_0 < 1. \quad (4)$$

We now provide a brief overview of this paper's purpose.

1^0) We will study the system of DDEs (3) by three results, when $Q(\ldots) = 0$ and $Q(\ldots) \ne 0$. To search these problems, we define an alternative auxiliary functional, i.e., an L-K-F, which is different from that in Chuan-Ke et al. ([4], [5]), Ge et al. [7], Lee et al. [16], Li et al. [17], Li et al. [18], Seuret and Gouaisbaut [19], Tian and Ren [22] and Wang et al. [35]).

2^0) Firstly, we will study the AS and subsequently the integrable solutions of the following unmodified system of DDEs by Theorems 1, 2, respectively:

$$\dot{x}(t) = -A(t)F(x(t)) - G(X(t)) + CH(x(t - d(t))). \quad (5)$$

The results to be given here are new and original.

3^0) Secondly, we will search conditions for the bounded solutions of the modified system (3), i.e. when $Q(\ldots) \ne 0$, see, Theorem 3. This result is also new and original.

4^0) In particular cases of the systems of DDEs (3) and (5), new examples will be given to justify the applications of Theorems 1-3.

PRELIMINARIES

We take into account the below system:

$$\frac{dx}{dt} = Z(t, x_t), \quad (6)$$

where $Z \in C(\mathrm{R} \times C_0, \mathrm{R}^n)$, $Z(t, 0) = 0$, and it converts bounded sets into bounded sets. Suppose that for some $\tau > 0$, $C_0 = C_0([-\tau, 0], \mathbb{R}^n)$ is the space the functions $\phi : [-\tau, 0] \to \mathbb{R}^n$, which are continuous

Let $x \in C_0([t_0 - \tau, t_0 + a], \mathbb{R}^n)$, where $a \ge 0$, $a \in \mathrm{R}$, $x_t = x(t + \theta)$ for $-\tau \le \theta \le 0$ and $t \ge t_0$. Let $z \in \mathbb{R}^n$. The norm $\|.\|$ is described by $\|z\| = \sum_{i=1}^{n} |z_i|$. Next, assume that $U \in \mathbb{R}^{n\times n}$. Then, $\|U\|$ is described by $\|U\| = \max_{1\le j\le n}\left(\sum_{i=1}^{n} |u_{ij}|\right)$.

Throughout this study, let $x(t)$ denote x.

For any $\phi \in C_0$, assume that

$$\|\phi\|_{C_0} = \sup_{\theta\in[-r,0]} \|\phi(\theta)\| = \|\phi(\theta)\|_{[-r,0]}$$

and

$$C_H = \{\phi : \phi \in C_0 \text{ and } \|\phi\|_{C_0} \le H < \infty\}.$$

It is also assumed that the functional Z has the conditions of the uniqueness of solutions of (6). We note that the systems (1), (2), (3) and (5) are particular cases of the system (6). Let $\wp = [t_0 - \tau, t_0]$ and $x(t) = x(t, t_0, \phi)$ be a solution of (6), where $\phi \in C(\wp, \mathbb{R}^n)$, i.e., it is an initial function, and $x(t) = \phi(t)$, $\forall t \in \wp$.

Let

$$\Pi(t,\phi) : \mathbb{R}^+ \times C_H \to \mathbb{R}^+$$

be a continuous L-K-F for its arguments and $\Pi(t,0) = 0$. Additionally, let $\frac{d}{dt}\Pi(t, x_t)$ represent the derivative of this L-K-F along the system (6) on the right.

STABILITY AND INTEGRABILITY PROPERTIES

We take $Q(.) = Q(t, x(t), x(t - d(t))) = 0$ in DDEs (3). Let's consider the system of DDEs (5). We now establish generalizations and improve some results mentioned above under new and more suitable conditions. Our results will be confirmed by the L-K-F approach. The first new result, Theorem 1, studies the asymptotic stability.

Theorem 1. Assume that

$(C1)$ $a_0 > 0$, a_0 from $(C1)$, is such that

$$\sum_{j=1, j\ne i}^{n} |a_{ji}(t)| - a_{ii}(t) \le -a_0 \text{ for all } t \in \mathbb{R}^+;$$

$(C2)$ $f_0 > 0$, $g_0 > 0$, $h_0 > 0$ and $\rho_0 > 0$ are real constants such that

$$F(0) = 0,\ \|F(x)\| \ge f_0 \|x\|,$$
$$G(0) = 0,\ \|G(x)\| \ge g_0 \|x\|,$$
$$H(0) = 0,\ \|H(x)\| \le h_0 \|x\|,\ \forall x \in \mathbb{R}^n,$$
$$(a_0 f_0 + g_0)(1 - d_0) - h_0 \|C\| \ge \rho_0,$$

where the constant $d_0 > 0$ is from (4). Then, trivial solution of (5) is asymptotically stable.

Proof. Let $\Pi_1 := \Pi_1(t, x_t)$ be an L-K-F given by

$$\Pi(t, x_t) := \|x(t)\| + \upsilon \int_{t-d(t)}^{t} \|H(x(s))\| ds, \qquad (7)$$

in which $\upsilon \in \mathbb{R}$, $\upsilon > 0$. In fact, the constant υ is arbitrary and it will be chosen in the proof, after a few steps.

The L-K-F (7) is converted to the subsequent form:

$$\Pi(t, x_t) := \sum_{i=1}^{n} |x_i(t)| + \upsilon \sum_{i=1}^{n} \int_{t-h(t)}^{t} |h_i(x(s))| ds.$$

Based on the information above, it is derived from the L-K-F $\Pi(t, x_t)$ that

$$\Pi(t, 0) = 0,$$
$$\Pi(t, x_t) \ge \sum_{i=1}^{n} |x_i(t)| = \|x(t)\|.$$

Considering the auxiliary functional $\Pi(t, x_t)$ in (7) and employing (3), we find

$$\frac{d}{dt}\Pi(t, x_t) = \sum_{i=1}^{n} x_i'(t) \operatorname{sgn}\left(x_i(t+0)\right) + \upsilon \|H(x(t))\| - \upsilon(1 - d'(t)) \|H(x(t - d(t)))\|. \qquad (8)$$

As for the next step, by virtue of the first term of (8), conditions $(C1)$, $(C2)$ and elementary calculations, we get:

$$\sum_{i=1}^{n} \operatorname{sgn} x_i(t+0) x_i'(t) \le \sum_{i=1}^{n} \left(\sum_{j=1, j\ne i}^{n} |a_{ji}(t)| - a_{ii}(t) \right) |F_i(x(t))| - \|G(x(t))\| + \|C\| \ \|H(x(t - d(t)))\|$$
$$\le -a_0 f_0 \|x(t)\| - g_0 \|x(t)\| + \|C\| \ \|H(x(t - d(t)))\|.$$

Substituting the result of the inequality above into (8) and using the conditions $(C1)$, $(C2)$ and $0 \le d'(t) \le d_0 < 1$, we derive:

$$\frac{d}{dt}\Pi(t,x_t) \le -a_0 f_0 \|x(t)\| - g_0\|x(t)\| + \|C\|\,\|H(x(t-d(t)))\| + \upsilon\|H(x(t))\| - \upsilon(1-d_0)\|H(x(t-d(t)))\|.$$

Choosing υ as $\upsilon = (1-d_0)^{-1}\|C\|$, we get

$$\begin{aligned}\frac{d}{dt}\Pi(t,x_t) &\le -[a_0 f_0 + g_0]\|x(t)\| + (1-d_0)^{-1}h_0\|C\|\,\|H(x(t))\|\\ &\le -\left[a_0 f_0 + g_0 - (1-d_0)^{-1}h_0\|C\|\right]\|x(t)\|\\ &= -(1-d_0)^{-1} \times\left[(a_0 f_0 + g_0)(1-d_0) - h_0\|C\|\right]\|x(t)\|.\end{aligned}$$

Using the condition $(C2)$, we reach that

$$\frac{d}{dt}\Pi(t,x_t) \le -(1-d_0)^{-1}\rho_0\|x(t)\| \le 0. \qquad (9)$$

Thus, $\frac{d}{dt}\Pi(t,x_t)$ is negative definite. Hence, the proof of the result of the AS is finished, (see, [5]).
The second new result, Theorem 2, studies integrability of solutions.

Theorem 2. Let conditions $(C1)$ and $(C2)$ be fulfilled. Then, the solutions of (5) satisfy

$$\int_{t_0}^{\infty}\|x(s)\|ds \le I < \infty,$$

where $I = (1-d_0)\rho_0^{-1}\Pi_0 > 0$, and $(1-d_0)\rho_0^{-1}\Pi_0 \in \mathbb{R}$.

Proof. Take into account the L-K-F $\Pi(t,x_t)$. Next, according to the conditions $(C1)$ and $(C2)$, it is obvious that $\frac{d}{dt}\Pi(t,x_t) \le -(1-d_0)^{-1}\rho_0\|x(t)\| < 0.$

The inequality above confirms that the L-K-F $\Pi(t,x_t)$ is decreasing.
Subsequently, integrating the above inequality, we get:

$$\int_{t_0}^{t}\|x(s)\|(1-d_0)^{-1}\rho_0 ds \le \Pi(t_0,\phi(t_0)),$$
$$t \ge t_0 \ge 0.$$

Let $\Pi(t_0,\phi(t_0)) = \Pi_0 \ge 0$, $\Pi_0 \in \mathbb{R}$.

Based on the above results, we have

$$\int_{t_0}^{t}\|x(s)\|ds \le (1-d_0)\rho_0^{-1}\Pi_0.$$

Let $t \to +\infty$. Then,

$$\int_{t_0}^{\infty}\|x(s)\|ds \le (1-d_0)\rho_0^{-1}\Pi_0 < \infty.$$

Because of this reason, the solutions of (5) are integrable. It should be noted that here integarbility is taken in meaning of Lebesgue on $\mathbb{R}^+$. Thereby, Theorem 2's proof was completed.

Example 1. We take into account to the beneath two dimensional system of DDEs with a variable delay:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = -\begin{pmatrix} 25+(1+t)^{-1} & -(1+t)^{-1} \\ -(1+t)^{-1} & 25+(1+t)^{-1}\end{pmatrix} \times \begin{pmatrix} 2|x_1| + x_1^4 \\ 2|x_2| + x_2^4 \end{pmatrix} - \begin{pmatrix} |x_1| + \frac{|x_1|}{1+\exp(x_1^2)} \\ |x_2| + \frac{|x_2|}{1+\exp(x_2^2)} \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1(t - \frac{1}{\pi}arctgt) \\ x_2(t-\frac{1}{\pi}arctgt)\end{pmatrix}, \qquad (10)$$

where $d(t) = \frac{1}{\pi}arctgt$ is the delay function and $t \ge 2^{-1}$.
As the next step, from the comparison of the systems of DDEs (10) and DDEs (5), we find the following relations:

$$A(t) = \begin{pmatrix} 25+(1+t)^{-1} & (1+t)^{-1} \\ (1+t)^{-1} & 25+(1+t)^{-1}\end{pmatrix},$$

$$-a_{ii}(t) + \sum_{j=1,\, j\ne i}^{n}|a_{ji}(t)| = -25 < -24 = -a_0$$

thanks to

$$a_{11}(t)+|a_{21}(t)|$$
$$=-25-\frac{1}{1+t}+\frac{1}{1+t}=-25<-24=-a_0,$$
$$a_{22}(t)+|a_{12}(t)|$$
$$=-25-\frac{1}{1+t}+\frac{1}{1+t}=-25<-24=-a_0;$$
$$F(x)=F(x_1,x_2)=\begin{pmatrix}2|x_1|+x_1^4\\2|x_2|+x_2^4\end{pmatrix},\ F(0)=0,$$
$$\|F(x)\|=\|F(x_1,x_2)\|=\left\|\begin{pmatrix}2|x_1|+x_1^4\\2|x_2|+x_2^4\end{pmatrix}\right\|$$
$$=2|x_1|+x_1^4+2|x_2|+x_2^4$$
$$\geq 2\|x\|,\ f_0=2;$$
$$G(x)=G(x_1,x_2)=\begin{pmatrix}|x_1|+\frac{|x_1|}{1+\exp(x_1^2)}\\|x_2|+\frac{|x_2|}{1+\exp(x_2^2)}\end{pmatrix},$$
$$\|G(x)\|=\|G(x_1,x_2)\|=\left\|\begin{pmatrix}|x_1|+\frac{|x_1|}{1+\exp(x_1^2)}\\|x_2|+\frac{|x_2|}{1+\exp(x_2^2)}\end{pmatrix}\right\|$$
$$=|x_1|+\frac{|x_1|}{1+\exp(x_1^2)}+|x_2|+\frac{|x_2|}{1+\exp(x_2^2)}$$
$$\geq |x_1|+|x_2|=\|x\|,\ g_0=2;$$
$$C=\begin{pmatrix}3&1\\1&3\end{pmatrix},\ \|C\|=4;$$
$$H(x_1(t),x_2(t))=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix},\ H(0)=0,$$
$$\|H(x)\|=\|H(x_1(t),x_2(t))\|=\left\|\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}\right\|$$
$$=|x_1|+|x_2|=\|x\|,\ h_0=1;$$
$$d(t)=\frac{1}{\pi}arctgt,$$
$$0\leq d(t)=\frac{1}{\pi}arctgt\leq\frac{1}{2},$$
$$0\leq d'(t)=\frac{1}{\pi(1+t^2)}\leq\frac{1}{\pi}=d_0<1;$$
$$(a_0f_0+g_0)(1-d_0)-h_0\|C\|$$
$$=(48+2)(1-\pi^{-1})-4=46-50\pi^{-1}>0.$$

According to the results above, the conditions $(C1)$ and $(C2)$ of the two proved results, i.e., Theorem 1 and Theorem 2, have been satisfied. Thanks to this fact, the solution $(x_1(t),x_2(t))=(0,0)$ of (10) is asymptotically stable, additionally, the solutions are integrable depending upon their norm.

BOUNDEDNESS

The third new result, Theorem 3, studies boundedness of solutions.
As for the bounded solutions of the system (3), we need the following conditions, which are modified forms of conditions in $(C2)$:

$(C3)$
$$F(0)=0,\ \|F(x)\|\geq f_0\|x\|,$$
$$G(0)=0,\ \|G(x)\|\geq g_0\|x\|,$$
$$H(0)=0,\ \|H(x)\|\leq h_0\|x\|,\ \forall x\in\mathbb{R}^n,$$
$$\|Q(t,x(t),x(t-d(t)))\|\leq|q_0(t)|\ \|x(t)\|$$

and

$$\left[(a_0f_0+g_0)(1-d_0)-h_0\|C\|-(1-d_0)|q_0(t)|\right]\geq 0,$$
$$\forall t\in\mathrm{R}^+,\ \forall x,\ x(t-d(t))\in\mathbb{R}^n,$$

where the constants $d_0>0$ is from (4), a_0 is from $(C1)$, $f_0>0,\ g_0>0,\ h_0>0,\ \rho_0>0$ are from $(C2)$, respectively, and $q_0\in C(\mathbb{R},\mathbb{R})$.

Theorem 3. Let the conditions $(C1)$ and $(C3)$ be fulfilled. Then the solutions of (3) are bounded.

Proof . Like the same as before, the L-K-F $\Pi(t,x_t)$ is the main tool here for the proof. The conditions $(C1)$ and $(C3)$ lead that

$$\frac{d}{dt}\Pi(t,x_t)\leq$$

$$-(1-d_0)^{-1}\left[(a_0 f_0 + g_0)(1-d_0) - h_0\|C\|\right]\times\|x(t)\|$$
$$+\|Q(t,x(t),x(t-d(t)))\|$$
$$\le -(1-d_0)^{-1}\left[(a_0 f_0 + g_0)(1-d_0) - h_0\|C\| - (1-d_0)|q_0(t)|\right]\|x(t)\|.$$

Hence, according to the condition $(C3)$, it is evident that

$$\frac{d}{dt}\Pi(t,x_t) \le 0. \tag{11}$$

Integrating, we obtain

$$\Pi(t,x_t) \le \Pi(t_0,\phi(t_0)) \equiv \Pi_0 > 0,\ \Pi_0 \in \mathbb{R}.$$

By the virtue of (11) and the auxiliary functional $\Pi(t,x_t)$, it follows that

$$\|x(t)\| \le (1-d_0)^{-1}\|C\|\int_{t-d(t)}^{t}\|F(x(s))\|ds + \|x(t)\|$$
$$= \Pi(t,x_t) \le \Pi(t_0,\phi(t_0)) \equiv \Pi_0 > 0.$$

Thereby, in the subsequent, we have

$$\|x(t)\| \le \Pi_0.$$

Next, it is obvious that

$$\lim_{t\to+\infty}\|\|x(t)\|\| \le \lim_{t\to+\infty}\Pi_0 = \Pi_0.$$

Then, the boundedness of solutions of (3) is observed depending upon $t\to+\infty$. Theorem 3's proof was completed.

Example 2. We have the following modified system:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = -\begin{pmatrix} 25-\frac{1}{1+t} & -\frac{1}{1+t} \\ -\frac{1}{1+t} & 25-\frac{1}{1+t} \end{pmatrix}$$
$$\times\begin{pmatrix} 2|x_1| + x_1^4 \\ 2|x_2| + x_2^4 \end{pmatrix}$$
$$-\begin{pmatrix} |x_1| + \frac{|x_1|}{1+\exp(x_1^2)} \\ |x_2| + \frac{|x_2|}{1+\exp(x_2^2)} \end{pmatrix}$$
$$+\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1(t-\frac{1}{\pi}arctgt) \\ x_2(t-\frac{1}{\pi}arctgt) \end{pmatrix}$$
$$+\begin{pmatrix} \frac{\exp(t)\sin x_1}{1+\exp(2t)+x_1^4(t-\frac{1}{\pi}arctgt)} \\ \frac{\exp(t)\sin x_2}{1+\exp(2t)+x_2^4(t-\frac{1}{\pi}arctgt)} \end{pmatrix}, \tag{12}$$

where $d(t) = \frac{1}{\pi}arctgt$ is time-varying retardation and $t \ge 2^{-1}$.

If we compare the systems (12) and (3), then it is evident that functions $A(t)$, F, G, H and the matrix C are the same as in Example 1. We do not need to show the calculations related to these functions again. Hence, we consider the subsequent function:

$$Q(t,x(t),x(t-\frac{1}{\pi}|\arctan(t)|))$$
$$= \begin{pmatrix} \frac{\exp(t)\sin x_1}{1+\exp(2t)+x_1^4(t-\frac{1}{\pi}arctgt)} \\ \frac{\exp(t)\sin x_2}{1+\exp(2t)+x_2^4(t-\frac{1}{\pi}arctgt)} \end{pmatrix}.$$

Then,

$$\left\|Q(t,x(t),x(t-\frac{1}{\pi}|\arctan(t)|))\right\|$$
$$= \left\|\begin{pmatrix} \frac{\exp(t)\sin x_1}{1+\exp(2t)+x_1^4(t-\frac{1}{\pi}arctgt)} \\ \frac{\exp(t)\sin x_2}{1+\exp(2t)+x_2^4(t-\frac{1}{\pi}arctgt)} \end{pmatrix}\right\|$$
$$\le (1+\exp(2t))^{-1}\left[\exp(t)|\sin x_1| + \exp(t)|\sin x_2|\right]$$
$$\le \exp(t)(1+\exp(2t))^{-1}\left[|x_1|+|x_2|\right] = |q_0(t)|\|x\|,$$

where

$$\left|q_0(t)\right| = \exp(t)\left(1+\exp(2t)\right)^{-1}, \ \left\|x\right\| = \left|x_1\right| + \left|x_2\right|.$$

By the virtue of Example 1 and the discussion above, we obtain

$$\left[(a_0 f_0 + g_0)(1-d_0) - h_0\left\|C\right\| - (1-d_0)\left|q_0(t)\right|\right]$$
$$= (48+2)(1-\pi^{-1}) - 4 - (1-\pi^{-1})\frac{\exp(t)}{1+\exp(2t)}$$
$$= 46 - 50\pi^{-1} - (1-\pi^{-1}) = 45 - 49\pi^{-1} > 0.$$

Thereby**,** the conditions $(C1)$ and $(C3)$ of Theorem 3 are satisfied. Therefore, we can reach that the solutions of (12) are bounded depending upon $t \to \infty$**.**

CONCLUSION

In this study, the unperturbed and perturbed systems of DDEs (3) and (5) were considered. We proved there new results, Theorems 1-3, on the behaviors of solutions of the non-perturbed system (3) as well as the perturbed system (5). These theorems, Theorems 1-3, include new sufficient conditions. As the main tool in the proofs, a Lyapunov- Krasovskiĭ type functional was constructed and used to prove Theorems 1-3. As for the applications of the results of this study, two examples were given to show applications of Theorems 1-3. The new outcomes of this study have new contributions to the qualitative theory of DDEs, and they extend and enhance a few related results in the past literature.

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